The symplectic duality of Hermitian symmetric spaces [*†]

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The unit disc $\Delta \subset \mathbb{C}$ (I).

The unit disc $\Delta = \{z \in \mathbb{C} : |z| < 1\}$ has two well-known symplectic forms ω_0 and ω_{hyp}:

$$\omega_0 = \frac{\mathrm{i}}{2} \, \mathrm{d}\, z \wedge \mathrm{d}\, \overline{z},$$
$$\omega_{\text{hyp}} = \frac{\omega_0}{(1 - |z|^2)^2} \; .$$

The plane $\mathbb{C}$ has also two symplectic forms. Namely,

$$\omega_0 = \frac{\mathrm{i}}{2} \, \mathrm{d}\, z \wedge \mathrm{d}\, \overline{z},$$
$$\omega_{\text{FS}} = \frac{\omega_0}{(1 + |z|^2)^2} \; .$$

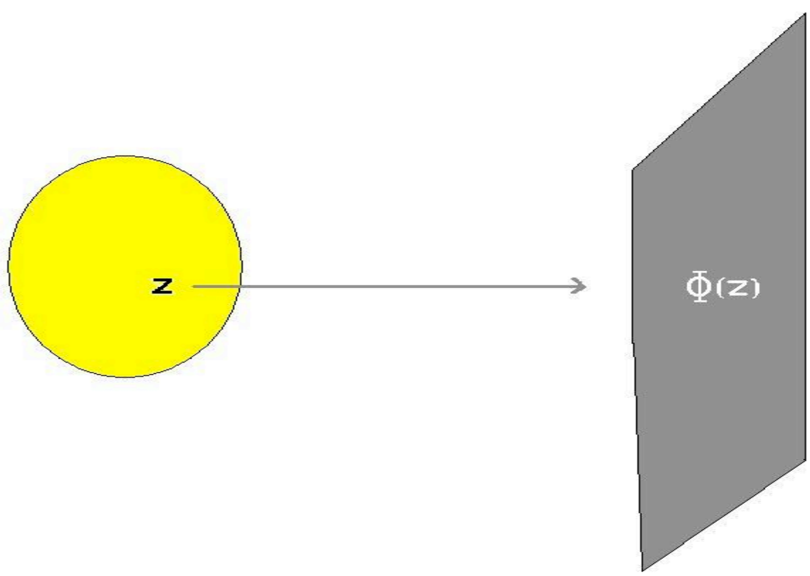
Actually, the Fubini-Study form ω_{FS} on $\mathbb{C}$ comes from the standard embedding $\mathbb{C} \subset \mathbb{C}P^1$, i.e. $z \hookrightarrow (z : 1)$.

Notice that $(\mathbb{C}P^1, \omega_{\text{FS}})$ is the compact dual of the unit disc $(\Delta, \omega_{\text{hyp}})$.

The unit disc $\Delta \subset \mathbb{C}$ (II).

Consider the map $\Phi : \Delta \to \mathbb{C}$ given by

$$\Phi(z) := \frac{z}{\sqrt{1 - |z|^2}}$$



We claim that: $\begin{cases} \Phi^* \omega_0 = \omega_{\mathrm{hyp}}, \\ \Phi^* \omega_{\mathrm{FS}} = \omega_0. \end{cases}$

A map Φ with the above properties is called a <u>bisymplectomorphism</u> of $(\Delta, \omega_{\mathrm{hyp}}, \omega_0)$ and $(\mathbb{C}, \omega_0, \omega_{\mathrm{FS}})$.

The unit disc $\Delta \in \mathbb{C}$ (III).

What about the uniqueness of the bisymplectomorphism Φ ?

Let $\Psi : \Delta \to \mathbb{C}$ be another bisymplectomorphism, i.e.

$$\begin{cases} \Psi^* \omega_0 = \omega_{\text{hyp}}, \\ \Psi^* \omega_{\text{FS}} = \omega_0. \end{cases}$$

Then the composition $f := \Phi^{-1} \circ \Psi$ is a bisymplectomorphism of $(\Delta, \omega_0, \omega_{\text{hyp}})$, i.e.

$$f^*(\omega_0) = \omega_0$$
$$f^*(\omega_{\text{hyp}}) = \omega_{\text{hyp}}$$

So we can introduce the group $\mathcal{B}(\Delta)$ of bisymplectomorphisms of the disc $(\Delta, \omega_0, \omega_{\text{hyp}})$.

