Recent developments in pseudo-Riemannian holonomy theory

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Abstract. We review recent results in the theory of holonomy groups of pseudo-Riemannian manifolds, i.e. manifolds with indefinite metrics. First we present the classification of Lorentzian holonomy groups, that is a list of possible groups and metrics which realise these as holonomy groups. This is followed by applications and some remarks about holonomy related structures on Lorentzian manifolds. Then we review partial results in signature $(2, n+2)$, in particular the classification of unitary holonomy groups, again presenting the groups and the realising metrics. Then we turn to results in neutral signature (n, n), focussing on the situation of a para-Kähler structure. Finally, the classification in signature $(2, 2)$ obtained by Bérard-Bergery and Ikemakhen is presented. As a new result we prove the existence of metrics in cases for which the realisation as holonomy group was left open in their article.

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Contents

1. Introduction

In the study of geometric structures of manifolds equipped with a non-degenerate metric the notion of a holonomy group turned out to be very useful. It links geometric and algebraic properties and allows to apply the tools of algebra to geometric questions. In particular, it enables us describe parallel sections in geometric vector bundles associated to the manifold, such as the tangent bundle, tensor bundles, or the spin bundle, as holonomy-invariant objects and by algebraic means. Hence, a classification of holonomy groups gives a framework in which geometric structures on semi-Riemannian manifolds can be studied.

Many interesting developments in differential geometry were initiated or driven by the study and the knowledge of holonomy groups, such as the study of so-called *special geometries* in Riemannian geometry. These developments were based on the classification result of Riemannian holonomy groups, which was achieved by the de Rham decomposition theorem [36] and the Berger list of irreducible pseudo-Riemannian holonomy groups [14] (see Section 2.2 of the present article).

For manifolds with indefinite metric this question was for a long time widely open and untackled, apart from a classification regarding 4-dimensional Lorentzian metrics by J. F. Schell [75] and R. Shaw [79] (presented in Section 3.7). The main difficulty in the case of pseudo-Riemannian manifolds is the situation that the holonomy group preserves a degenerate subspace of the tangent space. In this situation the de Rham theorem does not apply, and one cannot reduce the algebraic aspect of the classification problem to irreducible representations. However, the Wu theorem [86] and the results of Berger [14] reduce the task of classifying holonomy groups of pseudo-Riemannian manifolds to the case of indecomposably, non-irreducibly acting groups (any such group does not preserve any proper non-degenerate subspace of the tangent space, but preserves a proper isotropic subspace of the tangent space, see Section 2.2). In this article we want to present how this problem can be dealt with and which classifications results have been obtained recently by applying this method.

After explaining the basic properties of affine and semi-Riemannian holonomy groups in Section 2, we discuss the classification of the holonomy algebras (equivalently, connected holonomy groups) of Lorentzian manifolds in Section 3. The first step in this classification was done in 1993 by L. Bérard-Bergery and A. Ikemakhen who divided indecomposable, non-irreducible subalgebras of $\mathfrak{so}(1, n+1)$ into 4 types, see [11]. In [47] a more geometric proof of this result is given, which is presented in Section 3.2. To each indecomposable, non-irreducible subalgebra of $\mathfrak{h} \subset \mathfrak{so}(1, n+1)$ one can associate a subalgebra of $\mathfrak{so}(n)$, which is called the orthogonal part of $\mathfrak{h}$. In [66, 68, 69] (see also [70, 73]) it is proved that the orthogonal part of an indecomposable, non-irreducible holonomy algebra of a Lorentzian manifold is the holonomy algebra of a Riemannian manifold (see Section 3.3). In [48] metrics for all possible holonomy algebras of Lorentzian manifold were constructed (Section 3.4). This completes the classification of holonomy algebras for Lorentzian manifolds. There are many applications of holonomy theory for Lorentzian manifold such as the study of equations motivated by physics in relation to the possible holonomy groups. On the one hand these are the Einstein equations, on the other hand certain spinor field equations in supergravity theories (see e.g. [42]). In Section 3.5 we present the classification of holonomy groups of indecomposable Lorentzian manifolds which admit a parallel spinor. In Section 3.6 we describe holonomy related geometric structures on Lorentzian manifolds.

Widely open is the classification problem of holonomy groups in signatures other than Riemannian and Lorentzian apart from some results in certain signatures.

In Section 4 we discuss the holonomy of pseudo-Riemannian manifolds of index 2. For indecomposable, non-irreducible subalgebras of $\mathfrak{so}(2, n+2)$ that satisfy a certain condition Ikemakhen gave in [56] a distinction into different types similar to the Lorentzian case. In [44] the analog of the orthogonal part of an indecomposable, non-irreducible subalgebra in $\mathfrak{so}(2, n+2)$ is studied. The surprising result is that, unlike to the Lorentzian case, there is no additional condition on the subalgebra $\mathfrak{g} \subset \mathfrak{so}(n)$ induced by the Bianchi-identity and replacing the property which turned out to be essential in Lorentzian signature. Instead, any subalgebra of $\mathfrak{so}(n)$ can be realised as this part of a holonomy algebra, see Section 4.1. Furthermore, in [45] indecomposable, non-irreducible holonomy algebras of pseudo-Kählerian manifolds of index 2, i.e. holonomy algebras contained in $\mathfrak{u}(1, n+1) \subset \mathfrak{so}(2, 2n+2)$, were classified. We present this classification with the idea of the proof in Section 4.2. In Section 4.3 examples of 4-dimensional Lie groups with left-invariant pseudo-Kählerian metrics are given.

In Section 5 we review the results known about holonomy for metrics of neutral signature (n, n), again obtained by Bérard-Bergery and Ikemakhen in [12]. They also gave a list of possible holonomy groups in signature $(2, 2)$, and realised those in the list which leave invariant two complementary totally isotropic planes as holonomy algebras. Our results in [45] enables us to realise all subalgebras of $\mathfrak{u}(1, 1)$ from this list. As a new result we give metrics realising two algebras which were still in question to be realised. We should remark, that some of these metrics disprove claims made in [49] that the corresponding algebras cannot be realised as holonomy algebras. This result completes the classification of holonomy groups in signature (2,2), apart from one exception, for which we could not find a metric. Upon completion of this article we were informed that L. Bérard-Bergery and T. Krantz have developed a construction on the cotangent bundle of a surface which ensures that even this last group can be realised as a holonomy group [13].

2. Holonomy groups

2.1. Holonomy groups of linear connections. If a smooth manifold M of dimension m is equipped with a linear connection ∇ on the tangent bundle TM, we can parallel translate a tangent vector $X \in T_pM$ at a point $p \in M$ along any given curve $\gamma : [0, 1] \to M$ starting at p, i.e. $\gamma(0) = p$. The parallel displacement, denoted by $X(t)$, is a vector field along γ satisfying the equation $\nabla_{\dot{\gamma}(t)} X(t) = 0$ for all t in the domain of the curve. This is a linear ordinary differential equation, and thus, for any curve γ the map

$$\begin{array}{cccc} \mathcal{P}_{\gamma(t)} & : & T_{\gamma(0)}M & \to & T_{\gamma(t)}M \\ & & X & \mapsto & \mathcal{P}_{\gamma(t)}(X) := X(t) \end{array}$$

is a vector space isomorphism which is called *parallel displacement*. Hence, ∇ enables us to link the tangent spaces in different points, which is the reason why it bears the name *connection*. Then the *holonomy group* of ∇ at p is the group defined by parallel displacements along loops about this point,

$$Hol_p(M,\nabla) \quad := \quad \{\mathcal{P}_{\gamma(1)} \mid \gamma(0)=\gamma(1)=p\}.$$

This group is a Lie group which is connected if the manifold is simply connected. Its connected component is called *connected holonomy group*, denoted by $Hol_p^0(M,\nabla)$, and its the group generated by parallel displacements along homotopically trivial loops. Its Lie algebra $\mathfrak{hol}_p(M,\nabla)$ its called *holonomy algebra*. Obviously, both are given together with their representation on the tangent space T_pM which is usually identified with the $\mathbb{R}^m$. In this sense we have that $Hol_p(M,\nabla)\subset Gl(m,\mathbb{R})$, but only defined up to conjugation. Holonomy groups at different points in a connected component of the manifold are conjugated by an element in $Gl(m,\mathbb{R})$, which is obtained by the parallel displacement along a curve joining these different points. It is worthwhile to note that the holonomy group is closed if it acts irreducibly (for a proof of this fact see [82] or [38]). This is not true in general, there are examples of non-closed holonomy groups.

The calculation of holonomy groups uses the *Ambrose–Singer holonomy theorem*, which states that for a manifold M with linear connection ∇ the holonomy algebra $\mathfrak{hol}_p(M,\nabla)$ is equal to

$$\operatorname{span}\left\{ \mathcal{P}^{-1}_{\gamma(t)} \circ \mathcal{R}\left(\mathcal{P}_{\gamma(t)}X, \mathcal{P}_{\gamma(t)}Y\right) \circ \mathcal{P}_{\gamma}(t) \,\middle|\, \gamma(0)=p,\ X,Y\in T_pM \right\},$$

where $\mathcal{R}$ is the curvature of ∇, $\mathcal{R}(X,Y)=[\nabla_X,\nabla_Y]-\nabla_{[X,Y]}$. For connections that have no torsion, i.e. $\nabla_XY-\nabla_YX=[X,Y]$, the curvature satisfies the first Bianchi-identity

$$\mathcal{R}(X,Y)Z+\mathcal{R}(Y,Z)X+\mathcal{R}(Z,X)Y \quad = \quad 0,$$

which imposes very strong algebraic conditions on the holonomy algebra which can be described in terms of *curvature endomorphisms*. Let $\mathbb{K}$ be the real or complex numbers. The curvature endomorphisms of a subalgebra $\mathfrak{g}\subset\mathfrak{gl}(n,\mathbb{K})$ are defined as

$$\mathcal{K}(\mathfrak{g}) \ := \ \{R\in\Lambda^2(\mathbb{K}^n)^*\otimes\mathfrak{g} \mid R(x,y)z+R(y,z)x+R(z,x)y=0\}. \tag{1}$$

$\mathcal{K}(\mathfrak{g})$ is a $\mathfrak{g}$-module, and the space $\mathfrak{g}^{\mathcal{K}} \ := \ span\{R(x,y)\mid x,y\in\mathbb{K}^n, R\in\mathcal{K}(\mathfrak{g})\}$ is an ideal in $\mathfrak{g}$. One defines:

Definition 2.1. $\mathfrak{g}\subset\mathfrak{gl}(n,\mathbb{K})$ is called *Berger algebra* if $\mathfrak{g}^{\mathcal{K}}=\mathfrak{g}$.

Then, the Ambrose Singer holonomy theorem implies the following result.

Theorem 2.2. *The Lie algebra of a holonomy group of a torsion free connection on a smooth manifold is a Berger algebra.*

Hence, there are two steps involved in the classification of holonomy groups of torsion free connections. The first is to classify Berger algebras, and the second to find torsion free connections with the algebras obtained as holonomy algebras. The first problem can be solved in full generality when the Berger algebra acts *irreducibly*. This was done recently by S. Merkulov and L. Schwachhöfer in [74, 77, 78], also providing examples of torsion free connections realising all the algebras obtained. This classification extends the well-known *Berger list* of irreducible holonomy groups of pseudo-Riemannian manifolds (c.f. next section). However, the assumption of irreducibility is essential for these classification results because their proof uses the theory of irreducible representations of Lie algebras. An overview about results on irreducible holonomy groups is also given in [25] and [24].

As we will see later, the approach we have to take for connections of indefinite metrics has to deal with non-irreducible representations.

One should remark that a classification problem for holonomy groups only arises if one poses further conditions on the linear connection, such as conditions on the torsion because of the following result of J. Hano and H. Ozeki [54]: Any closed subgroup of $Gl(m,\mathbb{R})$ can be obtained as a holonomy group of a linear connection, but possibly a connection with torsion. They also gave examples of holonomy groups which were not closed. We will return to this question later.

Concluding this introductory section we want to point out a general principle in holonomy theory which says that any subspace which is invariant under the holonomy group corresponds to a distribution (i.e. a subbundle of the tangent bundle) which is invariant under parallel transport. Obviously, this distribution is obtained by parallel transporting the invariant subspace, and this procedure is independent of the chosen path because of the holonomy invariance of the subspace. This distribution is called *parallel*, which means that its sections are mapped onto its sections under ∇_X for any $X \in TM$.

2.2. Holonomy groups of semi-Riemannian manifolds. (M,g) is a semi-Riemannian manifold of dimension $m = r + s$ and signature (r,s) if g is a metric of signature (r,s). If the metric is positive definite ($r = 0$ in our convention) it is called *Riemannian*, otherwise *pseudo-Riemannian.* For a semi-Riemannian manifold, there exists a uniquely defined linear torsion-free connection $\nabla = \nabla^g$ which parallelises the metric, called *Levi-Civita connection.* The holonomy group of a semi-Riemannian manifold then is the holonomy group of this connection, $Hol_p(M,g) := Hol_p(M,\nabla)$. As the Levi-Civita connection is metric, the parallel displacement preserves the metric. This implies on the one hand that the holonomy group is a subgroup of $O(T_pM,g)$, and can be understood as a subgroup of $O(r,s)$ which is only defined up to conjugation in $O(r,s)$. On the other hand it ensures that for a subspace $\mathcal{V} \subset T_pM$ which is invariant under the holonomy group the orthogonal complement $\mathcal{V}^\perp$ is invariant as well.

