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Parallel spinors and the spinorial Weierstrass repr. in 3+1

Parallel Spinors
special holonomy
string theory

Cauchy-Problem
of general Relativity

surface theory
Weierstrass 1866
Kusner-Schmitt 1990's
Bär 1990's
Friedrich

conformal
geometry

Embedding property
Hitchin – 6+1, and 7+1
Salamon 5+1
Conti

Parallel spinors and the spinorial Weierstrass representation in 3+1 dimensions

(work in progress)

M a Riem. mfd., $SM \to M$ spinor bundle

$(e_1, \ldots, e_n)$ local frame field

$\rightsquigarrow$ local trivial. of TM ~~$\to M$~~

$\rightsquigarrow$ – – of SM ~~$\to M$~~

$$\nabla_{e_i} e_j = \sum \Gamma_{ij}^k e_k$$

● $\psi \in \Gamma(SM)$ $\quad \nabla^M_{e_i} \psi = \partial_{e_i}\psi + \frac{1}{4}\sum_{jk} \Gamma_{ij}^k e_j \cdot e_k \cdot \psi = \partial_{e_i}\psi + \frac{1}{2}\sum_{j<k} \Gamma_{ij}^k e_j e_k \psi$

$$D^M \psi = \sum_i e_i \nabla_{e_i} \psi \quad \text{Dirac op.}$$

Now $M^n \subseteq N^{n+1}$ ~~hyp~~ e_0 unit normal vector

$(e_1, \ldots, e_n)$ $(e_0, \ldots, e_n)$

As vector bdls $\quad$ ~~$SN_{|M} = \begin{cases} SM & \text{if } \dim M = n \text{ even} \\ SM \oplus SM & \text{if } n \text{ odd} \end{cases}$~~

○ ~~SN_+~~ $SM = \begin{cases} SN_{|M} & \text{if } \dim M = n \text{ even} \\ (SN_+)_{|M} & \text{if } \dim M = n \text{ odd} \end{cases}$

$$\sum_k \left(\Gamma_{ij}^k(N) e_k - \Gamma_{ij}^k(M) e_k\right) = \mathrm{II}_{ij} e_0$$

W: Weingarten map

$$\nabla^M_{e_i}\psi - \nabla^N_{e_i}\psi = \frac{1}{2}\sum_j \mathrm{II}_{ij} e_j e_0 \psi = \frac{1}{2} W(e_i) e_0 \psi; \quad \underbrace{\varphi = (1-e_0)\psi_{|M}}$$

Assume $\nabla^N \psi = 0 \Rightarrow$ ~~$D^M\psi = -\frac{n}{2} H e_0 \psi$~~ $\quad W = \frac{1}{2}A$ ok

$|\psi| = 1$ $\quad \nabla^M_{e_i}\psi = \frac{1}{2} W(e_i) e_0 \psi \Rightarrow \boxed{\nabla^M_{e_i}\psi = \tfrac{1}{2} A(e_i)\cdot\varphi} \quad (1)$

$H := \frac{1}{2} \operatorname{tr} W$

$$\Rightarrow \boxed{D^M \varphi = \tfrac{1}{2} H\varphi, \ |\varphi| = 1}. \quad (2)$$

Bär, Bär–Gauduchon–Moroianu

B②

Remarks:

N parall. spinor $\Rightarrow Ric = 0$

Most results also for N carrying Killing spinor

Def.: (1) $\Leftrightarrow$ φ is a generalized Killing spinor with $A \in End(TM)$ symmetric

Ref. ~~N parallel spinor~~
1) N has parallel spinor $\Rightarrow Ric = 0$
2)

Dirac operator & conformal change

$\hat{g} = f^2 g \qquad SM^g \xrightarrow{\cong} SM^{\hat{g}} \qquad$ s.t. $D^{\hat{g}} = \frac{1}{f} D^g$

$|\varphi|_{\hat{g}} = f^{-\frac{n-1}{2}} |\varphi|_g$

$D^g \varphi = h |\varphi|_g^{\frac{2}{n-1}} \varphi \qquad$ conformally invariant.