Thus, the map Φ is unique up to elements of $\mathcal{B}(\Delta)$.

The unit disc $\Delta \subset \mathbb{C}$ (IV).

The following theorem gives a description of $\mathcal{B}(\Delta)$.

Theorem 0.1. *The elements $f \in \mathcal{B}(\Delta)$ are the maps defined by*

$$f(z) = u\left(|z|^2\right) z \qquad (z \in \Delta)\,,$$

where u is a smooth function $u : [0,1) \to S^1 \simeq U(1)$.

In other words, the restriction of a bisymplectomorphism $f \in \mathcal{B}(\Delta)$ to a circle of radius r $(0 < r < 1)$ is the rotation $u\left(r^2\right)$.

Notice that if $f \in \mathcal{B}(\Delta)$ then $f(0) = 0$.

The unit disc $\Delta \in \mathbb{C}$ (Proof II).

<u>*Sketch of the Proof of Theorem 0.1 :*</u>

It is not difficult to show that the maps $f(z) = u\left(|z|^2\right) z$, where u is a smooth function $u : [0,1) \to S^1 \simeq U(1)$ are bisymplectomorphisms.

Conversely, assume now that f is a bisymplectomorphism.

- Since f preserves both symplectic forms then f preserves the quotient $\frac{\omega_0}{\omega_{hyp}} = (1-|z|^2)^2$. Thus,

  $$|f(z)| = |z|$$

  for $z \in \Delta$.

- A simple computation shows that $f(z) = v(|z|)z$ for $z \in \Delta \setminus \{0\}$ and $v : (0,1) \to U(1)$ smooth.

- A Whitney's Theorem can be used to show that $v(|z|) = u\left(|z|^2\right)$ for a smooth u. $\square$

The unit disc $\Delta \in \mathbb{C}$ (Proof I).

To prove that $\Phi^*(\omega_0) = \omega_{\text{hyp}}$ notice that:

$$\mathrm{d}\,\Phi - \mathrm{d}((1-|z|^2)^{-1/2})z = (1-|z|^2)^{-1/2}\ \mathrm{d}\,z.$$

So

$$\overline{\Phi}(\mathrm{d}\,\Phi - \mathrm{d}((1-|z|^2)^{-1/2}).z) = (1-|z|^2)^{-1}\overline{z}\,\mathrm{d}\,z\,.$$

then

$$-\frac{\mathrm{i}}{2}\,\mathrm{d}\,\Phi \wedge \mathrm{d}\,\overline{\Phi} = -\frac{\mathrm{i}}{2}\,\mathrm{d}(\overline{\Phi}(\mathrm{d}\,\Phi - \mathrm{d}((1-|z|^2)^{-1/2}).z)) =$$

$$= -\frac{\mathrm{i}}{2}\,\mathrm{d}((1-|z|^2)^{-1}\overline{z}\,\mathrm{d}\,z) = \omega_{\text{hyp}}\ ,$$

since $\overline{\Phi}\,\mathrm{d}((1-|z|^2)^{-1/2})z = (1-|z|^2)^{-1/2}\,\mathrm{d}((1-|z|^2)^{-1/2})|z|^2$ is exact.

Thus, we get

$$\omega_{\text{hyp}} = \Phi^*(\omega_0)\ .$$

The proof that $\Phi^*(\omega_{\text{FS}}) = \omega_0$ is similar.

The Cartan's domain $D_1[n]$ (I).