For a Riemannian metric the holonomy group acts completely reducibly, i.e. the tangent space decomposes into subspaces on which it acts trivially or irreducibly, but for indefinite metrics the situation is more subtle. We say that the holonomy group acts *indecomposably* if the metric is degenerate on any invariant proper subspace. In this case we also say that the manifold is *indecomposable.* Of course, for Riemannian manifolds, this is the same as irreducibility.

A remarkable property is that the holonomy group of a product of Riemannian manifolds (i.e. equipped with the product metric) is the product of the holonomy groups of these manifolds (with the corresponding representation on the direct sum). Even more remarkable is the fact that a converse of this statement is true in the following sense: Any pseudo-Riemannian manifold whose tangent space at a point admits a decomposition into *non-degenerate*, holonomy-invariant subspaces is locally isometric to a product of pseudo-Riemannian manifolds corresponding to the invariant subspaces, and moreover, the holonomy group is a product of the groups acting on the corresponding invariant subspaces. These groups are the holonomy groups of the manifolds in the local product decomposition if the original manifold is complete (see [16, Theorem 10.38 and Remark 10.42]). This was proven by A. Borel and A. Lichnerowicz [18], and the property that a decomposition of the representation space entails a decomposition of the acting group is sometimes called *Borel–Lichnerowicz property.* A global version of this statement was proven under the assumption that the manifold is simply-connected and complete by G. de Rham [36, for Riemannian manifolds] and H. Wu [86, in arbitrary signature]. Summarizing we have the following result:

Theorem 2.3 (G. de Rham [36] and H. Wu [86]). *Any simply-connected, complete pseudo-Riemannian manifold (M,g) is isometric to a product of simply connected, complete pseudo-Riemannian manifolds one of which can be flat and the others have an indecomposably acting holonomy group and the holonomy group of (M,g) is the product of these indecomposably acting holonomy groups.*

The other groundbreaking result in the holonomy theory of semi-Riemannian manifolds is the list of irreducible holonomy groups of non locally-symmetric pseudo-Riemannian manifolds, which was obtained by M. Berger [14]. This list as it appears here is a result of the efforts of several other authors, simplifying the proof in Riemannian signature [80], eliminating groups of locally symmetric metrics [1, 22], realising the exceptional groups as holonomy groups [23], eliminating groups which were not Berger algebras and finding missing entries (for an overview see [24]).

Theorem 2.4 (M. Berger [14]). *Let (M,g) be a simply connected pseudo-Riemannian manifold of dimension $m=r+s$ and signature (r,s), which is not locally-symmetric. If the holonomy group of (M,g) acts irreducibly, then it is either $SO_0(r,s)$ or one of the following (modulo conjugation in $O(r,s)$):*

$$\begin{array}{rcl}
U(p,q) \text{ or } SU(p,q) & \subset & SO(2p,2q),\ m\geq 4\\
Sp(p,q) \text{ or } Sp(p,q)\cdot Sp(1) & \subset & SO(4p,4q),\ m\geq 8\\
SO(r,\mathbb{C}) & \subset & SO(r,r),\ m\geq 4\\
Sp(p,\mathbb{R})\cdot Sl(2,\mathbb{R}) & \subset & SO(2p,2p),\ m\geq 8\\
Sp(p,\mathbb{C})\cdot Sl(2,\mathbb{C}) & \subset & SO(4p,4p),\ m\geq 16\\
G_2 & \subset & SO(7)\\
G^*_{2(2)} & \subset & SO(4,3)\\
G_2^{\mathbb{C}} & \subset & SO(7,7)\\
Spin(7) & \subset & SO(8)\\
Spin(4,3) & \subset & SO(4,4)\\
Spin(7)^{\mathbb{C}} & \subset & SO(8,8)
\end{array}$$

We should remark that a lot of progress has been made in constructing Riemannian manifolds with given holonomy group and certain *topological properties*. We only mention a few: compact manifolds with holonomy $Sp(q)$ have been constructed in [8], see also [16]. [64] constructed complete manifolds with holonomy $Sp(1)\cdot Sp(q)$, for compact examples with this holonomy group see [65]. Complete examples with exceptional holonomy G_2 and $Spin(7)$ were constructed in [27] and compact ones in [59, 58]. For an overview over related results see [24]. For indefinite metrics these global questions are widely open, apart from attempts in [7], where globally hyperbolic Lorentzian metrics with indecomposable, non-irreducible holonomy group (see next section) were constructed.

Returning to the classification problem, for Riemannian manifolds one combines these classification results with the de Rham decomposition in order to obtain a comprehensive holonomy classification.

Theorem 2.5. *Any simply-connected, complete Riemannian manifold (M,g) is isometric to a product of simply-connected, complete Riemannian manifolds one of which may be flat and the others are either locally symmetric or have one of the following groups as holonomy, $SO(n)$, $U(n)$, $SU(n)$, $Sp(n)$, $Sp(n)\cdot Sp(1)$, G_2, or $Spin(7)$. The holonomy group of (M,g) is a product of these groups.*

In case of symmetric spaces, the holonomy group is equal to the isotropy group of the symmetric space and in many cases determines the space up to duality. Simply connected pseudo-Riemannian symmetric spaces with irreducible holonomy groups were classified by E. Cartan [33, for Riemannian signature] and M. Berger [15, for arbitrary signature]. In the case of indecomposable, non-irreducible holonomy group the classification exists in the following cases: Lorentzian signature [30], signature $(2,q)$ [28, 29, 60], hyper-Kählerian manifolds of signature $(4,4q)$ [4, 61]. For more details see the overview [62].

For indefinite metrics there is the possibility that one of the factors in Theorem 2.3 is indecomposable, but non-irreducible. This means that the holonomy representation admits an invariant subspace on which the metric is degenerate, but no proper non-degenerate invariant subspace. An attempt to classify holonomy groups for indefinite metric has to provide a classification of these indecomposable, non-irreducible holonomy groups.

If a holonomy group $Hol_p(M,h) =: H \subset SO_0(p,q)$ acts indecomposably, but non-irreducibly, with an degenerate invariant subspace $\mathcal{V} \subset T_pM$, it admits a totally isotropic invariant subspace

$$\mathcal{I} := \mathcal{V} \cap \mathcal{V}^{\perp}.$$

This implies that H is contained in the stabiliser of this totally isotropic subspace,

$$H \subset SO_0(p,q)_{\mathcal{I}} := \{A \in SO_0(p,q) \mid A\mathcal{I} \subset \mathcal{I}\},$$

or in terms of the corresponding Lie algebras

$$\mathfrak{h} \subset \mathfrak{so}(p,q)_{\mathcal{I}} := \{X \in \mathfrak{so}(p,q) \mid X\mathcal{I} \subset \mathcal{I}\}.$$

In the following sections we will present results about the classification of holonomy groups contained in $\mathfrak{so}(p,q)_{\mathcal{I}}$ for $\mathcal{I}$ a totally isotropic subspace.

Finally we should mention that by the general principle, $\mathcal{I}$ defines a totally isotropic distribution on M, i.e. a subbundle of TM which is invariant under parallel transport. This parallel distribution ensures the existence of so-called *Walker co-ordinates* (first in [83, 84, 85], see also [37]).

Theorem 2.6 (Walker [83, 84, 85]). *Let (M,h) be a pseudo-Riemannian manifold of dimension n with a parallel r-dimensional totally isotropic distribution $\mathcal{I}$. The exist co-ordinates $(x_1,\dots,x_n)$ such that*

$$\left(h\left(\frac{\partial}{\partial x_i},\frac{\partial}{\partial x_j}\right)\right)_{i,j=1}^{n} \begin{pmatrix} 0 & 0 & \mathbb{I}_r \\ 0 & G & F \\ \mathbb{I}_r & F^t & H \end{pmatrix},$$

where F, G, H are matrices of smooth functions, G is a symmetric $(n-2r)\times(n-2r)$ matrix, H is a symmetric $r\times r$ matrix, and F is a $r\times(n-2r)$ matrix, such that G and F are independent of the co-ordinates $(x_1,\dots,x_r)$.

These co-ordinates will be useful in order to obtain metrics which realise the possible indecomposable, non-irreducible holonomy groups.

3. Lorentzian holonomy groups

3.1. The classification result. In this section we want to describe the classification of reduced holonomy groups of Lorentzian manifolds. First of all, the Berger list in Theorem 2.4 implies that the only *irreducible* holonomy group of Lorentzian manifolds is the full $SO_0(1,n)$. This is due to the algebraic fact that the only connected irreducible subgroup of $O(1,n)$ is $SO_0(1,n)$ which was proven by A. J. Di Scala and C. Olmos [39] (see also [20, 10, 38] for other proofs). Hence, if one is interested in Lorentzian manifolds with *special holonomy*, i.e. with proper subgroups of $SO_0(1,n)$ as holonomy but not being a product, one has to look at manifolds admitting a holonomy-invariant subspace. Using this fact, the decomposition in Theorem 2.3 gives the following result for Lorentzian manifolds.

Corollary 3.1. *Any simply-connected, complete Lorentzian manifold (M,h) is isometric to the following product of simply-connected complete pseudo-Riemannian manifolds,*

$$(N,h) \times (M_1,g_1) \times \ldots \times (M_k,g_k),$$

where the (M_i,g_i) are either flat or irreducible Riemannian manifolds and (N,h) is either $(\mathbb{R},-dt^2)$ or an indecomposable Lorentzian manifold, the holonomy of which is either $SO_0(1,n)$ or contained in the stabiliser $SO_0(1,n)_{\mathcal{I}}$ of a light-like line $\mathcal{I}$. The holonomy group of (M,h) is the product of the holonomy groups of (N,h) and the (M_i,g_i)'s.

Hence, we have to focus on the classification of Lorentzian holonomy groups which act indecomposably, but non-irreducibly, i.e. which are contained in the stabiliser in $SO_0(T_pM)$ of a light-like line $\mathcal{I}$ in T_pM. This stabiliser is the parabolic group $SO_0(1,n+1)_{\mathcal{I}}$ in the conformal group $SO_0(1,n+1)$ if $(n+2)$ is the dimension of M. To describe this stabiliser further we identify T_pM with the Minkowski space $\mathbb{R}^{1,n+1}$ of dimension $(n+2)$ and fix a basis $(X,E_1,\ldots,E_n,Z)$ in which the scalar product has the form

$$\begin{pmatrix} 0 & 0^t & 1 \\ 0 & \mathbb{I}_n & 0 \\ 1 & 0^t & 0 \end{pmatrix}, \tag{2}$$

where $\mathbb{I}_n$ is the n-dimensional identity matrix. The Lie algebra of the connected stabiliser of $\mathcal{I}=\mathbb{R}\cdot X$ inside the conformal group $SO_0(1,n+1)$ can be written as follows

$$\mathfrak{so}(1,n+1)_{\mathcal{I}} \left\{ \left. \begin{pmatrix} a & v^t & 0 \\ 0 & A & -v \\ 0 & 0^t & -a \end{pmatrix} \right| a\in\mathbb{R}, v\in\mathbb{R}^n, A\in\mathfrak{so}(n) \right\}. \tag{3}$$

This Lie algebra is a semi-direct sum in an obvious way, $\mathfrak{so}(1,n+1)_{\mathcal{I}} = (\mathbb{R}\oplus\mathfrak{so}(n)) \ltimes \mathbb{R}^n$, the commutator relations are given as follows:

$$[(a,A,v),(b,B,w)] = \Big(0,[A,B]_{\mathfrak{so}(n)},(A+a\,Id)\,w-(B+b\,Id)\,v\Big). \tag{4}$$

In this sense we will refer to $\mathbb{R}$, $\mathbb{R}^n$ and $\mathfrak{so}(n)$ as subalgebras of $\mathfrak{so}(1,n+1)_{\mathcal{I}}$. $\mathbb{R}$ is an Abelian subalgebra of $\mathfrak{so}(1,n+1)_{\mathcal{I}}$, commuting with $\mathfrak{so}(n)$, and $\mathbb{R}^n$ an Abelian ideal in $\mathfrak{so}(1,n+1)_{\mathcal{I}}$. $\mathfrak{so}(n)$ is the semisimple part and $\mathfrak{co}(n)=\mathbb{R}\oplus\mathfrak{so}(n)$ the reductive part of $\mathfrak{so}(1,n+1)_{\mathcal{I}}$. The corresponding connected Lie groups in $SO_0(1,n+1)_{\mathcal{I}}$ are $\mathbb{R}_+$, $SO(n)$, and $\mathbb{R}^n$, and $SO_0(1,n+1)_{\mathcal{I}}$ is equal to the semidirect product $(\mathbb{R}_+\times SO(n))\ltimes\mathbb{R}^n$.

Now one can assign to a subalgebra $\mathfrak{h}\subset\mathfrak{so}(1,n+1)_{\mathcal{I}}$ the projections $pr_{\mathbb{R}}(\mathfrak{h})$, $pr_{\mathbb{R}^n}(\mathfrak{h})$ and $pr_{\mathfrak{so}(n)}(\mathfrak{h})$. The subalgebra $\mathfrak{g}:=pr_{\mathfrak{so}(n)}(\mathfrak{h})$ associated to a $\mathfrak{h}$ is called *the orthogonal part of* $\mathfrak{h}$. Note that if $\mathfrak{h}\subset\mathfrak{so}(1,n+1)_{\mathcal{I}}$ acts indecomposably , then $pr_{\mathbb{R}^n}(\mathfrak{h})=\mathbb{R}^n$, and $\mathfrak{h}$ is Abelian if and only if $\mathfrak{h}=\mathbb{R}^n$. Moreover, $\mathfrak{h}$ has a trivial subrepresentation if and only if $pr_{\mathbb{R}}\mathfrak{h}=0$. In this case (M,h) admits a parallel light-like vector field.

