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1 Main results

Main results

[–, Di Scala]

• computed the holonomy group $\Phi^\perp$ of the normal connection of complex symmetric submanifolds of $\mathbb{C}P^n$.

• as a by-product, given a new proof of the classification of complex symmetric submanifolds of $\mathbb{C}P^n$ by using a normal holonomy approach

Then, we prove Berger type theorems for $\Phi^\perp$, namely,

[–, Di Scala, Olmos]

$M$ full, irreducible and complete

1. for $\mathbb{C}^n$, $\Phi^\perp$ acts transitively on the unit sphere of the normal space;

2. for $\mathbb{C}P^n$, if $\Phi^\perp$ does not act transitively, then $M$ is the complex orbit, in the complex projective space, of the isotropy representation of an irreducible Hermitian symmetric space of rank greater or equal to 3.

2 Submanifolds and Holonomy

2.1 Real submanifold geometry

Submanifolds of real space forms

$M \hookrightarrow \mathbb{R}^n, S^n, \mathbb{R}H^n$ with induced metric $\langle \cdot, \cdot \rangle$ and Levi-Civita connection $\nabla$

$$\nu_M: \text{normal bundle of } M \text{ with the normal connection } \nabla^\perp$$

$$\nu_0M = \text{maximal parallel and flat subbundle of } \nu_M$$

Notation

$\alpha$ second fundamental form

$A$ shape operator

$R^\perp$ normal curvature tensor

recall $\langle \alpha(X,Y), \xi \rangle = \langle A\xi, X \rangle$, which is symmetric in $X, Y$

Fundamental equations

Gauss: $\langle [\alpha(X,Z), W], Y \rangle = \langle \alpha(Y, W), \alpha(X,Z) \rangle - \langle \alpha(X,W), \alpha(Y,Z) \rangle$

Codazzi: $\{\tilde{\nabla}_X \alpha(Y,Z)\}$ are symmetric in $X, Y, Z$

Ricci: $\langle R^\perp \alpha(X,Y), \xi \rangle = \langle [A\xi, A\eta], X \rangle, Y \rangle$

Nullity: $\mathcal{N} = \cap \xi \ker A\xi$

2.2 Normal holonomy for submanifolds of real space forms

Normal holonomy for submanifolds of real space forms

(Restricted) Normal Holonomy $\Phi^\perp$ ($\Phi^\perp^*$):

(restricted) holonomy of the normal connection

on the normal bundle of a submanifold

Normal Holonomy Theorem [Olmos]

$M$ submanifold of a space form $\overline{M}$.

$\Rightarrow \Phi^\perp^* \text{ (at some point } p) \text{ is compact,}$

$\Phi^\perp$ acts (up to its fixed point set) as the isotropy representation of a Riemannian symmetric space ($s$-representation)

Consequences:

The Normal Holonomy Theorem is a very important tool for the study of submanifold geometry, especially in the context of submanifolds with “simple extrinsic geometric invariants”
e.g., isoparametric and homogeneous submanifolds

**Distinguished class:**

**orbits of s-representations = flag manifolds**

similar rôle as symmetric spaces in Riemannian geometry

**Special cases**

**Symmetric submanifolds: characterizations**  
[Ferus, Strübing]

- parallel second fundamental form (\( \nabla \alpha = 0 \))
- distinguished orbits of s-repr. (symmetric R-spaces)

\( K \) compact Lie group

\[ M = \text{Ad}(K)X \cong K/K_K \leftarrow (t, -B(, )) \]

**standard immersion of a cx flag manifold = cx orbit of s-repr**

## 2.3 Complex submanifolds

**Complex submanifolds**

\[ M \hookrightarrow \mathbb{C}^n, \mathbb{C}P^n, \mathbb{C}H^n \] complex submanifold

\( J \): complex structure (both on \( M \) and on the ambient space)

\[ \alpha(X, JY) = J\alpha(X, Y) \iff A\xi = -JA\xi = -A\xi \]

\[ \implies [A\xi, A\eta] = J[A\xi, A\eta] - 2JA\xi A\eta \]

for \( \eta = \xi \), by the Ricci equation

\[ \langle R^+ (X, Y)\xi, J\xi \rangle = \langle -2JA^2 \xi X, Y \rangle \]

**Consequence:** [Di Scala]

\( M \hookrightarrow \mathbb{C}^n \) is full (not contained in any proper affine hyperplane) \( \iff \nu_0 M \) is trivial