$D_1[n] \subset \mathbb{C}^{n^2} \cong M_n(\mathbb{C})$ is given by

$$D_1[n] := \{Z \in M_n(\mathbb{C}) \,|\, I_n - ZZ^* >> 0\}.$$

So $D_1[n]$ has two standard symplectic forms ω_0 and ω_{hyp} given by:

$$\omega_0 = \frac{\mathrm{i}}{2}\,\mathrm{d}\,Z \wedge \mathrm{d}\,\overline{Z},$$

$$\omega_{\text{hyp}} = -\frac{\mathrm{i}}{2}\partial\overline{\partial}\,\log\det(I_n - ZZ^*).$$

The complex euclidean space $\mathbb{C}^{n^2} \cong M_n(\mathbb{C})$ has two symplectic forms:

$$\omega_0 = \frac{\mathrm{i}}{2}\,\mathrm{d}\,Z \wedge \mathrm{d}\,\overline{Z},$$

$$\omega_{\text{FS}} = \frac{\mathrm{i}}{2}\partial\overline{\partial}\,\log\det(I_n + ZZ^*).$$

The Cartan's domain $D_1[n]$ (II).

Notice that

$$D_1[n] \subset \mathbb{C}^{n^2} \subset G_n(\mathbb{C}^{2n}) \hookrightarrow \mathbb{C}P^N \ .$$

The last arrow is the Plücker embedding

$$G_n(\mathbb{C}^{2n}) \hookrightarrow \mathbb{C}P^N \ ,$$

where $N = \begin{pmatrix} 2n \\ n \end{pmatrix} - 1$ and $G_n(\mathbb{C}^{2n})$ is the complex Grassmannian of complex n subspaces of $\mathbb{C}^{2n}$.

Notice that $G_n(\mathbb{C}^{2n})$ is the compact dual of $D_1[n]$.

Indeed, the form ω_{FS} on $\mathbb{C}^{n^2}$ comes as the pullback form of $(\mathbb{C}P^N \, , \, \omega_{FS})$ via the above embedding.

The Cartan's domain $D_1[n]$ (III).

Now we can ask the following two questions:

- Do there exist a bisymplectomorphism

  $$\Phi : (D_1[n], \omega_0, \omega_{\mathrm{hyp}}) \to (\mathbb{C}^{n^2}, \omega_{\mathrm{FS}}, \omega_0) \ , i.e.$$

  a diffeomorphism $\Phi : D_1[n] \to \mathbb{C}^{n^2}$ such that:

  $$\Phi^*(\omega_0) = \omega_{\mathrm{hyp}},$$

  $$\Phi^*(\omega_{\mathrm{FS}}) = \omega_0 \ ?$$

- It is possible to describe the group $\mathcal{B}(D_1[n])$ of diffeomorphisms f of $D_1[n]$ such that:

  $$f^*(\omega_0) = \omega_0$$

  $$f^*(\omega_{\mathrm{hyp}}) = \omega_{\mathrm{hyp}} \ ?$$

The Cartan's domain $D_1[n]$ (IV).

Claim: The map $\Phi : D_1[n] \to \mathbb{C}^{n^2} \cong M_n(\mathbb{C})$ given by

$$\Phi(Z) := (I_n - ZZ^*)^{-1/2} Z$$

is a bisymplectomorphism. That is to say, Φ is a diffeomorphism and :

$$\Phi^*(\omega_0) = \omega_{\text{hyp}},$$

$$\Phi^*(\omega_{\text{FS}}) = \omega_0 \ .$$

The Cartan's domain $D_1[n]$ (V).

First of all observe that we can write

$$\omega_{\text{hyp}} = -\frac{\mathrm{i}}{2}\partial\,\overline{\partial}\log\det(I_n - ZZ^*) = \frac{\mathrm{i}}{2}\,\mathrm{d}\,\partial\log\det(I_n - ZZ^*) =$$

$$= \frac{\mathrm{i}}{2}\,\mathrm{d}\ \partial\ \mathrm{tr}\ \log(I_n - ZZ^*) = \frac{\mathrm{i}}{2}\,\mathrm{d}\ \mathrm{tr}\ \partial\log(I_n - ZZ^*) =$$

$$= -\frac{\mathrm{i}}{2}\ \mathrm{d}\ \mathrm{tr}[Z^*(I_n - ZZ^*)^{-1}\,\mathrm{d}\,Z],$$

where we use the decomposition $\mathrm{d} = \partial + \bar{\partial}$ and the identity $\log\det A = \mathrm{tr}\ \log A$.