Finally $\mathfrak{g} := pr_{\mathfrak{so}(n)}(\mathfrak{h}) \subset \mathfrak{so}(n)$ is compact, i.e. there exists a positive definite invariant symmetric bilinear form on it. This implies that $\mathfrak{g}$ is *reductive*, i.e. its Levi decomposition is $\mathfrak{g} = \mathfrak{z} \oplus \mathfrak{g}'$ where $\mathfrak{z}$ is the center of $\mathfrak{g}$ and $\mathfrak{g}' := [\mathfrak{g}, \mathfrak{g}] = \mathfrak{z}^\perp$ is the derived Lie algebra, which is semisimple.

Now, the classification of indecomposable, non-irreducible Lorentzian holonomy algebras consists of two main results. The first is the distinction of indecomposable subalgebras of $\mathfrak{so}(1, n+1)_{\mathcal{I}}$ into four types due to the relation between their projections obtained by L. Bérard-Bergery and A. Ikemakhen [11].

Theorem 3.2 (Bérard-Bergery, Ikemakhen [11]). *Let $\mathfrak{h}$ be a subalgebra of $\mathfrak{so}(1, n+1)_{\mathcal{I}} = (\mathbb{R} \oplus \mathfrak{so}(n)) \ltimes \mathbb{R}^n$, acting indecomposably on $\mathbb{R}^{n+2}$, and let $\mathfrak{g} : pr_{\mathfrak{so}(n)}(\mathfrak{h}) = \mathfrak{z} \oplus \mathfrak{g}'$ be the Levi-decomposition of its orthogonal part. Then $\mathfrak{h}$ belongs to one of the following types.*

1. *If $\mathfrak{h}$ contains $\mathbb{R}^n$, then there are three types:*

   Type 1: *$\mathfrak{h}$ contains $\mathbb{R}$. Then $\mathfrak{h} = (\mathbb{R} \oplus \mathfrak{g}) \ltimes \mathbb{R}^n$.*

   Type 2: *$pr_{\mathbb{R}}(\mathfrak{h}) = 0$, i.e. $\mathfrak{h} = \mathfrak{g} \ltimes \mathbb{R}^n$.*

   Type 3: *Neither type 1 nor type 2. In this case there exists an epimorphism $\varphi : \mathfrak{z} \to \mathbb{R}$, such that*
   $$\mathfrak{h} = (\mathfrak{l} \oplus \mathfrak{g}') \ltimes \mathbb{R}^n,$$
   where $\mathfrak{l} := graph\ \varphi = \{(\varphi(T), T) | T \in \mathfrak{z}\} \subset \mathbb{R} \oplus \mathfrak{z}$. Or, written in matrix form:
   $$\mathfrak{h} = \left\{ \left(\begin{array}{ccc} \varphi(A) & v^t & 0 \\ 0 & A+B & -v \\ 0 & 0 & -\varphi(A) \end{array} \right) \middle| A \in \mathfrak{z}, B \in \mathfrak{g}', v \in \mathbb{R}^n \right\}.$$

2. *In the case where $\mathfrak{h}$ does not contain $\mathbb{R}^n$ we have* **Type 4:** *There exists a non-trivial decomposition $\mathbb{R}^n = \mathbb{R}^k \oplus \mathbb{R}^l$, $0 < k, l < n$ and a epimorphism $\psi : \mathfrak{z} \to \mathbb{R}^l$, such that $\mathfrak{g} \subset \mathfrak{so}(k)$ and $\mathfrak{h} = (\mathfrak{l} \oplus \mathfrak{g}') \ltimes \mathbb{R}^k \subset \mathfrak{so}(1, n+1)_{\mathcal{I}}$ where $\mathfrak{l} := \{(\varphi(T), T) | T \in \mathfrak{z}\} = graph\ \psi \subset \mathbb{R}^l \oplus \mathfrak{z}$. Or, written in matrix form:*
   $$\mathfrak{h} = \left\{ \left(\begin{array}{cccc} 0 & \psi(A)^t & v^t & 0 \\ 0 & 0 & 0 & -\psi(A) \\ 0 & 0 & A+B & -v \\ 0 & 0 & 0 & 0 \end{array} \right) \middle| A \in \mathfrak{z}, B \in \mathfrak{g}', v \in \mathbb{R}^k \right\}.$$

The distinction of Theorem 3.2 obviously gives four types for the corresponding connected, indecomposable groups in the parabolic $SO_0(1, n+1)_{\mathcal{I}}$. We should remark that these types are independent of conjugation within $O(1, n+1)$.

The second result gives a classification of the orthogonal part and was proved in [66, 68, 69, 48].

Theorem 3.3. *Let H be a connected subgroup of $SO_0(1, n+1)$ which acts indecomposably and non-irreducibly. Then H is a Lorentzian holonomy group if and only if its orthogonal part is a Riemannian holonomy group.*

Naturally, the proof of this theorem consists of two main steps. The first is to show that the orthogonal part of H has to be a Riemannian holonomy group. This involves the notion of weak-Berger algebras and their classification, which is explained in Section 3.3. This step uses similar methods as the classification of irreducible holonomy groups of torsion free connections and was done in [66, 68, 69]. The second step consists of showing that each of the arising groups can actually be realised as holonomy group. This is easy for the Types 1 and 2 in Theorem 3.2 (see for example [67, 70]) but more involved for the coupled Types 3 and 4 and was achieved recently in [48]. This method is explained in Section 3.4.

3.2. Indecomposable subalgebras of $\mathfrak{so}(1,n+1)_{\mathcal{I}}$. The proof of Theorem 3.2 given by L. Bérard-Bergery and A. Ikemakhen was purely algebraic. We describe now a more geometric proof of this theorem given in [47], which works directly for the groups and provides a geometric interpretation for the different types. Therefore we equip $\mathbb{R}^n$ with the Euclidean scalar product. Denote by $\mathrm{Sim}(n)$ the connected component of the Lie group of similarity transformations of $\mathbb{R}^n$. Then $\mathbb{R}_+$, $SO(n)$, and $\mathbb{R}^n$ in $\mathrm{Sim}(n)$ are the connected identity components of the Lie groups of homothetic transformations, rotations and translations, respectively. We obtain for $\mathrm{Sim}(n)$ the same decomposition as for $SO_0(1,n+1)_{\mathcal{I}}$, i.e. we have a Lie group isomorphism $\Gamma : P \to \mathrm{Sim}(n)$. The isomorphism Γ can be defined geometrically. For this consider the vector model of the real hyperbolic space $\mathbf{H}^{n+1} \subset \mathbb{R}^{1,n+1}$ and its boundary $\partial\mathbf{H}^{n+1} \subset \mathbb{PR}^{1,n+1}$ that consists of isotropic lines of $\mathbb{R}^{1,n+1}$ and is isomorphic to the n-dimensional sphere. Any element $f \in P$ induces a transformation $\Gamma(f)$ of the Euclidean space $\partial\mathbf{H}^{n+1}\backslash\{\mathbb{R}\cdot X\} \simeq \mathbb{R}^n$. In fact, $\Gamma(f)$ is a similarity transformation of $\mathbb{R}^n$. This defines the isomorphism Γ. Now, in [47] we have proven that *a connected Lie subgroup $H \subset P$ is indecomposable if and only if the subgroup $\Gamma(H) \subset \mathrm{Sim}(n)$ acts transitively on $\mathbb{R}^n$.* Then, using a description for connected transitive subgroups of $\mathrm{Sim}(n)$ given in [2] and [3], one can show the following theorem.

Theorem 3.4. *A connected Lie subgroup $H \subset \mathrm{Sim}(n)$ is transitive if and only if H belongs to one of the following types for which $G \subset SO(n)$ is a connected Lie subgroup:*

Type 1: $H = (\mathbb{R}_+ \times G) \ltimes \mathbb{R}^n$;

Type 2: $H = G \ltimes \mathbb{R}^n$;

Type 3. $H = (\mathbb{R}_+^{\Phi} \times G) \ltimes \mathbb{R}^n$, *where* $\Phi : \mathbb{R}_+ \to SO(n)$ *is a non-trivial homomorphism and*

$$\mathbb{R}_+^{\Phi} = \{a \cdot \Phi(a) | a \in \mathbb{R}_+\} \subset \mathbb{R}_+ \times SO(n)$$

is a group of screw dilations of $\mathbb{R}^n$ *that commutes with* G;

Type 4. $H = (G \times (\mathbb{R}^{n-m})^{\Psi}) \ltimes \mathbb{R}^m$, *where* $0 < m < n$, $\mathbb{R}^n = \mathbb{R}^m \oplus \mathbb{R}^{n-m}$ *is an orthogonal decomposition,* $\Psi : \mathbb{R}^{n-m} \to SO(m)$ *is a homomorphism with* $\ker d\Psi = \{0\}$, *and*

$$(\mathbb{R}^{n-m})^{\Psi} = \{\Psi(u) \cdot u | u \in \mathbb{R}^{n-m}\} \subset SO(m) \times \mathbb{R}^{n-m}$$

is a group of screw isometries of $\mathbb{R}^n$ *that commutes with* G.

The indecomposable Lie algebras of the corresponding Lie subgroups of $SO_0(1,n+1)_{\mathcal{I}}$ have the same type as in Theorem 3.2.

Now we want to describe the curvature endomorphisms $\mathcal{K}(\mathfrak{h})$ for a subalgebra $\mathfrak{h} \subset \mathfrak{so}(1,n+1)_{\mathcal{I}} \subset \mathfrak{so}(1,n+1)$ with respect to these four types. In addition to the space $\mathcal{K}(\mathfrak{h})$ defined in (1) we define another kind of curvature endomorphisms. Let $\mathbb{K}$ be the real or complex numbers. For a subalgebra $\mathfrak{g} \subset \mathfrak{so}(n,\mathbb{C})$ or $\mathfrak{g} \subset \mathfrak{so}(r,s)$ we set

$$\mathcal{B}(\mathfrak{g}) \ := \ \{Q \in (\mathbb{K}^n)^* \otimes \mathfrak{g}) \mid \langle Q(x)y,z\rangle + \langle Q(y)z,x\rangle + \langle Q(z)x,y\rangle = 0\}, \qquad (5)$$

where $\langle .,. \rangle$ is the corresponding scalar product. $\mathcal{B}(\mathfrak{g})$ is a $\mathfrak{g}$-modules of *curvature endomorphisms*. In order to distinguish it from $\mathcal{K}(\mathfrak{g})$, one may call $\mathcal{B}(\mathfrak{g})$ the space of *weak curvature endomorphisms*. In [46] the following theorem was proved, for which we fix a basis $(X, E_1, \ldots, E_n, Z)$ of $\mathbb{R}^{1,n+1}$ as in the previous section.

Theorem 3.5. *Let $\mathfrak{h}$ be a subalgebra of the parabolic algebra $\mathfrak{so}(1, n+1)_{\mathcal{I}}$ in $\mathfrak{so}(1, n+1)$ and $\mathfrak{g} = \mathfrak{z} \oplus \mathfrak{g}'$ its orthogonal part. Then it holds:*

(1) Any $R \in \mathcal{K}(\mathfrak{h})$ is uniquely given by

$$\lambda \in \mathbb{R},\ L \in (\mathbb{R}^n)^*,\ Q \in \mathcal{B}(\mathfrak{g}),\ R_0 \in \mathcal{K}(\mathfrak{g}),\ \text{and } T \in End(\mathbb{R}^n) \text{ with } T^* = T$$

in the following way,

$$\begin{aligned} R(X, Z) &= (\lambda, 0, L^*(1)), \qquad & R(U, V) &= (0, R_0(U, V), -\frac{1}{2}Q^*(U \wedge V)) \\ R(U, Z) &= (L(U), Q(U), T(U)), \qquad & R(X, U) &= 0, \end{aligned}$$

where $U, V \in \mathrm{span}(E_1, \ldots, E_n)$.

(2) If $\mathfrak{h}$ is indecomposable of Type 2, any $R \in \mathcal{K}(\mathfrak{h})$ is given as in (1) with $\lambda = 0$ and $L = 0$.

(3) If $\mathfrak{h}$ is indecomposable of Type 3 defined by the epimorphism $\varphi : \mathfrak{z} \to \mathbb{R}$, any $R \in \mathcal{K}(\mathfrak{h})$ is given as in (1) with $\lambda = 0$, $L = \tilde{\varphi} \circ Q$ and $R_0 \in \mathcal{K}(\mathfrak{g}' \oplus \ker \varphi)$, where $\tilde{\varphi}$ is the extension of φ to $\mathfrak{g}$ set to zero on $\mathfrak{g}'$.

(4) If $\mathfrak{h}$ is of Type 4 defined by the epimorphism $\psi : \mathfrak{z} \to \mathbb{R}^{n-k}$, any $R \in \mathcal{K}(\mathfrak{h})$ is given as in (1) with $\lambda = 0$, $L = 0$, $\mathrm{pr}_{\mathbb{R}^{n-k}} \circ T = \tilde{\psi} \circ Q$ and $R_0 \in \mathcal{K}(\mathfrak{g}' \oplus \ker \psi)$, where $\tilde{\psi}$ is the extension of ψ to $\mathfrak{g}$ set to zero on $\mathfrak{g}'$.

Here $U \wedge V$ denotes the identification of Λ^2 with $\mathfrak{so}(n)$ and the * denotes the adjoint with respect to the scalar product $\langle .,. \rangle$ in $\mathbb{R}^n$ and the Killing form in $\mathfrak{so}(n)$. In particular T is a symmetric matrix and $Q^* : \mathfrak{g} \to \mathbb{R}^n$ is given by $Q^*(U \wedge V) = 2\langle Q(E_i)U, V \rangle E_i$.