[Indeed if \( \xi \) is a section of \( \nu_0 M \), \( R^+ (X, Y)\xi = 0 \implies A\xi = 0 \implies M \) not full]

## 2.4 Normal holonomy for submanifolds of complex space forms

**Normal holonomy for complex (Kähler) submanifolds**

- \( M \hookrightarrow \mathbb{C}^n \)

[Di Scala]: \( M \) is irreducible (up a totally geodesic factor) \( \iff \Phi^+ \) acts irreducibly

(extrinsic analogue of the de Rham decomposition theorem)

- \( M \hookrightarrow \mathbb{C}P^n, \mathbb{C}H^n \)

**Theorem** [Alekseevsky-Di Scala]

If \( \Phi^+ \) acts irreducibly on \( \nu_p M \) \( \implies \Phi^+ \) is linear isomorphic to the holonomy group of an irreducible Hermitian symmetric space.

\( M \) full & \( \mathcal{N} = \{0\} \implies \Phi^+ \) acts irreducibly
Homogeneous Kähler submanifolds

**Calabi rigidity theorem** of complex submanifolds $M \hookrightarrow \mathbb{CP}^N \implies$ isometric and holomorphic immersions are equivariant: any intrinsic isometry can be extended to $\mathbb{CP}^N$.

**Borel-Weil construction**

$G$ simple Lie group, $d$ positive integer

$\rho : G \rightarrow \mathfrak{gl}(\mathbb{C}^N)$ irreducible representation of $G$ with highest weight $d\Lambda_j$

$(\Lambda_j$ fundamental weight corresponding to the simple root $\alpha_j)$

Induces a unitary representation of $G$

$M = G/[p] \hookrightarrow \mathbb{CP}^N$

$d$-th canonical embedding of $M$

$\Phi_\perp : \nu_{\perp} \big(\mathbb{C}^N\big) \rightarrow \dim \mathbb{C}\big(\nu_{\perp}(M)\big) = \dim \mathbb{C}(H/S)$

$\Phi_\perp$ acts on $\nu_{\perp}(M)$ as the isotropy repr. of $S$ on $T_{\mathbb{CP}^N}(H/S)$.

$\Phi_\perp$ computation of the 3rd column in the Table

Symmetric complex submanifolds $M \subset \mathbb{CP}^n$

$M \subset \mathbb{CP}^n$ symmetric $\iff$ $\nabla \alpha = 0$

Arise as unique complex orbits in $\mathbb{CP}^n$ of the isotropy representation of an irreducible Hermitian symmetric space

[Nakagawa-Takagi]

$\bullet$ computed the holonomy group of the normal connection of complex symmetric submanifolds of the complex projective space.

$\bullet$ as a by-product, given a new proof of the classification of complex symmetric submanifolds by using a normal holonomy approach

Symmetric complex submanifolds $M \subset P(T|_{\mathfrak{k}}/G/K)$

[Nakagawa-Takagi]:

$\iff$ computation of the 3rd column in the Table

Idea of the proof

Use [Alekseevsky-Di Scala] to get

**Lemma 1.** $M = G/K$ Hermitian symmetric space

$M \hookrightarrow \mathbb{CP}^N$ full embedding with $\nabla \alpha = 0$

$\implies \exists$ an irreducible Hermitian symmetric space $H/S$ such that

$\Phi_{\perp} = S = K/\mathfrak{i}$ where $I \subset K$ is a normal subgroup,

$\dim_{\mathbb{C}}(\nu_{\perp}(M)) = \dim_{\mathbb{C}}(H/S)$ and

$\Phi_{\perp}$ acts on $\nu_{\perp}(M)$ as the isotropy repr. of $S$ on $T_{\mathbb{CP}^N}(H/S)$. 

$\iff$ computation of the 3rd column in the Table
Alternate proof of classification of complex symmetric submanifolds of $\mathbb{CP}^N$

A tool is Theorem 2. Let $f_d : G/K \to \mathbb{CP}^N$ be the $d$-th canonical embedding of $G/K$.

If $\nabla \alpha = 0$ and $f_d$ is not the Veronese embedding, $f_d$ is the first canonical embedding $f_1$.

$\Rightarrow$ look at 1st canonical embeddings only.

The following theorem gives a sharp description.

Theorem 3. If the first canonical embedding $f_1$ of an irreducible Hermitian symmetric space $M$ of higher rank ($\geq 1$) has $\nabla \alpha = 0$, $\Rightarrow \text{rank}(M) = 2$.

Remark: list of images of the 1st can. embedding of an irreducible Hermitian symmetric space of rank two.