By substituting $X = (I_n - ZZ^*)^{-\frac{1}{2}}Z$ in the last expression one gets:

$$-\frac{\mathrm{i}}{2}\,\mathrm{d}\,\mathrm{tr}[Z^*(I_n - ZZ^*)^{-1}\,\mathrm{d}\,Z] =$$

$$= -\frac{\mathrm{i}}{2}\,\mathrm{d}\,\mathrm{tr}(X^*dX) + \frac{\mathrm{i}}{2}\,\mathrm{d}\,\mathrm{tr}\{X^*d[(I_n - ZZ^*)^{-\frac{1}{2}}]Z\ .$$

Finally, notice that the 1-form

$$\mathrm{tr}[X^*\,\mathrm{d}(I_n - ZZ^*)^{-\frac{1}{2}}Z]$$

is exact being equal to $\mathrm{d}\ \mathrm{tr}(\frac{C^2}{2} - \log C)$, where $C = (I_n - ZZ^*)^{-\frac{1}{2}}$.

So $\Phi^*(\omega_0) = \omega_{\text{hyp}}$.
The proof that $\Phi^*(\omega_{\text{FS}}) = \omega_0$ is similar.

The general picture (I).

Let Ω be a symmetric bounded domain and let Ω^* be its compact dual. Assume $dim_{\mathbb{C}}(\Omega) = n$.

The following inclusions are well-known:

$$\Omega \subset \mathbb{C}^n \subset \Omega^* \hookrightarrow \mathbb{C}P^N ,$$

where the last arrow is the Borel-Weil embedding.

So the compact dual Ω^* and $\mathbb{C}^n$ can be endowed with the pullback form of the Fubini-Study form ω_{FS} of $\mathbb{C}P^N$.

Thus, we can regard $\mathbb{C}^n$ as a complex euclidean space equipped with two symplectic forms ω_0 and ω_{FS}.

The general picture (II).

We can ask about the existence and uniqueness of a symplectic duality map Φ. Namely,

- Do there exist a bisymplectomorphism

  $$\Phi : (\Omega, \omega_0, \omega_{\mathrm{hyp}}) \to (\mathbb{C}^n, \omega_{\mathrm{FS}}, \omega_0) \ \ , i.e.$$

  a diffeomorphism $\Phi : \Omega \to \mathbb{C}^n$ such that:

  $$\Phi^*(\omega_0) = \omega_{\mathrm{hyp}},$$

  $$\Phi^*(\omega_{\mathrm{FS}}) = \omega_0 \ \ ?$$

- It is possible to describe the group $\mathcal{B}(\Omega)$ of diffeomorphisms f of Ω such that:

  $$f^*(\omega_0) = \omega_0$$

  $$f^*(\omega_{\mathrm{hyp}}) = \omega_{\mathrm{hyp}} \ \ ?$$

Related results (I).

The existence of a symplectomorphism:

$$\psi : (\Omega, \omega_{\text{hyp}}) \to (\mathbb{C}^n, \omega_0)$$

was proved by D. McDuff in *The symplectic structure of Kähler manifolds of non-positive curvature* , J. Diff. Geometry 28 (1988), pp. 467-475.

As a conclusion it follows that the symplectic struture ω_{hyp} on $\mathbb{R}^{2n}$ is not <u>exotic</u>.

Related results (II).

- Notice that our question is stronger. Namely, we ask about the existence of a BISYMPLECTOMORPHISM , i.e. :

$$\Phi^*(\omega_0) = \omega_{\text{hyp}},$$

$$\Phi^*(\omega_{\text{FS}}) = \omega_0 \ \ ?$$

- Observe that McDuff's theorem is *existencial* ,i.e. there is not given an explicit symplectomorphism.

  Actually, we are going to give an explicit formula for our bisymplectomorphism Φ.

  Moreover, we are going to give an explicit description of all bysimplectomorphism Φ's.

Bounded Symmetric Domains and Hermitian Jordan Triple systems (I).