Now we can apply these results to the (connected) holonomy group $H := Hol_p(M, h)$ of an indecomposable, non-irreducible Lorentzian manifold (M, h). H belongs to one of the four types corresponding to the characterization of the Lie algebra $\mathfrak{h}$ in Theorem 3.2. The Lie group corresponding to the orthogonal part $\mathfrak{g}$ is denoted by $G \subset SO(n)$. If $\mathfrak{h}$ is of uncoupled Type 1 or 2, then we have either

$$H = (\mathbb{R}_+ \times G) \ltimes \mathbb{R}^n, \quad \text{or} \quad H = G \ltimes \mathbb{R}^n, \tag{6}$$

respectively. If $\mathfrak{h}$ is of one of the coupled types 3 or 4 it is defined by a epimorphism $\varphi : \mathfrak{z} \to \mathbb{R}$ or $\psi : \mathfrak{z} \to \mathbb{R}^l$ where $\mathfrak{z}$ is the center of $\mathfrak{g}$ due to Theorem 3.2. For Type 3 or 4 we have that

$$H = L \cdot G' \ltimes \mathbb{R}^n, \quad \text{or} \quad H = L \cdot G' \ltimes \mathbb{R}^{n-l}, \tag{7}$$

where G' is the Lie group corresponding to the derived Lie algebra $\mathfrak{g}'$ of $\mathfrak{g}$ and L is the Lie group corresponding to the graph of φ or ψ, respectively. We get some immediate consequences.

Proposition 3.6. *A Lorentzian manifold with indecomposable, non-irreducible holonomy group H admits a parallel light-like vector field if and only if $pr_{\mathbb{R}_+}(H) = 0$, i.e. if and only if its Lie algebra is of Type 2 or 4.*

Regarding Lorentzian Einstein manifolds, we get another consequence by (1) of Theorem 3.5 which implies that the Ricci-trace $\mathrm{Ric} = \mathrm{tr}_{(1,4)} R$ is given by

$$\begin{array}{rcl} \mathrm{Ric}(X,Z) &=& -\lambda, \\ \mathrm{Ric}(U,V) &=& \mathrm{Ric}_0(U,V), \text{ where } \mathrm{Ric}_0 = \mathrm{tr}_{(1,4)} R_0 \\ \mathrm{Ric}(U,Z) &=& -L(U) - \sum_{i=1}^{n} \langle Q(E_i)U, E_i \rangle \\ \mathrm{Ric}(Z,Z) &=& \mathrm{tr}(T) \end{array} \tag{8}$$

for $U, V \in \mathrm{span}(E_1, \ldots, E_n)$. Evaluating these formulas we get in [43] the following consequence.

Theorem 3.7. *Let (M,h) be an indecomposable non-irreducible Lorentzian Einstein manifold. Then the holonomy of (M,h) is of uncoupled type 1 or 2. If the Einstein constant of (M,h) is non-zero, then the holonomy of (M,h) is of type 1.*

3.3. Lorentzian holonomy, weak-Berger algebras, and their classification. In this section we want to present the classification of possible Lorentzian holonomy groups. In particular, we want to describe how to prove the following Theorem.

Theorem 3.8. *Let H be the connected holonomy group of an indecomposable, non-irreducible Lorentzian manifold. Then its orthogonal part $G := \mathrm{pr}_{SO(n)}(H)$ is a Riemannian holonomy group.*

To this end we will explain the notion of weak-Berger algebras, which was introduced and studied in [66]. As for the definition of $\mathcal{K}(\mathfrak{g})$ in Section 2.1, let $\mathbb{K}$ be the real or complex numbers. For $\mathfrak{g} \subset \mathfrak{so}(n,\mathbb{C})$ or $\mathfrak{g} \subset \mathfrak{so}(r,s)$ and $\mathcal{B}(\mathfrak{g})$ the space of weak curvature endomorphism we define:

$$\mathfrak{g}^{\mathcal{B}} \;:=\; span\{Q(x) \mid x \in \mathbb{K}^n, Q \in \mathcal{B}(\mathfrak{g})\}.$$

Just as $\mathfrak{g}^{\mathcal{K}}$ in Definition 2.1, $\mathfrak{g}^{\mathcal{B}}$ is an ideal in $\mathfrak{g}$.

Definition 3.9. $\mathfrak{g} \subset \mathfrak{so}(n,\mathbb{C})$ or $\mathfrak{g} \in \mathfrak{so}(r,s)$ is a *weak-Berger algebra* if $\mathfrak{g}^{\mathcal{B}} = \mathfrak{g}$.

Equivalent to the (weak-)Berger property is the fact that there is no proper ideal $\mathfrak{h}$ in $\mathfrak{g}$ such that $\mathcal{K}(\mathfrak{h}) = \mathcal{K}(\mathfrak{g})$ (resp. $\mathcal{B}(\mathfrak{h}) = \mathcal{B}(\mathfrak{g})$). One easily verifies that the vector space $\mathcal{R}(\mathfrak{g})$ spanned by $\{R(x,.) \in \mathcal{B}(\mathfrak{g}) \mid R \in \mathcal{K}(\mathfrak{g}),\ x \in \mathbb{K}^n\}$ is a $\mathfrak{g}$-submodule of $\mathcal{B}(\mathfrak{g})$. This implies $\mathfrak{g}^{\mathcal{K}} \subset \mathfrak{g}^{\mathcal{B}}$, and thus:

Proposition 3.10. *Every orthogonal Berger algebra is a weak-Berger algebra.*

For a weak-Berger algebra the Bianchi-identity which defines $\mathcal{B}(\mathfrak{g})$ yields a decomposition property similar to the Borel–Lichnerowicz property mentioned in Section 2.2.

Theorem 3.11. *Let $\mathfrak{g} \subset \mathfrak{so}(n)$ be a weak-Berger Algebra. To the decomposition of $\mathbb{R}^n$ into invariant subspaces $\mathbb{R}^n = E_0 \oplus E_1 \oplus \ldots \oplus E_k$, where E_0 is a trivial submodule and the E_i are irreducible for $i = 1, \ldots, k$, corresponds a decomposition of $\mathfrak{g}$ into ideals $\mathfrak{g} = \mathfrak{g}_1 \oplus \ldots \oplus \mathfrak{g}_r$ such that $\mathfrak{g}_i$ acts irreducibly on E_i and trivially on E_j for an $i \neq j$. Each of the $\mathfrak{g}_i \subset \mathfrak{so}(\mathrm{dim} E_i)$ is a weak Berger algebra and it holds that $\mathcal{B}(\mathfrak{g}) = \mathcal{B}(\mathfrak{g}_1) \oplus \ldots \oplus \mathcal{B}(\mathfrak{g}_k)$.*

We should point out that the same statement holds for orthogonal Berger algebras for a decomposition of $\mathbb{R}^n$ into $\mathfrak{g}$-invariant orthogonal subspaces. This corresponds to the algebraic aspect of Theorem 2.3. Using the description of the curvature endomorphisms in Theorem 3.5 and by Proposition 3.10 one obtains the following consequence, the ‘only if’-direction of which was proved in [66] independently of the description of the space of curvature endomorphisms by restricting the Bianchi-identity to the space $\mathrm{span}(E_1, \ldots, E_n)$.

Corollary 3.12. *An indecomposable subalgebra* $\mathfrak{h} \subset \mathfrak{p}$ *is a Berger algebra if and only if its orthogonal part* $\mathfrak{g} \subset \mathfrak{so}(n)$ *is a weak-Berger algebra.*

Holonomy algebras of torsion free connections are Berger algebras but the $\mathfrak{so}(n)$-projection of an indecomposable, non-irreducible Lorentzian manifold *a priori* is no holonomy algebra, and therefore not necessarily a Berger algebra. But from Corollary 3.12 and the Ambrose–Singer holonomy theorem it follows that it is a weak-Berger algebra.

Theorem 3.13. *The orthogonal part* $\mathfrak{g}$ *of an indecomposable, non-irreducible Lorentzian holonomy algebra is a weak-Berger algebra. In particular,* $\mathfrak{g}$ *decomposes into irreducibly acting weak-Berger algebras as in Theorem 3.11.*

This theorem has several important consequences. It not only gives an algebraic criterion for the orthogonal part from which a classification attempt can start, it also provides a proof of the Borel–Lichnerowicz decomposition property for the orthogonal part proved by L. Bérard-Bergery and A. Ikemakhen [11, Theorem II], which is given in our Theorem 3.11. This ensures that we are at a similar point as in the Riemannian situation, that means left with the task of classifying irreducible weak-Berger algebras instead of Berger algebras.

But Theorem 3.13 also has implication for algebras of coupled type 4. They were defined by an epimorphism $\psi : \mathfrak{z} \to \mathbb{R}^p$ for $0 < p < n$ where $\mathfrak{z}$ is the center of the orthogonal part $\mathfrak{g}$. If $\mathbb{R}^n$ decomposes as

$$\mathbb{R}^n = \mathbb{R}^{n_0} \oplus \mathbb{R}^{n_1} \oplus \ldots \oplus \mathbb{R}^{n_s},$$

where $\mathfrak{g}$ acts irreducibly on the $\mathbb{R}^{n_i}$ and trivial on $\mathbb{R}^{n_0}$, inducing the decomposition of $\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_s$ as in Theorem 3.11, then, first of all, we have that $0 < p \leq n_0 < n - 1$. Moreover, as the $\mathfrak{g}_i$'s act irreducibly, their center has to be at most one-dimensional. Since ψ is surjective this implies that $0 < p \leq s$. In particular, Type 4 only occurs for $n \geq 3$, i.e. $\dim M \geq 5$.

Before we explain the classification of weak-Berger algebras we want to present implications of Theorem 3.13 about the conditions under which indecomposable, non-irreducible holonomy groups are *closed*. Let H be an indecomposable, non-irreducible Lorentzian holonomy group and, as above, let G be its orthogonal part. For the uncoupled types 1 and 2 it depends only on G if H is closed. But due to Theorem 3.13 G is a product of irreducibly and orthogonally acting Lie groups, which are closed. Therefore G, and thus H is closed in this case. If H is of one of the coupled types 3 or 4 it is closed if and only if L as in formulae (7) is closed. But L is closed if and only if its intersection with the Torus Z, which is the center of G, is closed. This can be summarised in the following result obtained in [11].

Corollary 3.14. *If the Lie algebra of an indecomposable, non-irreducible Lorentzian holonomy group* H *is of Type 1 and 2, then it is closed. If it is of Type 3 or 4, defined by an epimorphism* φ *, then it is closed if and only if the Lie group generated by the subalgebra* $\ker(\varphi)$ *is a compact subgroup of the torus.*

This corollary implies that holonomy groups of Lorentzian manifolds of dimension less or equal to 5 are closed.

Now we turn to the classification of irreducible weak-Berger algebras obtained in [66, 68, 69]. As we use representation theory of complex semisimple Lie algebras, we have to describe the transition of a real weak-Berger algebra to its complexification. The spaces $\mathcal{K}(\mathfrak{g})$ and $\mathcal{B}_h(\mathfrak{g})$ for $\mathfrak{g} \subset \mathfrak{so}(r+s)$ are defined by the following exact sequences:

$$\begin{array}{ccccccc} 0 & \to & \mathcal{K}(\mathfrak{g}) & \hookrightarrow & \Lambda^2(\mathbb{R}^n)^* \otimes \mathfrak{g} & \stackrel{\lambda}{\to} & \Lambda^3(\mathbb{R}^n)^* \otimes \mathbb{R}^n \\ 0 & \to & \mathcal{B}(\mathfrak{g}) & \hookrightarrow & (\mathbb{R}^n)^* \otimes \mathfrak{g} & \stackrel{\lambda^*}{\to} & \Lambda^3(\mathbb{R}^n)^*, \end{array}$$

where the map λ is the skew-symmetrization and λ^* the dualization by the scalar product and the skew-symmetrization. If we consider a real Lie algebra $\mathfrak{g}$ acting orthogonally on $\mathbb{R}^n$, then the scalar product extends by complexification to a complex-linear scalar product which is invariant under $\mathfrak{g}^{\mathbb{C}}$, i.e. $\mathfrak{g}^{\mathbb{C}} \subset \mathfrak{so}(r+s,\mathbb{C})$. The complexification of the above exact sequences gives

$$\mathcal{K}(\mathfrak{g})^{\mathbb{C}} = \mathcal{K}(\mathfrak{g}^{\mathbb{C}}) \quad \text{and} \quad \mathcal{B}(\mathfrak{g})^{\mathbb{C}} = \mathcal{B}(\mathfrak{g}^{\mathbb{C}}) \tag{9}$$

and leads to the following statement.

Proposition 3.15. $\mathfrak{g} \subset \mathfrak{so}(r,s)$ *is a (weak-) Berger algebra if and only if* $\mathfrak{g}^{\mathbb{C}} \subset \mathfrak{so}(r+s,\mathbb{C})$ *is a (weak-) Berger algebra.*

Thus complexification preserves the weak-Berger as well as the Berger property. But irreducibility is *not* preserved under complexification. In order to deal with this problem we have to recall the following distinctions (for details of the following see [66] or [73]). Let $\mathfrak{g} \subset \mathfrak{so}(n)$ be a real orthogonal Lie algebra which acts irreducibly on $\mathbb{R}^n$. Then one can consider the complexification of this representation, i.e. the representation of the real Lie algebra $\mathfrak{g}$ on $\mathbb{C}^n$ given by $\mathfrak{g} \subset \mathfrak{so}(n) \subset \mathfrak{so}(n,\mathbb{C})$. This representations can still be irreducible, in which case we say that $\mathfrak{g}$ is of *real type*, or it can be reducible, and we say it is of *unitary type*. In the second case $n = 2k$ has to be even and $\mathbb{C}^n$ decomposes into two $\mathfrak{g}$-invariant subspaces, $\mathbb{C}^n = \mathbb{C}^k \oplus \overline{\mathbb{C}^k}$ for which we obtain that $\mathfrak{g} \subset \mathfrak{u}(k)$ is unitary and irreducible. Note that this implies that $\mathfrak{g} \not\subset \mathfrak{so}(k,\mathbb{C})$. This distinction was made by E. Cartan in [32] (see also [57] and [51]) for arbitrary irreducible real representations, where these are called "representations of first type" and of "second type".