Higher canonical embedding and holonomy

Theorem [-, Di Scala]

Let $f_d : G/K \hookrightarrow \mathbb{CP}^N$ be the $d$-th canonical embedding of an irreducible Hermitian symmetric space. If $d \geq 2$ then the normal holonomy group is the full unitary group of the normal space (unless it is the Veronese embedding $\text{Ver}_2$).

Motivated by the above theorem we have the following

Question

$M \hookrightarrow \mathbb{CP}^N$ complete (connected) and full (i.e. not contained in a proper hyperplane) complex submanifold. Is it true in general that if the normal holonomy group is not the full unitary group, then $M$ has parallel second fundamental form?

The answer is YES.

3 Geometry of parallel focal manifolds and holonomy tubes

A Berger type Theorem

[-, Di Scala, Olmos]

$M \hookrightarrow \mathbb{C}P^r$, $\mathbb{C}P^\alpha$ full, irreducible and complete

1. For $\mathbb{C}P^r$, $\Phi_r^+$ acts transitively on the unit sphere of the normal space; $\Rightarrow \Phi_r^+ = U_0(\nu_0M)$, since it acts as an $s$-representation.

2. For $\mathbb{C}P^\alpha$, if $\Phi_\alpha^-$ does not act transitively, then $M$ is the complex orbit, in the complex projective space, of the isotropy representation of an irreducible Hermitian symmetric space of rank greater or equal to 3.

( $\Rightarrow$ it is extrinsic symmetric)
False if $M$ is non complete (counterexamples)

The methods in the proofs rely heavily on the singular data of appropriate holonomy tubes (after lifting the submanifold to the complex Euclidean space, in the $\mathbb{C}P^n$ case) and basic facts of complex submanifolds.

Some geometry needed in the proof

Endpoint map

$$t_\xi : M \to \mathbb{R}^n$$

$$x \mapsto x + \xi(x) = \exp(\xi(x))$$

$\xi$ focal point in direction $\xi = \text{critical value of } t_\xi$.

$$x + \xi(x) \text{ focal point in dir. of } \xi \iff \ker(id - A_\xi(x)) \text{ is non trivial}$$

3.1 Parallel focal manifolds

Parallel focal manifolds

$\xi$ parallel normal field, $\text{im}(t_\xi) = M_\xi = \{x + \xi(x) | x \in M\}$

- if 1 is not an eigenvalue of $A_\xi$, parallel manifold
- if 1 is a constant eigenvalue of $A_\xi$, parallel focal manifold

$$T_xM = T_{x+\xi(x)}M_\xi \oplus \ker(id - A_\xi(x))$$

integrable

$\pi : M \to M_\xi : x \mapsto x + \xi(x)$,

$\pi^{-1}(p)$ isoparametric in $\nu_p M_\xi$

(by Olmos’ Normal Holonomy Theorem)

3.2 Holonomy tubes

Holonomy tube

$$M_{\eta_p} = \{c(1) + \eta(1)\} = \{c(1) + \nabla^\perp_{c(1)} \eta_p\},$$

where $c : [0, 1] \to M$ is an arbitrary curve starting at $p$ and $\eta(t)$ is the $\nabla^\perp$-parallel transport of $\eta_p$ along $c(t)$.

Proposition

$M_{\eta_p}$ has flat normal bundle ($\iff$ full holonomy tube)

$\Phi_{\pi(p)} : (p - \pi(p))$ is maximal dimensional

$\mathcal{H}^p$ horizontal subspace of $\pi : N = M_{\eta_p} \to N_\eta = M$

tube formulae
\[ A_M^N = A_M^N (\text{id} - A_M^N (x))^{-1}, \quad \xi_x \in V_N \]

\[ A_M^N |_{\mathcal{H}_x} = A_M^N (\text{id} - A_M^N (x))^{-1}, \quad \xi_x \in V_N \]

### 3.3 The canonical foliation

The canonical foliation

\[ N \to \mathbb{R}^n, \text{ take } M = N_{\xi_p} \text{ full holonomy tube} \]

Assume:

- \( \Phi^\top \) acts irreducibly and not transitively on \( V_p N \)
- \( 0 \) is a constant eigenvalue of \( A_M^N \),
  i.e. \( E_{\xi}^0 \) is non-trivial