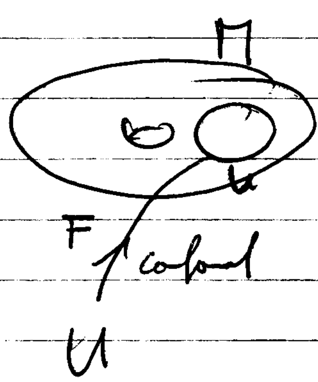
Conclusion

parallel spinor on N, $M^n \subseteq N^{n+1}$, W Weingarten map $\quad \overset{A = \frac{1}{2}W}{\Rightarrow} \quad$ ~~Solution~~ gen. Killing spinor on M $\Rightarrow$ $D\varphi = H\varphi$, $|\varphi| = 1$ $\Rightarrow$ $D^{\hat{g}}\varphi = H|\varphi|_{\hat{g}}^{\frac{2}{n-1}}\varphi$

$\Leftarrow$? ③ $\qquad \Leftarrow$? ② $n \le 3$ $\qquad \Leftarrow$? ① ✓ $\qquad \Uparrow$ ④ Conf. geom.

$n=2$: N^3 is flat, say $N^3 = \mathbb{R}^3$

Weierstrass 1866



$dF_{\mathbb{C}}|_z : \mathbb{C} \to \mathbb{C}^3$

$\searrow \{x_1^2 + x_2^2 + x_3^2 = 0\} \subseteq$ conformal

$$dF_{\mathbb{C}}|_z = \begin{pmatrix} \varphi_+ \otimes \varphi_+ + \overline{\varphi_-} \otimes \overline{\varphi_-} \\ i\varphi_+ \otimes \varphi_+ - i\overline{\varphi_-} \otimes \overline{\varphi_-} \\ 2i\varphi_+ \otimes \overline{\varphi_-} \end{pmatrix}$$

M minimal $\Leftrightarrow$ $\varphi_+, \overline{\varphi_-}$ holomorphic $\Leftrightarrow D\varphi = 0$.

B ③

Kusner-Schmitt 1996

Conformal immersion $\tilde{M} \hookrightarrow \mathbb{R}^3$ / Translations $\overset{1:1}{\Longleftrightarrow}$ Solutions of $D\varphi = H|\varphi|^2\varphi$ on M / $\{\pm 1\}$.

Abresch, T. Friedrich!

Converse direction

①: Assume $D^g\varphi = H|\varphi|^{\frac{2}{n-1}}\varphi$ on (M,g)

$\tilde{g} := |\varphi|^{\frac{4}{n-1}} g \Rightarrow D^{\tilde{g}}\varphi = H\varphi$, $|\varphi|_{\tilde{g}} = 1$. on $M \setminus \varphi^{-1}(0)$

②: Prop 1: A ~~(n=3)~~

Let $n=3$, $D\varphi = H\varphi$, $|\varphi| = 1$

Then $\exists$ $A \in \mathrm{End}(TM)$, s.t. $\nabla_X\varphi = A(X)\cdot\varphi$

e_1, e_2, e_3 frame, $e_1 \cdot e_2 \cdot e_3 = 1$ $\quad$ $\langle\,,\rangle$ real scal prod.