Now we complexify also the Lie algebra $\mathfrak{g}$ because we want to use the tools of the theory of irreducible representations of complex Lie algebras. Of course, it holds that $\mathfrak{g} \subset \mathfrak{so}(n)$ is irreducible if and only if $\mathfrak{g}^{\mathbb{C}} \subset \mathfrak{so}(n,\mathbb{C})$ is irreducible. Hence, for an irreducible $\mathfrak{g} \subset \mathfrak{so}(n)$ we end up with two cases: If $\mathfrak{g}$ is of real type, then $\mathfrak{g}^{\mathbb{C}} \subset \mathfrak{so}(n,\mathbb{C})$ is irreducible, or if $\mathfrak{g}$ is of unitary type, i.e. $n = 2k$, then $\mathfrak{g}^{\mathbb{C}} \subset \mathfrak{gl}(k,\mathbb{C})$ with $\mathfrak{g}^{\mathbb{C}} \not\subset \mathfrak{so}(k,\mathbb{C})$.

For a *unitary weak-Berger algebra* $\mathfrak{g}_0 \subset \mathfrak{u}(k) \subset \mathfrak{so}(2k)$ in [66] it is shown that there is an isomorphism between the complexified weak curvature endomorphisms $\mathcal{B}(\mathfrak{g}_0^{\mathbb{C}})$ and the *first prolongation*

$$\mathfrak{g}^{(1)} \quad = \quad \{Q \in Hom(\mathbb{C}^k, \mathfrak{g}) \mid Q(u)v = Q(v)u\},$$

where $\mathfrak{g} \simeq \mathfrak{g}_0^{\mathbb{C}}$ is the complexification of $\mathfrak{g}_0$ restricted to the irreducible module $\mathbb{C}^k$, as explained above, i.e. $\mathfrak{g} \subset \mathfrak{gl}(k,\mathbb{C})$ irreducibly. We should point out that an analogous result can be obtained for Berger algebras leading to a classification of irreducible Berger algebras of unitary type. In this situation one can use the classification of irreducible complex linear Lie algebras with non-vanishing first prolongation, which is due to E. Cartan [31], and S. Kobayashi and T. Nagano [63] (see also the list in [78]). Checking the entries in this list one finds that all but one are complexifications of Riemannian holonomy algebras, either of non-symmetric Kählerian ones or of hermitian symmetric spaces. The only exception is $\mathbb{C} \oplus \mathfrak{sp}(k/2,\mathbb{C})$, but it can be shown that this is not a weak-Berger algebra. Hence, in this case we end up with:

Proposition 3.16. *If $\mathfrak{g} \subset \mathfrak{u}(k) \subset \mathfrak{so}(2k)$ is an irreducible weak-Berger algebra of unitary type, then it is a Riemannian holonomy algebra, in particular a Berger algebra.*

Now we turn to the case of a *weak-Berger algebra of real type*, i.e. an irreducible real Lie algebra $\mathfrak{g}_0 \subset \mathfrak{so}(n)$ such that $\mathfrak{g} := \mathfrak{g}^{\mathbb{C}} \subset \mathfrak{so}(n,\mathbb{C})$ is irreducible as well. The first thing to notice is that by the Schur lemma, $\mathfrak{g}$ has no center, and thus $\mathfrak{g}$ is not only reductive but semisimple. Considering the four different types of indecomposable, non-irreducible holonomy algebras from Theorem 3.2, this fact already yields the observation that the $\mathfrak{so}(n)$-projection of an indecomposable, non-irreducible Lorentzian holonomy algebra can only be of coupled type 3 or 4 if at least one of the irreducibly acting ideals of $\mathfrak{g} \subset \mathfrak{so}(n)$ is of non-real type.

As $\mathfrak{g}$ is semisimple, the weak-Berger property can be transformed into conditions on roots and weights of the corresponding representation as follows (for the proofs see [68, 69, 73]). Let $\mathfrak{t}$ be the Cartan subalgebra of $\mathfrak{g}$, $\Delta \subset \mathfrak{t}^*$ be the roots of $\mathfrak{g}$, and set $\Delta_0 := \Delta \cup \{0\}$. $\mathfrak{g}$ decomposes into its root spaces $\mathfrak{g}_\alpha := \{A \in \mathfrak{g} | [T, A] = \alpha(T) \cdot A \text{ for all } T \in \mathfrak{t}\} \neq \{0\}$. If $\Omega \subset \mathfrak{t}^*$ are the weights of $\mathfrak{g} \subset \mathfrak{so}(n,\mathbb{C})$, then $\mathbb{C}^n$ decomposes into weight spaces $V_\mu := \{v \in V | T(v) = \mu(T) \cdot v \text{ for all } T \in \mathfrak{t}\} \neq \{0\}$, which satisfy that $V_\mu \perp V_\lambda$ if and only if $\lambda \neq -\mu$. In particular, if μ is a weight, then $-\mu$ too. If we denote by Π the weights of the $\mathfrak{g}$–module $\mathcal{B}(\mathfrak{g})$ we define a subset of $\mathfrak{t}^*$ by

$$\Gamma := \left\{ \mu + \phi \;\middle|\; \begin{array}{l} \mu \in \Omega,\ \phi \in \Pi \text{ and there is an } u \in V_\mu \\ \text{and a } Q \in \mathcal{B}_\phi \text{ such that } Q(u) \neq 0 \end{array} \right\} \subset \mathfrak{t}^*.$$

It is not difficult to see that $\Gamma \subset \Delta_0$. But for a weak-Berger algebra, the fact that $Q_\phi(u_\mu) \in \mathfrak{g}_{\phi+\mu}$, implies that

$$\mathfrak{g}^{\mathcal{B}} = span\{Q_\phi(u_\mu) \mid \phi + \mu \in \Gamma\} \subset \bigoplus_{\beta \in \Gamma} \mathfrak{g}_\beta.$$

Hence for a weak-Berger algebra of real type we have even that

$$\Gamma = \Delta_0.$$

This property then can be tested for the representations of all the simple [68], and based on this, the semisimple Lie algebras [69], with the following result.

Proposition 3.17. *Let $\mathfrak{g} \subset \mathfrak{so}(n)$ be an irreducible weak-Berger algebra of real type. Then $\mathfrak{g}$ is a Riemannian holonomy algebra, and in particular a Berger algebra.*

Propositions 3.16 and 3.17 yield Theorem 3.8 at the beginning of this section.

3.4. Metrics realizing all possible Lorentzian holonomy groups.

In this section we shall present Lorentzian metrics that realise all possible groups obtained in the previous section. But first we want to specify the Walker co-ordinates of Theorem 2.6 to the Lorentzian situation.

For an indecomposable, non-irreducible Lorentzian manifold, the holonomy-invariant light-like line $\mathcal{I} \subset T_pM$ corresponds to a distribution Ξ of light-like lines which are invariant under parallel transport. Locally, this distribution is spanned by a *recurrent light-like vector field*. A vector field X is called *recurrent* if there is a one-form ξ such that $\nabla X = \xi \otimes X$. If $d\xi = 0$, e.g. if the length of X is not zero, X can be re-scaled to a parallel vector field. Now Theorem 2.6 reads as follows.

Proposition 3.18. *Let (M, h) be a Lorentzian manifold of dimension $(n + 2)$.*

1. *(M,h) admits recurrent, light-like vector field if and only if there exists co-ordinates $(x,(y_i)_{i=1}^n,z)$ such that*

$$h = 2\ dxdz + \sum_{i=1}^{n} u_i dy_i dz + f dz^2 + \sum_{i,j1}^{n} g_{ij} dy_i\ dy_j \tag{10}$$

with $\frac{\partial g_{ij}}{\partial x} = \frac{\partial u_i}{\partial x} = 0$, $f \in C^\infty(M)$. The recurrent vector field is parallel if and only if $\frac{\partial f}{\partial x} = 0$. In this case the co-ordinates are called Brinkmann co-ordinates *[21, 41].*

2. *(M,h) is a Lorentzian manifold with parallel light-like vector field if and only if there exists co-ordinates $(x,(y_i)_{i=1}^n,z)$ such that*

$$h = 2\ dxdz + \sum_{i,j=1}^{n} g_{ij} dy_i\ dy_j\ ,\ with\ \frac{\partial g_{ij}}{\partial x} = 0. \tag{11}$$

These co-ordinates are due to R. Schimming [76].

In these co-ordinates the vector field $\frac{\partial}{\partial x}$ corresponds to the recurrent/parallel light-like vector field. We should remark that if f is sufficient general (e.g. $\frac{\partial f}{\partial y_i} \neq 0$ for all $i = 1,\ldots,n$), then (M,h) is indecomposable.

Walker co-ordinates define n-dimensional submanifolds $W_{(x,z)}$ through a point p with co-ordinates $(x,y_1,\ldots,y_n,z)$ by varying only the y_i and keeping x and z constant. One can understand the g_{ij} as coefficients of a family of Riemannian metrics g_z and the u_i as coefficients of a family of 1-forms ϕ_z on $W_{(x,z)}$ depending on a parameter z. A direct calculation shown in [55] gives a relation between the holonomy of these Riemannian metrics and the orthogonal component of the Lorentzian holonomy, namely

$$Hol_{(x,y,z)}(W_{(x,z)}, g_z) \quad \subset \quad \mathrm{pr}_{\mathfrak{so}(n)}\left(Hol_{(x,y,z)}(M,h)\right).$$

Led by this description, for a given Riemannian holonomy group G, it is not difficult to construct an indecomposable, non-irreducible Lorentzian manifold having G as orthogonal component of its holonomy. In fact, the following is true [67, 70].

Proposition 3.19. *Let (N,g) be a n-dimensional Riemannian manifold with holonomy group G and let $f \in C^\infty(\mathbb{R}\times N)$ a smooth function on M also depending on the parameter x, and φ a smooth real function of the parameter z. Then the Lorentzian manifold $\left(M := \mathbb{R}\times N\times\mathbb{R}, h = 2dxdz + fdz^2 + \mathrm{e}^{2\varphi}g\right)$ has holonomy $(\mathbb{R}\times G)\ltimes\mathbb{R}^n$ if f is sufficiently generic, and $G\ltimes\mathbb{R}^n$ if f does not depend on x.*

This obviously gives a construction method for any Lorentzian holonomy group of uncoupled type 1 or 2. This procedure was used in physics literature to construct examples of Lorentzian manifolds in special cases [42].

Although in [11] some examples of metrics with holonomy of coupled types 3 and 4 were constructed in order to verify that there are metrics of this type, after the classification of possible Lorentzian holonomies, the following question arose: *Given a Riemannian holonomy group G with Lie algebra $\mathfrak{g}$ having a non-trivial center $\mathfrak{z}$, and given an epimorphism $\varphi : \mathfrak{z} \to \mathbb{R}^l$, for $0 < l < n$, does there exist a Lorentzian manifold with holonomy algebra of type 3 or 4 defined by φ?* In [48] this question was set in the affirmative by providing a unified construction of local polynomial metrics realizing all possible indecomposable, non-irreducible holonomy algebras of Lorentzian manifolds. We will now sketch this method.

Let $\mathfrak{g} \subset \mathfrak{so}(n)$ be the holonomy algebra of a Riemannian manifold. As seen above we have an orthogonal decomposition $\mathbb{R}^n = \mathbb{R}^{n_0} \oplus \mathbb{R}^{n-n_0}$, where $\mathfrak{g}$ acts trivially on $\mathbb{R}^{n_0}$ and $\mathbb{R}^{n-n_0}$ decomposes further into irreducible modules. We choose an orthonormal basis $e_1, \dots, e_n$ of $\mathbb{R}^n$ compatible with this decomposition. Obviously, $\mathfrak{g} \subset \mathfrak{so}(n-n_0)$ does not annihilate any proper subspace of $\mathbb{R}^{n-n_0}$. If $\mathfrak{h}$ is an indecomposable subalgebra of $\mathfrak{so}(1, n+1)_{\mathcal{I}}$ with orthogonal part $\mathfrak{g}$ having center $\mathfrak{z}$ and of coupled type 3 defined by an epimorphism $\varphi : \mathfrak{z} \to \mathbb{R}$, then we denote $\mathfrak{h}$ by $\mathfrak{h}(\mathfrak{g}, \varphi)$. If $\mathfrak{h}$ is of coupled type 4 defined by an epimorphisms $\psi : \mathfrak{z} \to \mathbb{R}^p$ for $0 < p < n$ we denote $\mathfrak{h}$ by $\mathfrak{h}(\mathfrak{g}, \psi, p)$. Note that in the latter case we have $0 < p \leq n_0 < n$.