The canonical foliation

\[ \text{def.}: x \sim y \text{ if } \exists \text{ curve } \gamma \text{ in } M \text{ from } x \text{ to } y: \dot{\gamma}(t) \perp E_{\xi}^0, \forall t \]

\[ H_{\xi}^\top (x) = \{ y \in M : x \sim y \} \]

the orthogonal distribution \( \mathcal{V}_x^\top \) to the foliation \( H_{\xi}^\top (p) \) is integrable.

\[ \Sigma_x \xi(x) \text{: leaf of } \mathcal{V}_x^\top \text{ through } x \]

Note: \( \mathcal{V}_x^\top \subseteq N = \bigcap \ker A_M^N \text{ (nullity)} \Rightarrow \text{the foliation is indep. on } \xi \mid E_{\xi}^0 \neq \{0\} \)

The canonical foliation

\[ \mathcal{H} : \text{the horizontal distribution in } M \text{ (w. r. to } \pi : M \to N) \]

#### Technical Lemma

Assume that \( \exists \text{ parallel } \xi, \xi' \text{ such that } \mathcal{H} \subseteq (\ker A_M^N + \ker A_M^N) \]

\[ \Rightarrow \forall x \in M, H_{\xi}^\top (x) = H_{\xi'}^\top (x) \text{ is an isoparametric submanifold.} \]

(we are around a generic point s. t. (\( \ker A_M^N + \ker A_M^N \)) is a distribution of \( M \))

Projecting down to \( N \),

\[ N = \bigcup_{y \in \pi \Sigma_x (x)} (\pi(H_{\xi}^\top (x)))_{y - \pi(x)} \]

Using Thorbergsson Theorem

Corollary of the Technical Lemma

\( \exists \text{ compact group } K \text{ of isometries of } \mathbb{R}^n \text{ acting as an irred. } s\text{-representation s. t. (loc) } K \cdot \pi(x) = \pi(H_{\xi}^\top (x)), \)

for all \( x \in M \).

\[ \Rightarrow N \text{ is locally given, around a generic point } q, \text{ as} \]

\[ N = \bigcup_{y \in (V_q(K \cdot q))_0} (K \cdot q)_y. \]

Moreover the nullity space of \( N \) at \( p \) is \( A_{N_p}^N = (V_q(K \cdot p))_p \).

### 4 Complex submanifold geometry

#### 4.1 Complex submanifolds of \( \mathbb{C}^n \)

Complex submanifolds of \( \mathbb{C}^n \)

\[ N \to \mathbb{C}^n \text{ full, irreducible complex submanifold for which } \Phi^\top \text{ does not act transitively} \]

on the unit sphere of the normal space

Choose \( \xi_0^1 \in V_q N \mid \Phi_q^\top \cdot [\xi_0^1] \in \mathbb{C}P(V_q N) \text{ (unique) complex orbit} \]
Lemma 2

Complex submanifolds of O’Neill’s type formula

Proof of the Berger-type Theorem for submanifolds of symmetric space such that Hermitian

4.2 Complex submanifolds of C

Assume that M ↪ C

Now choose 0 ≠ ξ2 ∈ (ξ1) ⊕ ∩ vξΦ−1 ξ1 · ξ1 q cx subsp. (non trivial by non-transitivity!)

Since RξX ∈ L(Φq), \( 0 = \langle [A_N^N, A_N^N], X, Y \rangle \),

The same is true if we replace ξ2/q by Jξq, \( |A_N^N, A_N^N| = 0 \).

By complex geometry \( \overline{A_N^N A_N^N} = A_N^N A_N^N = 0 \)

Take the holonomy tube \( \overline{M} := (Nξ1)_\overline{ξ1} = N\overline{ξ1} + \overline{ξ1} \)

\( \overline{ξ1} \overline{ξ2} \sim \parallel v \cdot f \). \( \overline{ξ1} \overline{ξ2} \) on M

Tube formula \( \Rightarrow \overline{A_M^N A_M^N} = 0 \) \( \Rightarrow \mathcal{H} \subset (\ker A_M^N + \ker A_M^N) \)

⇒ Technical Lemma and its corollary apply

Complex submanifolds of \( \mathbb{C}^n \)

⇒ \( \exists \) compact group K of isometries of \( \mathbb{C}^n \), which acts as the isotropy representation of an irreducible

Hermitian symmetric space such that

\[ N = \bigcup_{v \in (v_0(K, q))_q} (K \cdot q)_v \]

Moreover \( \mathcal{A}^N = (v_0(K, p))_p \).

⇒

Proof of the Berger-type Theorem for submanifolds of \( \mathbb{C}^n \)

We assume that 0 is the fixed point of K.