We use the approach "via" Jordan Algebras, due to Max Koecher, to construct all the symmetric bounded domains $\Omega \subset \mathbb{C}^n$ by starting with a **Hermitian Positive Jordan Triple System** $(V, \{,,\})$:

- $V = \mathbb{C}^n$ and $\{,,\} : V^3 \to V$,
- $\{x, y, z\}$ is $\mathbb{C}$-bilinear in (x, z) and $\mathbb{C}$-anti-linear in y.
- satisfying the *Jordan identity :*

$$\{x, y, \{u, v, w\}\} - \{u, v, \{x, y, w\}\} =$$
$$= \{\{x, y, u\}, v, w\} - \{u, \{v, x, y\}, w\}.$$

- the sesquilinear form $(x \,|\, y) := traceD(x, y)$ is positive, where $D(x, y)(\cdot) := \{x, y, \cdot\}$.

Bounded Symmetric Domains and Hermitian Jordan Triple systems (II).

Each element $x \in V$ has a *spectral decomposition* :

$$x = \lambda_1 c_1 + \lambda_2 c_2 + \cdots + \lambda_r c_r \ ,$$

where $\lambda_1 \geq \lambda_2 \geq \cdots \geq 0$ and $(c_1, c_2, \cdots, c_r)$ is a *frame* a maximal system of mutually orthogonal tripotents, i.e. $\{c_i, c_i, c_i\} = 0$ and $D(c_i, c_j) = 0$ for $i \neq j$. Unique just for elements $x \in V$ of maximal rank r.

There exist polynomials $m_1, \ldots, m_r$ on $\mathcal{M} \times \overline{\mathcal{M}}$, homogeneous of respective bidegrees $(1,1), \ldots, (r,r)$, such that for $x \in \mathcal{M}$, the polynomial

$$m(T, x, y) = T^r - m_1(x, y)T^{r-1} + \cdots + (-1)^r m_r(x, y)$$

satisfies

$$m(T, x, x) = \prod_{i=1}^{r} (T - \lambda_i^2),$$

where x is the spectral decomposition of $x = \sum \lambda_j c_j$.
The inohomogeneous polynomial

$$N(x, y) = m(1, x, y)$$

is called the *generic norm.*

Bounded Symmetric Domains and Hermitian Jordan Triple systems (III).

Construction of the bounded domain Ω.

- The **Spectral Norm** $|z|$ of $z \in V$ is defined as

$$|z|^2 := \frac{\|D(z,z)\|}{2}$$

where $\|\cdot\|$ is the operator norm in V endowed with $(\cdot|\cdot)$.

- The bounded domain attached to the HPJTS $(V, \{,,\})$ is given by:

$$\Omega := \{z \in V : \ |z| < 1\} .$$

That is to say, Ω is the unit sphere w.r.t. the Spectral Norm .

Bounded Symmetric Domains and Hermitian Jordan Triple systems (IV).

Construction of the symplectic forms $\omega_0, \omega_{\text{hyp}}$ of Ω:

- Here is the hyperbolic form ω_{hyp} of Ω:

$$\omega_{\text{hyp}} := -\frac{\mathrm{i}}{2}\partial\,\overline{\partial}\log\mathcal{N}(z)\ ,$$

where $\mathcal{N}(z) = N(z, z)$.

- Here is the flat form ω_0 of Ω:

$$\omega_0 := \frac{\mathrm{i}}{2}\partial\,\overline{\partial}\,m_1(z, z)\ \ .$$

Remark: If Ω is irreducible, (i.e. $(V, \{,,\})$ is simple), then:

- $\omega_{\text{hyp}} = \frac{\omega_{Berg}}{g}$,
- $\omega_0 = \frac{\frac{\mathrm{i}}{2}\partial\,\overline{\partial}\ trD(z,z)}{g} = \frac{\frac{\mathrm{i}}{2}\partial\,\overline{\partial}\ (z\,|\,z)}{g}$.

The number g above is called the *genus* of the bounded domain and is a natural number, e.g. $g = 2$ for the unit disc.

The symplectic duality (I).

The *Bergman operator* $\mathrm{B}(u,v) : V \to V$ is given by

$$\mathrm{B}(u,v) := \mathrm{id} - D(u,v) + Q(u)Q(v) \ ,$$

where $2Q(u)(v) := \{u, v, u\}$.