$$\langle \nabla_{e_1}\varphi, e_2\varphi\rangle = \langle e_1 \cdot \nabla_{e_1}\varphi, \overbrace{e_1 \cdot e_2}^{-e_3}\varphi\rangle = \langle e_2\nabla_{e_2}\varphi, e_2\varphi\rangle + \langle e_3\nabla_{e_3}\varphi, e_3\varphi\rangle - \underbrace{\langle H\varphi, e_3\varphi\rangle}_{=0}$$

$$= \cancel{-\langle e_3 e_2} = -\langle \nabla_{e_2}\varphi, e_2e_3\varphi\rangle + \underbrace{\langle \nabla_{e_3}\varphi, \varphi\rangle}_{\frac{1}{2}\partial_{e_3}|\varphi|^2 = 0} - H\underbrace{\langle\varphi, e_3\varphi\rangle}_{=0}$$

$$= \langle \nabla_{e_2}\varphi, e_1\varphi\rangle$$

$\langle A(X), Y\rangle := \langle \nabla_Y\varphi, X\cdot\varphi\rangle$ $\quad$ $A \in \mathrm{End}(TM)$ symmetric. $\square$

$\boxed{n=2, \text{Friedrich}}$

$e_1 \cdot \varphi, e_2 \cdot \varphi, e_3 \cdot \varphi$ is a basis of $\varphi^\perp$

$$\Rightarrow \langle \nabla_X \varphi, e_i \cdot \varphi \rangle = \sum_j \langle A(X), e_j \rangle \underbrace{\langle e_j \cdot \varphi, e_i \cdot \varphi \rangle}_{\delta_{ij}} = \langle A(X) \varphi, e_i \cdot \varphi \rangle$$

$\Rightarrow \varphi$ ~~generalized~~ gKS

$n=2$: Friedrich.

known

(3): a) $A = \lambda \, \mathrm{Id}$ Killing spin. Bär '93 $g^N = t^2 g$ M cell t^2

b) $\nabla^M A \equiv 0$ B. Morel Nancy '03 Epinal, generalized By M.-A. Pailleseau '06

c) $(\nabla_X^M A)(Y) = (\nabla_Y^M A)(X)$ Bär-Moroianu-Gauduchon ~~Codazzi-equation~~.

d) $n=2$

e) $n = 6, 7$ + analytic Mitchell
embedding problem est. diff.-system

f) $n=5$ S. Salamon, Conti

Th 1 A. (Embedding property)

Let (M^n, g) be analytic with φ gKS analytic. $\nabla_X \varphi = A(X) \varphi$.

Then there is a Riem. mfd N^{n+1} carrying a parallel spinor $\Psi \neq 0$.

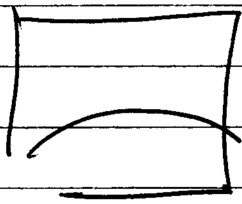
1) $M^n \subseteq N^{n+1}$ with $W = \frac{1}{2} A$

2) Ψ induces φ.

Rem: 1) N in general not complete

2) Analyticity necessary

Assume $H = \frac{1}{2} \mathrm{tr}\, W = \mathrm{const}$



N^{n+1} Ric $= 0 \Rightarrow$ analytic structure.

M^n M solution of an analytic non-linear PDE.

$\Rightarrow M$ analytic.

Prop 2 A. Let $[g]$ be a non-~~real~~ analytic conf. class on $\mathbb{RP}^3$, close to $[g_{can}]$. Then there is a ~~normal~~ non-analytic $g \in S_{par}(\mathbb{RP}^3, \tilde{g})$ s.th.

- $\tilde{g} \in [g]$.
- φ ... ($\Rightarrow$ No embedding).

Pf later

Idea for the proof of Thm 1.

$$\nabla_X \varphi = A(X) \cdot \varphi \qquad W := 2A$$

Calculate

$(-)$ $\mathrm{Scal}^M = (\mathrm{tr}\, W)^2 - (\mathrm{tr}\, W^2)$ Hamiltonian constraint

$d\, \mathrm{tr}\, W = \mathrm{div}\, W$ momentum constraint

ECE Einstein constraint equations

Lorentzian case $=$ - - Sign

(lECE)

Fact 1.) Any Hypersurface in a Ric=0 – Riem. mfd satisfies ECE

2.) - Riem. - - - Lorentz mfd - lECE

Th (Choquet-Bruhat 1952) $n=3$

If W^n satisfies (CECE) $\Rightarrow$ $\exists$ Ricci-flat Lorentz manifold N_s^{n+1} st

$M^n \hookrightarrow N^{n+1}$ with Weingarten map W

Now: Wick rotation

N Ricci flat

extend φ by parallel transp along ~~spat~~ radial geodesics

Show $\nabla\varphi = 0$.