First, for a weak-Berger algebra $\mathfrak{g} \subset \mathfrak{so}(n)$ one fixes weak curvature endomorphisms $Q_A \in \mathcal{B}(\mathfrak{g})$ for $A = 1, \dots, N$ such that $\{Q_A\}_{A=1\dots N}$ span $\mathcal{B}(\mathfrak{g})$. Now one defines the following polynomials on $\mathbb{R}^{n+1}$,

$$u_i(y_1, \dots, y_n, z) \;:=\; \sum_{A=1}^{N} \sum_{k,l=1}^{n} \frac{1}{3(A-1)!} \underbrace{\Big\langle Q_A(e_k)e_l + Q_A(e_l)e_k, e_i \Big\rangle}_{=:Q^i_{Akl}} y_k y_l z^A. \tag{12}$$

Then we define the following Lorentzian metric on $\mathbb{R}^{n+2}$,

$$h \;=\; 2dxdz + f dz^2 + 2\sum_{i=1}^{n} u_i dy_i dz + \sum_{k=1}^{n} dy_k^2, \tag{13}$$

where f is a function on $\mathbb{R}^{n+2}$ to be specified. If $\mathfrak{h}$ is of Type 3 defined by an epimorphism $\varphi : \mathfrak{z} \to \mathbb{R}$, i.e. $\mathfrak{h} = \mathfrak{h}(\mathfrak{g}, \varphi)$, first we extend φ to the whole of $\mathfrak{g}$ by setting it to zero on $\mathfrak{g}'$, i.e. we set $\widetilde{\varphi}(Z + U) = \varphi(Z)$ for $Z \in \mathfrak{z}$ and $U \in \mathfrak{g}'$. Then, for $A = 1, \dots, N$ and $i = n_0 + 1, \dots, n$ we define the numbers

$$\varphi_{Ai} = \frac{1}{(A-1)!} \widetilde{\varphi}(Q_A(e_i)).$$

If $\mathfrak{h}$ is of Type 4 defined by an epimorphism $\psi : \mathfrak{z} \to \mathbb{R}^p$, i.e. $\mathfrak{h} = \mathfrak{h}(\mathfrak{g}, \psi, p)$, again we extend ψ to an epimorphism $\widetilde{\psi}$ to the whole of $\mathfrak{g}$ as above, and define the following numbers,

$$\psi_{Aib} : \frac{1}{(A-1)!} \Big\langle \widetilde{\psi}(Q_A(e_i)), e_b \Big\rangle,$$

for $A = 1, \dots, N$, $i = n_0 + 1, \dots, n$, and $b = 1, \dots, p$. Then in [48] the following is proved.

Theorem 3.20. *Let $\mathfrak{h} \subset \mathfrak{so}(1, n+1)$ be indecomposable and non-irreducible with a Riemannian holonomy algebra $\mathfrak{g}$ as orthogonal part. If $\mathfrak{h}$ is given by the left-hand-side of the following table, then the holonomy algebra in the origin $0 \in \mathbb{R}^{n+2}$ of the Lorentzian metric h given in (12) is equal to $\mathfrak{h}$ if the function f is defined as in the right-hand side* *of the table:*

$\mathfrak{h}$	f
Type 1: $\mathfrak{h} = (\mathbb{R} \oplus \mathfrak{g}) \ltimes \mathbb{R}^n$	$x^2 + \sum_{i=1}^{n_0} y_i^2$
Type 2: $\mathfrak{h} = \mathfrak{g} \ltimes \mathbb{R}^n$	$\sum_{i=1}^{n_0} y_i^2$
Type 3: $\mathfrak{h} = \mathfrak{h}(\mathfrak{g}, \varphi)$	$2x \sum_{A=1}^{N} \sum_{i=n_0+1}^{n} \varphi_{Ai} y_i z^{A-1} + \sum_{k=1}^{n_0} y_k^2$
Type 4: $\mathfrak{h} = \mathfrak{h}(\mathfrak{g}, \psi, p)$	$2 \sum_{A=1}^{N} \sum_{i=n_0+1}^{n} \sum_{b=1}^{p} \psi_{Aib} y_i y_b z^{A-1} + \sum_{k=p+1}^{n_0} y_k^2$

Obviously, this theorem implies the 'if'-direction of the main classification result of Theorem 3.3. The idea of its proof is the following: The metric h given in (12) with a function f given as in the theorem is analytic. Hence, its holonomy at $0 \in \mathbb{R}^{n+2}$ is generated by the derivations of the curvature tensor at 0. But the metric is constructed in a way such that the only non-vanishing $\mathfrak{so}(n)$-parts of the curvature and its derivatives satisfy at $0 \in \mathbb{R}^{n+2}$:

$$pr_{\mathfrak{so}(n)}\Big[\big(\underbrace{\nabla_{\partial_z} \dots \nabla_{\partial_z}}_{(A-1)-\text{times}} \mathcal{R}\big)\left(\partial_i, \partial_z\right)_0\Big] \;=\; Q_A(e_i), \tag{14}$$

for $A = 1, \dots, N$, $i = n_0 + 1, \dots, n$, and writing ∂_z for $\frac{\partial}{\partial z}$ and ∂_i for $\frac{\partial}{\partial y_i}$. But $Q_1, \dots, Q_N$ span $\mathcal{B}(\mathfrak{g})$, hence, the derivatives of the curvature will span $\mathfrak{g}$, since this is a weak-Berger algebra. Therefore the orthogonal part $\mathfrak{g}$ of $\mathfrak{h}$ we started with is the orthogonal part of $\mathfrak{hol}_0(\mathbb{R}^{n+2}, h)$. A more detailed analysis also shows that (14) implies that $\mathbb{R}^{n-n_0}$ is contained in $\mathfrak{hol}_0(\mathbb{R}^{n+2}, h)$. But, more importantly, one can show that for the different choices of the function f the derivatives of the curvature generate holonomy algebras of the corresponding types.

We want to conclude this section with some remarks and examples constructed by the method of Theorem 3.20.

First of all one notices that the resulting Lorentzian manifolds are of a special type, introduced in [72] as *manifolds with light-like hypersurface curvature.* They are defined by the condition that their curvature tensor $\mathcal{R}$ vanishes on $\Xi^\perp \times \Xi^\perp \times \Xi^\perp \times \Xi^\perp$. It is remarkable that this rather strong condition on the curvature does not prevent these manifolds from having any possible indecomposable, non-irreducible Lorentzian holonomy. All the following examples, including the ones with non-closed holonomy, will be of this type.

The method of Theorem 3.20 works for any Riemannian holonomy algebra, as soon as one is able to calculate $\mathcal{B}(\mathfrak{g})$. Sometimes it is not necessary to calculate the whole of $\mathcal{B}(\mathfrak{g})$ but a submodule which is sufficient to generate the Lie algebra $\mathfrak{g}$. This could be the sub-module $\mathcal{R}(\mathfrak{g})$. For instance, in [72] a Riemannian symmetric space G/K with $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ is considered. The curvature endomorphisms of $\mathfrak{k}$ satisfy $\mathcal{K}(\mathfrak{k}) = \mathbb{R} \cdot [.,.]$, where $[.,.]$ is the commutator of $\mathfrak{g}$. Since $\mathfrak{k}$ is the holonomy algebra of this space we get $\mathfrak{k} = \text{span}\{[X, Y] \mid X, Y \in \mathfrak{m}\}$. Hence for a basis $X_1, \dots, X_n$ of $\mathfrak{m}$, the $Q_j := \text{ad}(X_j)$ are spanning the submodule $\mathcal{R}(\mathfrak{k})$ in $\mathcal{B}(\mathfrak{k})$ and generate the whole Lie algebra $\mathfrak{k}$. In this situation, the polynomials u_i defined in (12) can be written in terms of the basis X_i and the Killing form of $\mathfrak{g}$,

$$\begin{aligned} u_i^{(G,K)}(y_1,\dots,y_n,z) &:= \\ &\sum_{j,k,l=1}^{n} \frac{1}{3(j-1)!}\Big(B\big([X_j,X_k],[X_l,X_i]\big)+B\big([X_j,X_l],[X_k,X_i]\big)\Big)y_k y_l z^j, \end{aligned}$$

where $[.,.]$ is the commutator in $\mathfrak{g}$ and B the Killing form. In this way one obtains a Lorentzian manifold with the isotropy group K of a symmetric space G/K as orthogonal part of the holonomy. Examples where the orthogonal part is given by the Riemannian symmetric pair $\mathfrak{so}(3)\subset\mathfrak{so}(5)$ have been constructed in [55], [70], and [48].

For non-symmetric Riemannian holonomy algebras, $\mathcal{K}(\mathfrak{g})$ can be very big and thus the calculations complicated. As sketched in [72], another way is to use other, easier submodules of $\mathcal{B}(\mathfrak{g})$. This methods works if $\mathfrak{g}$ is simple, since any sub-module of $\mathcal{B}(\mathfrak{g})$ generates a non-trivial ideal in $\mathfrak{g}$ which has to be equal to $\mathfrak{g}$ if $\mathfrak{g}$ is $\mathfrak{g}$ simple. For example, in the case of the exceptional Lie algebra $\mathfrak{g}_2\subset\mathfrak{so}(V)$, with $V=\mathbb{R}^7$, the $\mathfrak{g}_2$-module $Hom(V,\mathfrak{g}_2)$ which contains $\mathcal{B}(\mathfrak{g}_2)$ splits into the direct sum of $V_{[1,1]}$, $\odot^2_0V^*$ and V, where $V_{[1,1]}$ is the 64-dimensional $\mathfrak{g}_2$-module of highest weight $(1,1)$, and $\odot^2_0V^*$ is the 27-dimensional module of highest weight $(2,0)$. Since $\mathcal{B}(\mathfrak{g}_2)$ is the kernel of the skew-symmetrization

$$\begin{array}{ccccc} \lambda & : & Hom(V,\mathfrak{g}_2) & \longrightarrow & \Lambda^3V^* \\ & & // & & \backslash\backslash \\ & & V_{[1,1]}\oplus\odot^2_0V^*\oplus V & & \odot^2_0V^*\oplus V\oplus\mathbb{R} \end{array}$$

a dimension analysis shows that $\mathcal{B}(\mathfrak{g})$ must contain $V_{[1,1]}$. Thus, by choosing a basis of $V_{[1,1]}$ a metric of the form (13) with coefficients as in (12) can be defined and one obtains a Lorentzian manifold with orthogonal holonomy part G_2.

Finally, we want to return to the question of closedness of holonomy groups. In [11] Lorentzian manifolds with indecomposable, non-irreducible holonomy of coupled type 3 and 4 are constructed which have a *non-closed* holonomy group. These examples use a dense immersion of the real line into the 2-torus. They are constructed similar to our construction method. Consider the metric

$$\begin{aligned} h &= 2dxdz-\sum_{i=1}^{4}dy_i^2+2x\,(y_1y_2+\alpha\ y_3y_4)\,dz^2 \\ &\quad +2\left(y_2^2y_1dy_1-y_1^2y_2dy_2+y_4^2y_3dy_3-y_3^2y_4dy_4\right)dz \end{aligned}$$

on $\mathbb{R}^6$ depending on the parameter α. For this metric one can show that it is of coupled type 3 defined by an epimorphism $\varphi:\mathfrak{z}\to\mathbb{R}$, its orthogonal part is the torus T^2, and that the kernel of φ defines a closed subgroup in T^2 if and only if α is rational. Hence, for α irrational, the holonomy group of h is not closed in $SO_0(1,5)$. Similarly, the metric

$$\begin{aligned} h &= 2dxdz-\sum_{i=1}^{5}dy_i^2+dz^2 \\ &\quad +2\left(y_2^2y_1dy_1-y_1^2y_2dy_2+y_4^2y_3dy_3-y_3^2y_4dy_4+z\,(y_1y_2+\alpha\ y_3y_4)\,dy_5\right)dz \end{aligned}$$

on $\mathbb{R}^7$ has a holonomy group of coupled type 4, with T^2 as orthogonal part, and which is non-closed if α is irrational.

3.5. Applications to parallel spinors. The existence of a parallel spinor field on a Lorentzian spin manifold (M,h) implies the existence of a parallel vector field in the following way: To a spinor field φ, one may associate a vector field X_φ, defined by the equation $h(V_\varphi,U) = \langle U\cdot\varphi,\varphi\rangle$ for any $U\in TM$, where $\langle .,.\rangle$ is the inner product on the spin bundle and $\cdot$ is the Clifford multiplication. X_φ sometimes is referred to as *Dirac current*. Now, the vector field associated to a spinor in this way is light-like or time-like. If the spinor field is parallel, so is the vector field. In the case where it is time-like, the manifold splits by the de-Rham decomposition theorem into a factor $(\mathbb{R},-dt^2)$ and Riemannian factors which are flat or irreducible with a parallel spinor, i.e. with holonomy $\{1\}$, G_2, $Spin(7)$, $Sp(k)$ or $SU(k)$.

In the case where the parallel vector field is light-like we have a Lorentzian factor which is indecomposable, but with parallel light-like vector field (and parallel spinor) and flat or irreducible Riemannian manifolds with parallel spinors. Hence, in this case one has to know which indecomposable Lorentzian manifolds admit a parallel spinor. The existence of the light-like parallel vector field forces the holonomy of such a manifold with parallel spinor to be contained in $SO(n)\ltimes\mathbb{R}^n$ i.e. to be of type 2 or 4.

Furthermore, the spin representation of the orthogonal part $\mathfrak{g}\subset\mathfrak{so}(n)$ of $\mathfrak{h}$ must admit a trivial subrepresentation. In fact, the dimension of the space of parallel spinor fields is equal to the dimension of the space of spinors which are annihilated by $\mathfrak{g}$ [67]. But for the coupled type 4, the orthogonal part $\mathfrak{g}$ has to have a non-trivial center. Due to the decomposition of $\mathfrak{g}$ into irreducible acting ideals at least one irreducible acting ideal is equal to $\mathfrak{u}(p)$. But a direct calculation shows that $\mathfrak{u}(p)$ cannot annihilate a spinor. Hence we obtain the following consequence.