N is complete \( \Rightarrow \) if \( p \in N \), the line \( \{ t \mapsto tp \} \subset N \)

\( \forall t \), \( T_pN = T_pN \), as subspaces of \( \mathbb{C}^n \) \( \Rightarrow \)

the isotropy \( K_p \) must leave this subspace invariant.

A contradiction for \( t = 0 \), since K acts irreducibly.

Thus the normal holonomy group must be transitive.

4.2 Complex submanifolds of \( \mathbb{C}P^n \)

Complex submanifolds of \( \mathbb{C}P^n \)

Let \( M \hookrightarrow \mathbb{C}P^n \) be a full complex submanifold.

Consider \( \pi : \mathbb{C}^{n+1}\setminus\{0\} \rightarrow \mathbb{C}P^n \)

\( \widetilde{M} : \text{lift } M \text{ to } \mathbb{C}^{n+1}\setminus\{0\}, \text{i.e. } \widetilde{M} := \pi^{-1}(M) \)

\( \mathcal{V} \) : vertical distribution of the submersion \( \pi : \widetilde{M} \rightarrow M \).

It is standard to show that \( \mathcal{V} \subset \mathcal{A}^M \).

If X is a tang. vector to \( M \) we let \( \tilde{X} \) be its horiz. lift to \( \mathbb{C}^{n+1}\setminus\{0\} \).

\( \pi : \mathbb{C}^{n+1}\setminus\{0\} \rightarrow \mathbb{C}P^n \) is not a Riemannian submersion. Anyway.

O’Neill’s type formula

Let \( X, Y \in \Gamma(\mathbb{C}^{n+1}\setminus\{0\}) \) be the horizontal lift of the vector fields \( X, Y \in \Gamma(\mathbb{C}P^n) \). Then,

\[ (D_X \widetilde{Y})_\widetilde{p} = (\nabla_X^\mathcal{V} Y)_\widetilde{p} + \mathcal{O}(\tilde{X}, \tilde{Y}) \]

where \( \mathcal{O}(\tilde{X}, \tilde{Y}) \in \mathcal{V} \) is vertical.

Complex submanifolds of \( \mathbb{C}P^n \)

Lemma 1

\( M \subset \mathbb{C}P^n \), \( \widetilde{M} \subset \mathbb{C}^{n+1} \) be its lift to \( \mathbb{C}^{n+1} \).

Assume that the tangent vector \( \overline{v}_p \in T_p\widetilde{M} \) is not a complex multiple of the position vector \( \overline{p} \).

If \( \overline{v}_p \in \mathcal{A}^M \Rightarrow v_p \in \mathcal{A}^M \).

Lemma 2

Assume that \( M \subset \mathbb{C}P^n \) is full and \( \Phi^1 = M \) does not act transitively on \( \mathcal{V}_p(M) \).
\[ \Phi \cdot \tilde{M} \text{ does not act transitively on } \nu_{\tilde{p}}(\tilde{M}), \text{ where } \pi(\tilde{p}) = p. \]

**Important fact (special case of a Theorem in [Abe-Magid])**

Let \( M \subset \mathbb{C}P^n \) complete full with \( \Phi \cdot \nu \) not transitive

\[ \Rightarrow \forall M = \{0\} \]

**Proof of the Berger-type Theorem for submanifolds of \( \mathbb{C}P^n \)**

\[ N = \tilde{M} \subset \mathbb{C}^{n+1} \Rightarrow \tilde{M} = \bigcup_{v \in (\nu_D(K \cdot q))_v} (K \cdot q)_v \]
(\( K \) is the isotropy group of a irreducible Hermitian symmetric space)

Observe also that \( \nu_D(K \cdot q)_q \) is a complex subspace (= \( N \cdot M \))

Then Lemma 1 and special case of Abe-Magid \( \Rightarrow \dim_{\mathbb{C}}(\nu_D(K \cdot q)_q) = 1 \), otherwise the nullity of the second fundamental form of \( M \) would be not trivial.

Since \( M \) is full \( \Rightarrow \) the unique fixed point of \( K \) is \( 0 \in \mathbb{C}^{n+1} \).

So the leaves of the nullity distribution \( \nu \cdot M \) are just the complex lines given by the fibers of the submersion \( \pi : \tilde{M} \to M \).

Thus, \( K \) acts transitively on the complex submanifold \( M \subset \mathbb{C}P^n \).

Therefore, \( M \) is a complex orbit of the projectivization of an irreducible Hermitian s-representation.

**References**

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