Let us introduced a map called Φ as follows:

$$\Phi : \Omega \to V, \ z \mapsto \mathrm{B}(z,z)^{-\frac{1}{4}} z \ ,$$

The map Φ is a (real analytic) *diffeomorphism* and its inverse Φ^{-1} is given by:

$$\Phi^{-1} : V \to \Omega, \ z \mapsto \mathrm{B}(z,-z)^{-\frac{1}{4}} z \, ;$$

The symplectic duality (II).

<u>**Theorem I :**</u> *The diffeomorphism* $\Phi(z) = \mathrm{B}(z,z)^{-\frac{1}{4}}z$ *is a bisymplectomorphism of* $(\Omega, \omega_{hyp}, \omega_0)$ *and* $(V, \omega_0, \omega_{FS})$. *That is to say:*

$$\Phi^*(\omega_0) = \omega_{hyp} \ ;$$

$$\Phi^*(\omega_{FS}) = \omega_0 \ ;$$

Remark : When $\Omega = D_1[n]$ then the above map Φ agree with the map $Z \mapsto (I_n - ZZ^*)^{-1/2}Z$ given in the previous example.

The symplectic duality (III).

Moreover, Φ has also the following properties:

(H) The map Φ is *hereditary* in the following sense: for any $(\Omega', 0) \stackrel{i}{\hookrightarrow} (\Omega, 0)$ complex and totally geodesic embedded submanifold $(\Omega', 0)$ through the origin 0, i.e. $i(0) = 0$ one has:

$$\Phi_{|_{\Omega'}} = \Phi.$$

Moreover

$$\Phi(\Omega') = V' \subset V,$$

where V' is the Hermitian positive Jordan triple system associated to Ω';

(I) Φ is a (non-linear) *interwining* map w.r.t. the action of the isotropy group $K \subset \mathrm{Iso}(\Omega)$ at the origin, where $\mathrm{Iso}(\Omega)$ is the group of isometries of Ω, i.e. for every $\tau \in K$

$$\Phi \circ \tau = \tau \circ \Phi;$$

About the uniqueness of Φ.

Definition 0.2. *A* bisymplectomorphism *of Ω is a diffeomorphism $f : \Omega \to \Omega$ which satisfies*

$$f^*\omega_0 = \omega_0 \ ,$$

$$f^*\omega_{hyp} = \omega_{hyp} \, .$$

That is to say, f preserves both symplectic forms ω_0 and ω_{hyp}.

Denote with $\mathcal{B}(\Omega)$ the group of bisymplectomorphisms of the bounded domain Ω.

Notice that the bisymplectomorphism

$$\Phi : (\Omega, \omega_{\text{hyp}}, \omega_0) \to (V, \omega_0, \omega_{FS})$$

is unique up to elements of $\mathcal{B}(\Omega)$.

The group of bisymplectomorphisms.

A bisymplectomorphism $f \in \mathcal{B}(\Omega)$ can be described by using the Bergman operator $\mathrm{B}(z) := \mathrm{B}(z, z)$. Namely,

Proposition 0.3. *Let Ω be a bounded domain. Then a diffeomorphism $f \in \mathrm{Diff}(\Omega)$ is a bisymplectomorphism if and only if it satisfies:*

- $f^*\omega_0 = \omega_0,$
- $\mathrm{B}\left(f(z)\right) \circ \mathrm{d}\, f(z) = \mathrm{d}\, f(z) \circ \mathrm{B}(z) \qquad (z \in \Omega).$

Notice that the second condition means that f preserves the Bergman operator $\mathrm{B}(z)$, i.e. $f^*\,\mathrm{B} = \mathrm{B}$

The rank one case (I).

Let $D_n \subset \mathbb{C}^n$ be the open unit ball of the standard Hermitian space $\mathbb{C}^n$, with Hermitian scalar product

$$(z \mid t) = \sum_{j=1}^{n} z_j \bar{t}_j$$

and associated norm $|z|$. That is to say,

$$D_n := \{z \in \mathbb{C}^n : |z| < 1\} .$$

Here is the description of the bisymplectomorphisms.