Corollary 3.21. *Let (M,h) be an indecomposable Lorentzian spin manifold of dimension $n+2>2$ with holonomy group H admitting a parallel spinor field. Then it is $H=G\ltimes\mathbb{R}^n$ where G is the holonomy group of an n-dimensional Riemannian manifold with parallel spinor, i.e. G is a product of $SU(p)$, $Sp(q)$, G_2 or $Spin(7)$.*

This generalises a result of R. L. Bryant in [26] (see also [42]) where it is shown up to $n\le 9$ that the maximal subalgebras of the parabolic algebra admitting a trivial subrepresentation of the spin representation are of type (Riemannian holonomy)$\ltimes\mathbb{R}^n$. Combining Corollary 3.21 with the de Rham–Wu decomposition theorem we obtain the following conclusion.

Theorem 3.22. *Let (M,h) be a simply connected, complete Lorentzian spin manifold which admits a parallel spinor. Then (M,h) is isometric to a product $(M',h')\times(N_1,g_1)\times\ldots\times(N_k,g_g)$, where the (N_i,g_i) are flat or irreducible Riemannian manifolds with a parallel spinor and (M',h') is either $(\mathbb{R},-dt)$ or it is an indecomposable, non-irreducible Lorentzian manifold of dimension $n+2>2$ with holonomy $G\ltimes\mathbb{R}^n$ where G is the holonomy group of a Riemannian manifold with parallel spinor. In particular, the holonomy group of (M,h) is the following product*

$$(G\ltimes\mathbb{R}^n)\times\widehat{G},$$

for some $n\ge 0$, and G and $\widehat{G}$ being holonomy groups of Riemannian manifolds admitting a parallel spinor, i.e. both being a product of the possible factors $\{1\}$, $SU(p)$, $Sp(q)$, G_2, or $Spin(7)$ (with G trivial if $n<2$).

3.6. Holonomy related geometric structures. In this section we want to present some remarks about the geometric structures corresponding to the possible holonomy groups. The parallel distribution Ξ of light-like lines equips a Lorentzian manifold

(M, g) with further, holonomy related structure. As Ξ is light-like, it is contained in its orthogonal complement $\Xi^\perp$. Hence, the tangent bundle admits a filtration

$$\Xi \subset \Xi^\perp \subset TM, \tag{15}$$

which enables us to define a vector bundle $\mathcal{S}$ whose fibers are the quotients $\Xi_p^\perp/\Xi_p$, and equip it with a metric $g^{\mathcal{S}}$ induced by the Lorentzian metric g. Since both distributions are parallel, the Levi-Civita connection of g equips $\mathcal{S}$ also with a metric connection $\nabla^{\mathcal{S}}$.

Definition 3.23. If (M, g, Ξ) is a Lorentzian manifold with a parallel distribution Ξ of light-like lines, the vector bundle $(\mathcal{S}, g^{\mathcal{S}}, \nabla^{\mathcal{S}})$ is called *screen bundle* of (M, g, Ξ). The holonomy group of the vector bundle connection $\nabla^{\mathcal{S}}$ is called *screen holonomy group.*

The screen holonomy was introduced in [70] and studied further in [72], where it is shown that the orthogonal part of an indecomposable, non-irreducible Lorentzian holonomy group is equal to the screen holonomy. By the above results we know that the screen holonomy has to be a Riemannian holonomy group. More importantly, algebraic properties of the orthogonal part of the holonomy now can be described by invariant structures of the screen bundle. E.g. if the orthogonal part is contained in the unitary group, then there is a parallel complex structure on the screen bundle, if the orthogonal part is contained in G_2 the screen bundle admits a parallel 3-form, etc. Moreover, we have seen that its fiber $\mathcal{S}_p = \Xi_p^\perp/\Xi_p$ decomposes into subspaces which are invariant under the orthogonal component $\mathfrak{g}$ of the holonomy algebra $\mathfrak{h}$, $\mathcal{S}_p = E_0 \oplus E_1 \oplus \ldots \oplus E_s$, such that $\mathfrak{g}$ acts trivial on E_0 and irreducibly on E_i for $1 \leq i \leq s$. Now, let the spaces Υ_p^i be the pre-image under the canonical projection $\Xi_p^\perp \to \mathcal{S}_p$ of those E_i. They have common intersection Ξ_p. and are holonomy invariant. Therefore they are the fibers of parallel distributions $\Upsilon^0, \ldots, \Upsilon^k$ on M with

$$\Xi = \Upsilon^0 \cap \ldots \cap \Upsilon^s. \tag{16}$$

All the foliations $\Xi \subset \Upsilon^i \subset \Xi^\perp$ are parallel, hence, they are involutive and therefore integrable. I.e. for every point $p \in M$, there are integral manifolds $\mathcal{X}_p$, $\mathcal{Y}_p^i$ and $\mathcal{X}_p^\perp$ of Ξ and $\Xi^\perp$ passing through it. Each leaf of $\mathcal{Y}^i$ and $\mathcal{X}^\perp$ again is foliated in leaves of $\mathcal{X}$, the latter being light-like geodesic lines. One can prove the existence of co-ordinates which respect this foliation. This is done by C. Boubel in [19], where also an additional condition was found under which these co-ordinates are unique. We should also point out that the definition of a screen bundle does not require a choice as it does the notion of a screen distribution introduced in [9]. The relation of the screen bundle to the preferred choices in [6] and [40] is not yet studied. Finally, in [17] the relation between the light-like hypersurfaces and the four types of holonomy groups is studied.

In the reminder of this section we want to characterise Lorentzian manifolds for which the screen holonomy is trivial and some of their generalizations. These results were obtained in [71] and [72]. A Lorentzian manifold with parallel light-like vector field is called *Brinkmann wave.* A Brinkmann wave admits co-ordinates as in Proposition 3.18. A Brinkmann-wave is called *pp-wave* if its curvature tensor $\mathcal{R}$ satisfies the trace condition $tr_{(3,5)(4,6)}(\mathcal{R} \otimes \mathcal{R}) = 0$. R. Schimming [76] proved that an $(n+2)$-dimensional pp-waves admits co-ordinates $(x, (y_i)_{i=1}^n, z)$ such that

$$h = 2\, dxdz + f dz^2 + \sum_{i1}^{n} dy_i^2 \text{ , with } \tfrac{\partial f}{\partial x} = 0. \tag{17}$$

In [71] we gave another equivalence for the definition which seems to be simpler than any of the trace conditions and which allows for generalizations.

Proposition 3.24. *A Brinkmann-wave (M,h) with parallel light-like vector field X and induced parallel distributions Ξ and $\Xi^\perp$ is a pp-wave if and only if its curvature tensor satisfies*

$$\mathcal{R}(U,V) : \Xi^\perp \longrightarrow \Xi \text{ for all } U,V \in TM, \tag{18}$$

or equivalently $\mathcal{R}(Y_1,Y_2) = 0$ for all $Y_1,Y_2 \in \Xi^\perp$.

From this description one obtains easily that a pp-wave is *Ricci-isotropic*, which means that the image of the Ricci-endomorphism is totally light-like, and has vanishing scalar curvature. But it also enables us to introduce a generalization of pp-waves by supposing (18) but only the existence of a recurrent light-like vector field. Assuming that the abbreviation 'pp' stands for 'plane fronted with parallel rays' we call them *pr-waves*, 'plane fronted with recurrent rays'.

Definition 3.25. A Lorentzian manifold with recurrent light-like vector field X its called *pr-wave* if $\mathcal{R}(U,V) : \Xi^\perp \longrightarrow \Xi$ for all $U,V \in TM$, or equivalently $\mathcal{R}(Y_1,Y_2) = 0$ for all $Y_1,Y_2 \in X^\perp$.

Since X is not parallel the trace condition which was true for a pp-wave, fails to hold for a pr-wave. But similar to a pp-wave, a Lorentzian manifold (M,h) is a pr-wave if and only if there are co-ordinates $(x,(y_i)_{i=1}^n,z)$ such that

$$h = 2\, dxdz + fdz^2 + \sum_{i1}^{n} dy_i^2 \text{ , with } f \in C^\infty(M). \tag{19}$$

Regarding the vanishing of the screen holonomy the following result can be obtained by the description of Proposition 3.24 and the definition of a pr-wave.

Proposition 3.26. *A Lorentzian manifold (M,h) with recurrent light-like vector field is a pr-wave if and only if the following equivalent conditions are satisfied:*

(1) The screen holonomy of (M,h) is trivial.

(2) (M,h) has solvable holonomy contained in $\mathbb{R} \ltimes \mathbb{R}^n$.

In addition, (M,h) is a pp-wave if and only if its holonomy is Abelian, i.e. contained in $\mathbb{R}^n$.

Finally, in [71] it is proved that a pr-wave is a pp-wave if and only if it is Ricci-isotropic. There are very important subclasses of pp-waves. The first are the *plane waves* which are pp-waves with quasi-recurrent curvature, i.e. $\nabla\mathcal{R} = \xi \otimes \tilde{\mathcal{R}}$ where $\xi = h(X,.)$ and $\tilde{\mathcal{R}}$ a $(4,0)$-tensor. For plane waves the function f in the local form of the metric is of the form $f = \sum_{i,j=1}^n a_{ij} y_i y_j$ where the a_{ij} are functions of z. A subclass of plane waves are the Lorentzian symmetric spaces with solvable transvection group, the so-called *Cahen-Wallach spaces* (see [30], also [11]). For these the function f satisfies $f = \sum_{i,j=1}^n a_{ij} y_i y_j$ where the a_{ij} are constants. Manifolds with light-like hypersurfaces curvature mentioned above are further generalizations of pp-waves [72].

3.7. Holonomy of space-times.

To conclude this section about Lorentzian holonomy we want to recall results about the holonomy of space-times, i.e. 4-dimensional Lorentzian manifolds of signature $(-+++)$. J. F. Schell [75] and R. Shaw [79] (see also [52] and [53]) found that there are 14 types of possible space-times holonomy groups. These 14 types can be derived by the following case study, in which we will also give examples of metrics realizing these groups. Let H be the connected holonomy group of a 4-dimensional Lorentzian manifold.

1. H acts irreducibly, i.e. $H = SO_0(1,3)$, which can be realised by the 4-dimensional de Sitter space $S^{1,3}$.
2. H acts indecomposably, but non-irreducibly. Then H is either:
   (a) $H = (\mathbb{R}_+ \times SO(2)) \ltimes \mathbb{R}^2$,
   (b) $H = SO(2) \ltimes \mathbb{R}^2$,
   (c) H is of Type 3, i.e. $H = L \ltimes \mathbb{R}^2$ with L given by the graph of an epimorphism $\varphi : \mathfrak{so}(2) \to \mathbb{R}$,
   (d) $H = \mathbb{R}^2$, i.e. the holonomy of a 4-dimensional pp-wave,
   (e) $H = \mathbb{R} \ltimes \mathbb{R}^2$, i.e. the holonomy of an 4-dimensional pr-wave.

   In all these cases the previous section gives examples of metrics realizing H.
3. H acts decomposably. Then H is either:
   (a) $H = SO(2)$, i.e. the holonomy of the product of the 2-sphere S^2 with the 2-dimensional Minkowski space $\mathbb{R}^{1,1}$,
   (b) $H = SO(1,1)$, i.e. the holonomy of the product of the 2-dimensional de Sitter space $S^{1,1}$ with the flat $\mathbb{R}^2$,
   (c) $H = SO(3)$, i.e. the holonomy of the product of $(\mathbb{R}, -dt^2)$ with the 3-sphere S^3,
   (d) $H = SO(1,2)$, i.e. the holonomy of the product of the line $\mathbb{R}$ with the 3-dimensional de Sitter space $S^{1,2}$,
   (e) $H = SO(1,1) \times SO(2)$, i.e. the holonomy of the product of the 2-dimensional de Sitter space $S^{1,1}$ with the 2-sphere S^2 ,
   (f) $H = \mathbb{R} \ltimes \mathbb{R}$. This is the holonomy of the product of $\mathbb{R}$ with a 3-dimensional Lorentzian manifold with a recurrent but not parallel light-like vector field, i.e. with a 3-dimensional *pr*-wave metric. The latter is of the form $h = 2dxdz + g(y)dy^2 + f(x,y,z)dz^2$.
   (g) $H = \mathbb{R}$. This is the holonomy of the product of $\mathbb{R}$ with a 3-dimensional Lorentzian manifold with a parallel light-like vector field, i.e. with a 3-dimensional *pp*-wave metric. The latter is of the form $h = 2dxdz + g(y)dy^2 + f(y,z)dz^2$.
   (h) H is trivial, i.e. the holonomy of the flat Minkowski space $\mathbb{R}^{1,3}$.

We should point out that there is another type of subgroup in $SO(1,3)$, which is a one-parameter subgroup of $SO(1,1) \times SO(2)$, not equal to either of the factors. But this cannot be a holonomy of a Lorentzian manifold because it does not satisfy the de Rham–Wu decomposition of Theorem 2.3. This is explained in [16, Section 10.J], where also the question is asked whether there is a space-time with holonomy of coupled type 3, in [16] denoted by B^3_θ. This question is answered affirmatively by A. Ikemakhen in [56, Section 4.2.3] by the metric

$$h = 2dxdz - dy_1^2 + dy_2^2 + 4\alpha y_1 y_2 dy_1 dz + 2xy_1 dz^2,$$

and by the general method given in [48] described in Theorem 3.20 in Section 3.4. Further results can be found in [50].

4. Holonomy in signature $(2, n+2)$

In this section we discuss the holonomy algebras of pseudo-Riemannian manifolds of signature $(2, n+2)$. From the Berger list one reads off that in this signature the only non-symmetric irreducible holonomy groups are $U(1, \frac{n}{2}+1)$ and $SU(1, \frac{n}{2}+1)$.