Exact proof: $N = M \times (-\varepsilon, \varepsilon)$

$$g^N = g_t^M + dt^2 \qquad g_0^M = g^M$$

$$\dot g_0^M = -2\,\mathrm{II} = -2W^b$$

~~Ric (TM × {0})~~ $\quad X, Y \in T(M \times \{t\})$

$\mathrm{Ric}^N(X,Y) = 0 \ \forall X,Y \Leftrightarrow$ 2nd order ODE on $\Gamma(\mathrm{Sym}^2(TM))$

$$\ddot g_t^M = \ldots$$

$0 = \mathrm{Ric}^N(X, \partial_t) \Leftrightarrow$ mom. constr.

$0 = \mathrm{Ric}^N(\partial_t, \partial_t) \Leftrightarrow$ Ham. constraint

} preserved ?

$\Rightarrow \mathrm{Ric}^N = 0 \qquad \Rightarrow \nabla\varphi$ as before

How to get ~~that~~ solutions of $D\varphi = H|\varphi|^{\frac{2}{n-1}}\varphi$? B7 8

M^n compact, $[g]$ ~~conf. class~~ $= \{ \underbrace{f^2 g}_{\tilde g} \mid \int f^n dv = 1 \}$

~~spec(D^{g})~~

spec$(D^{\tilde g})$

$$\lambda_1^-(\tilde g) < 0 = \dots = 0 < \lambda_1^+(\tilde g) \le \lambda_2^+(\tilde g) \dots$$

~~Minimize $\lambda_1^+(\tilde g)$ in~~

$$\mu(M,[g]) = \inf_{\tilde g \in [g]} \lambda_1^+(\tilde g) = \inf_{0 < \int \langle D\varphi, \varphi \rangle dv} \underbrace{\frac{\|D\varphi\|^2_{L^q(g)}}{\int_M \langle D\varphi, \varphi \rangle dv}}_{\mathcal{F}(\varphi)} \qquad q = \frac{2n}{n+1}$$

Thm (A.) 2003

If $\mu(M,[g]) < \mu(S^n) = \frac{n}{2}\omega_n^{\frac{1}{n}}$, then both infima are attained.

Euler-Lagrange-equations

$$\boxed{D^g\varphi = \lambda|\varphi|_g^{\frac{2}{n-1}}\varphi}$$

$$\boxed{\begin{array}{l} D^{\tilde g}\varphi = \lambda_1^+(\tilde g)\,\varphi \\ |\varphi|_{\tilde g} = 1 \end{array}}$$

Example

$M = \mathbb{RP}^3, g_{can}$ with 2-dim space of Killing spinors $\nabla_X \varphi = \frac{1}{2} X \cdot \varphi$.

Problem: $\mu(\mathbb{RP}^3, [g_{can}]) = \frac{1}{\sqrt[3]{2}} \mu(S^3, [g_{can}])$

$g \approx g_{can}$: Obtain solution $D^g \varphi = \lambda |\varphi| \varphi$. $|\varphi| \neq 0$.

$\tilde{g} = |\varphi|^2 g$ $\qquad D^{\tilde{g}} \varphi = \lambda \varphi$.

$\frac{2n}{n-1} - 2 = \frac{2}{n-1}$

$\frac{n-1}{2}$

$|\varphi|_{\tilde{g}} = 1$

$\nabla_X \varphi = A(X) \cdot \varphi$ $\qquad$ solution

If g is analytic, then obtain N.

If g is nonanalytic, N cannot exist.