Theorem 0.4. *The bisymplectomorphisms $f \in \mathcal{B}(D_n)$ are the maps defined by*

$$f(z) = \gamma\left(|z|^2\right) u(z) \qquad (z \in D_n) ,$$

where $u \in U(n)$ and γ is a smooth function $\gamma : [0,1) \to S^1 \simeq U(1)$.

The rank one case (II).

Sketch of the Proof of Theorem 0.4 :

The Bergman operator $\mathrm{B}(z)$ is given by:

$$\mathrm{B}(z)(w) := 2(1 - |z|^2)(w - z(w \mid z))\,.$$

In particular, **notice** that for fixed $z \in D_n$ the operator $B(z) : \mathbb{C}^n \to \mathbb{C}^n$ has two eigenspaces. Namely, $V_z := \mathbb{C}.z$ and $V_z^{\perp}$.

That is to say

$$\mathbb{C}^n = V_z \oplus V_z^{\perp}\ ,$$

where V_z and $V_z^{\perp}$ are $B(z)$-invariant.

Then Proposition 0.3 implies that f must infinitesimally preserve such decomposition.

The rank one case (III).

So if $f \in \mathcal{B}(D_n)$ we get:

- $|f(z)| = |z|$. Thus, $\mathrm{d}\, f(0)$ is unitary, i.e. $\mathrm{d}\, f(0) \in U(n)$.

- $\mathrm{d}\, f(z)$ preserves the complex line $l_z \subset T_z D_n$ spanned by z, i.e. $l_z := \{w \in T_z D_n : w = \lambda\, z\}$.

- Indeed, f takes complex lines through the origen into complex lines through the origen.

Notice that the complex lines through the origen are the *complex totally geodesic discs* Δ of the symmetric domain, i.e. the complexifications of the flats.

Now we can restrict f to the discs $\Delta \subset D_n$ to finish the proof.

The higher rank case (I).

The rank one case show that the description of $\mathcal{B}(\Omega)$ depends upon a good algebraic description of the Bergman operator $\mathrm{B}(z)$.

The theory of **Jordan Algebras** gives an algebraic description of the Bergman operator $\mathrm{B}(z)$ of all Bounded symmetric domains Ω.

A principal role is played by the so called **Peirce simultaneous decomposition** relative to $z \in \Omega$. That is exactly the generalization of the decomposition in eigenspaces $\mathbb{C}^n = V_z \oplus V_z^{\perp}$ for the Bergman operator of the rank one case.

The higher rank case (II).

Let $\Omega \subset V = \mathbb{C}^n$ be an irreducible bounded symmetric domain attached to the HPJTS $(V, \{,,\})$ of rank r.

Let us call **radial** a bisymplectomorphism $f \in \mathcal{B}(\Omega)$ such

$$f(\Delta^r) = \Delta^r \ ,$$

for all polydiscs $\Delta^r \subset \Omega$ generated by the frames $(c_1, c_2, \cdots, c_r)$, i.e. $\Delta^r = \Omega \bigcap \mathbb{C}c_1 \oplus \cdots \oplus \mathbb{C}c_r$.

<u>**Theorem II**</u> *Any* $f \in \mathcal{B}(\Omega)$ *is of the form*

$$f = u \circ R \ ,$$

where R *is a radial bisymplectomorphism and* $u = \mathrm{d}\, f(0) \in K$, *where* K *is the isotropy group at* $0 \in \Omega$ *of* Ω.

<u>**Theorem III**</u> *Let* R *be a radial bisymplectomorphism. Then there exists a function* $h \in C^\infty[0,1)$ *such that*

$$R(z) = e^{\mathrm{i}\, h(\lambda_1^2)}\lambda_1 e_1 + e^{\mathrm{i}\, h(\lambda_2^2)}\lambda_2 e_2 + \cdots + e^{\mathrm{i}\, h(\lambda_r^2)}\lambda_r e_r$$

for all $z \in M$, *where* $z = \lambda_1 e_1 + \lambda_2 e_2 + \cdots + \lambda_r e_r$ *is the spectral decomposition of* $z \in M$.