Again we consider indecomposable, non-irreducible holonomy algebras of manifolds of signature $(2, n+2)$. As explained in Section 2.2, if $\mathfrak{h} \subset \mathfrak{so}(2, n+2)$ is an indecomposable, non-irreducible subalgebra, then it preserves a proper degenerate subspace $\mathcal{V} \subset \mathbb{R}^{2,n+2}$ and the non-trivial isotropic subspace $\mathcal{I} := \mathcal{V} \cap \mathcal{V}^\perp \subset \mathbb{R}^{2,n+2}$. Obviously, $\dim(\mathcal{I}) = 1$ or 2. Thus for an indecomposable, non-irreducible subalgebra $\mathfrak{h} \subset \mathfrak{so}(2, n+2)$ we have two possibilities:

(1) $\mathfrak{h}$ preserves an isotropic plane;

(2) $\mathfrak{h}$ preserves an isotropic line and does not preserve any isotropic plane;

Until now only the first case has been considered. To explain the results in this case, let $\mathbb{R}^{2,n+2}$ be an $n+4$-dimensional real vector space endowed with a symmetric bilinear form $\langle\, ,\, \rangle$ of signature $(2, n+2)$. We fix a basis $X_1, X_2, E_1, ..., E_n, Z_1, Z_2$ of $\mathbb{R}^{2,n+2}$ such that the Gram matrix of $\langle\, ,\, \rangle$ has the form

$$\begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & \mathbb{I}_n & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}.$$

Let $\mathfrak{so}(2, n+2)_{\mathcal{I}} \subset \mathfrak{so}(2, n+2)$ be the subalgebra that preserves the isotropic plane $\mathcal{I} = \operatorname{span}\{X_1, X_2\}$. $\mathfrak{so}(2, n+2)_{\mathcal{I}}$ can be identified with the following matrix algebra:

$$\mathfrak{so}(2, n+2)_{\mathcal{I}} = \left\{ \left(\begin{array}{cc|c|cc} \multicolumn{2}{c|}{B} & \begin{matrix} X^t \\ Y^t \end{matrix} & \begin{matrix} 0 \\ c \end{matrix} & \begin{matrix} -c \\ 0 \end{matrix} \\ 0 & 0 & A & -X & -Y \\ 0 & 0 & 0 & \multicolumn{2}{c}{\multirow{2}{*}{$-B^t$}} \\ 0 & 0 & 0 & & \end{array} \right) \middle| \begin{array}{c} B \in \mathfrak{gl}(2, \mathbb{R}), \\ A \in \mathfrak{so}(n), \\ X, Y \in \mathbb{R}^n, \\ c \in \mathbb{R} \end{array} \right\}.$$

In [56] A. Ikemakhen classified indecomposable, non-irreducible subalgebras of $\mathfrak{so}(2, n+2)_{\mathcal{I}}$ that contain the ideal

$$\mathcal{A} := \mathbb{R} \begin{pmatrix} \multicolumn{2}{c}{0} & \begin{matrix} 0 \\ 0 \end{matrix} & \begin{matrix} 0 & -1 \\ 1 & 0 \end{matrix} \\ 0 & 0 & 0 & 0 \quad 0 \\ 0 & 0 & 0 & \multirow{2}{*}{0} \\ 0 & 0 & 0 & \end{pmatrix} \subset \mathfrak{so}(2, n+2)_{\mathcal{I}}.$$

As in the Lorentzian case, to each subalgebra $\mathfrak{h} \subset \mathfrak{so}(2, n+2)_{\mathcal{I}}$ one can associate its projection onto $\mathfrak{so}(n) \subset \mathfrak{so}(2, n+2)_{\mathcal{I}}$. In [56] it was noted that such projection $\mathfrak{g}$ of the holonomy algebra may not be a holonomy algebra of a Riemannian manifold. Moreover, in the next section we will see that there is no additional condition on the subalgebra $\mathfrak{g} \subset \mathfrak{so}(n)$ induced by the Bianchi-identity and replacing the weak-Berger property.

4.1. The orthogonal part of indecomposable, non-irreducible subalgebras of $\mathfrak{so}(2, n+2)$.

Let us consider one type of indecomposable, non-irreducible subalgebras of $\mathfrak{so}(2, n+2)$. For any subalgebra $\mathfrak{g} \subset \mathfrak{so}(n)$ define the Lie algebra

$$\mathfrak{h}^{\mathfrak{g}} = \left\{ \begin{pmatrix} 0 & 0 & X^t & 0 & -c \\ 0 & 0 & Y^t & c & 0 \\ 0 & 0 & A & -X & -Y \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \middle| A \in \mathfrak{g}, X, Y \in \mathbb{R}^n, c \in \mathbb{R} \right\} \subset \mathfrak{so}(2, n+2)_{\mathcal{I}}.$$

We identify an element of the Lie algebra $\mathfrak{h}^{\mathfrak{g}}$ with the 4-tuple (A,X,Y,Z). It is easy to see that the subalgebra $\mathfrak{h}^{\mathfrak{g}}\subset\mathfrak{so}(2,n+2)_{\mathcal{I}}$ is indecomposable. The following theorem was proved in [44].

Theorem 4.1. *For any subalgebra $\mathfrak{g}\subset\mathfrak{so}(n)$, the Lie algebra $\mathfrak{h}^{\mathfrak{g}}$ can be realised as the holonomy algebra of a pseudo-Riemannian manifold of signature $(2,n+2)$.*

Let $\mathfrak{g}\subset\mathfrak{so}(n)$ be a subalgebra. To prove Theorem 4.1, in [44] a polynomial metric on $\mathbb{R}^{n+4}$ is constructed, the holonomy algebra of which at the point 0 is exactly $\mathfrak{h}^{\mathfrak{g}}$. This was done in the following way. First we will explain why for any subalgebra $\mathfrak{g}\subset\mathfrak{so}(n)$, the Lie algebra $\mathfrak{h}^{\mathfrak{g}}$ is a Berger algebra. For $u,v\in E:=\mathrm{span}\{E_1,...,E_n\}$, any symmetric linear maps $T_1,T_2:E\to E$ and any linear map $S:E\to E$ such that $S-S^*\in\mathfrak{g}$ let

$$\begin{aligned} R^{T_1}(Z_1,u)&=(0,T_1(u),0,0), & R^{T_2}(Z_2,u)&=(0,0,T_2(u),0),\\ R^S(Z_1,Z_2)&=(S-S^*,0,0,0), & R^S(Z_1,u)&=(0,0,S(u),0),\\ R^S(Z_2,u)&=(0,S^*(u),0,0), & R^S(u,v)&=(0,0,0,\langle S(u),v\rangle-\langle u,S(v)\rangle)\end{aligned}$$

and extend these linear maps to $\mathbb{R}^{2,n+2}\otimes\mathbb{R}^{2,n+2}$ in the trivial way. It easy to check that $R^{P_1},R^{P_2},R^S\in\mathcal{K}(\mathfrak{h}^{\mathfrak{g}})$. Obviously, curvature endomorphisms of these forms generate $\mathfrak{h}^{\mathfrak{g}}$.

Now let $\dim\mathfrak{g}=N$ and let $A_1,\dots,A_N$ be a basis of the vector space $\mathfrak{g}$. Denote by $A^i_{j\alpha}$ the elements of the matrices A_α. Let $(x_1,x_2,y_1...,y_n,z_1,z_2)$ be the canonical co-ordinates on $\mathbb{R}^{n+4}$. Consider the following metric on $\mathbb{R}^{n+4}$:

$$g=2dx_1dz_1+2dx_2dz_2+\sum_{i=1}^n(dy_i)^2+2\sum_{i=1}^n u_idy_idz_2+f(dz_1)^2,$$

where

$$u_i=\sum_{\alpha=1}^N\sum_{j=1}^n\frac{1}{\alpha!}A^i_{j\alpha}y_jz_1^\alpha\quad\text{and}\quad f=\sum_{i=1}^n(y_i)^2.$$

This metric is constructed in such a way that

$$pr_{\mathfrak{so}(n)}\Big[\big(\underbrace{\nabla_{\partial_{z_1}}\dots\nabla_{\partial_{z_1}}}_{(r-1)-\text{times}}\mathcal{R}\big)\left(\partial_{z_1},\partial_{z_2}\right)_0\Big]=S_r-S_r^*=A_r,\tag{20}$$

for $r=1,\dots,N$, where $S_r=\frac{1}{2}A_r$, and the images of other covariant derivatives of $\mathcal{R}$ are contained in $\mathfrak{h}^{\mathfrak{g}}$.

4.2. Holonomy groups of pseudo-Kählerian manifolds of index 2.

A pseudo-Riemannian manifold (M,g) is called *pseudo-Kählerian* if there exists a parallel smooth field of endomorphisms J of the tangent bundle of M that satisfies $J^2=-\mathrm{id}$ and $g(JX,JY)=g(X,Y)$ for all vector fields X and Y on M. The holonomy algebra of a pseudo-Kählerian manifold of signature $(2,2n+2)$ is contained in $\mathfrak{u}(1,n+1)$.

From Theorem 2.3 it follows that the holonomy algebra of a pseudo-Kählerian manifold of $(2,2n+2)$ is a direct sum of irreducible holonomy algebras of Kählerian manifolds and of the indecomposable holonomy algebra of a pseudo-Kählerian manifold of signature $(2,2k+2)$. If the last algebra is irreducible, The Berger list in Theorem 2.4 implies that it is either $\mathfrak{u}(1,k+1)$ or $\mathfrak{su}(1,k+1)$, or that it is the holonomy algebra of an irreducible hermitian symmetric space and listed in [15]. If the last algebra is not irreducible, again we are left with the problem of classifying indecomposable, non-irreducible holonomy algebras in $\mathfrak{u}(1, k+1)$.

In order to describe this classification we denote by $\mathbb{R}^{2,2n+2}$ the vector space $\mathbb{R}^{2n+4}$ endowed with a complex structure J and with a J-invariant metric $\langle\, ,\, \rangle$ of signature $(2, 2n+2)$, i.e. $\langle Jx, Jy\rangle = \langle x, y\rangle$ for all $x, y \in \mathbb{R}^{2,2n+2}$. We fix a basis X_1, X_2, E_1,...,E_n, F_1,...,F_n, Z_1, Z_2 of $\mathbb{R}^{2,2n+2}$ such that the Gram matrix of the metric $\langle\, ,\, \rangle$ and the complex structure J have the form

$$\begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & \mathbb{I}_{2n} & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix} \text{ and } \begin{pmatrix} 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\mathbb{I}_n & 0 & 0 \\ 0 & 0 & \mathbb{I}_n & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \text{ respectively.}$$

We denote by $\mathfrak{u}(1, n+1)_{\mathcal{I}}$ the subalgebra of $\mathfrak{u}(1, n+1)$ that preserves the J-invariant 2-dimensional isotropic subspace $\mathcal{I} = \mathbb{R}X_1 \oplus \mathbb{R}X_2 \subset \mathbb{R}^{2,2n+2}$. The Lie algebra $\mathfrak{u}(1, n+1)_{\mathcal{I}}$ can be identified with the following matrix algebra

$$\mathfrak{u}(1, n+1)_{\mathcal{I}} = \left\{ \left(\begin{array}{cccccc} a_1 & -a_2 & -z_1^t & -z_2^t & 0 & -c \\ a_2 & a_1 & z_2^t & -z_1^t & c & 0 \\ 0 & 0 & B & -C & z_1 & -z_2 \\ 0 & 0 & C & B & z_2 & z_1 \\ 0 & 0 & 0 & 0 & -a_1 & -a_2 \\ 0 & 0 & 0 & 0 & a_2 & -a_1 \end{array} \right) \middle| \begin{array}{c} a_1, a_2, c \in \mathbb{R}, \\ z_1, z_2 \in \mathbb{R}^n, \\ \left(\begin{smallmatrix} B & -C \\ C & B \end{smallmatrix}\right) \in \mathfrak{u}(n) \end{array} \right\}.$$

Recall that

$$\mathfrak{u}(n) = \left\{ \left(\begin{smallmatrix} B & -C \\ C & B \end{smallmatrix}\right) \middle| B \in \mathfrak{so}(n), C \in \mathfrak{gl}(n, \mathbb{R}), C^t = C \right\}$$

and

$$\mathfrak{su}(n) = \left\{ \left(\begin{smallmatrix} B & -C \\ C & B \end{smallmatrix}\right) \in \mathfrak{u}(n) \middle| \operatorname{tr} C = 0 \right\}.$$

We identify an element of $\mathfrak{u}(1, n+1)_{\mathcal{I}}$ with the 4-tuple

$$(a_1 + ia_2, B + iC, z_1 + iz_2, c).$$

The non-vanishing components of the Lie brackets in $\mathfrak{u}(1, n+1)_{\mathcal{I}}$ are the following:

$$\begin{aligned} [(0, B+iC, 0, 0), (0, B_1 + iC_1, 0, 0)] &= (0, [B+iC, B_1 + iC_1]_{\mathfrak{u}(n)}, 0, 0), \\ [(a_1, 0, 0, 0), (0, 0, z_1 + iz_2, c)] &= (0, 0, a_1(z_1 + iz_2), 2a_1 c), \\ [(ia_2, 0, 0, 0), (0, 0, z_1 + iz_2, 0)] &= (0, 0, a_2(z_2 - iz_1), 0), \\ [(0, B+iC, 0, 0), (0, 0, z_1 + iz_2, 0)] &= (0, 0, Bz_1 - Cz_2 + i(Cz_1 + Bz_2), 0), \\ [(0, 0, z_1 + iz_2, 0), (0, 0, w_1 + iw_2, 0)] &= (0, 0, 0, 2(-z_1 w_2^t + z_2 w_1^t)). \end{aligned}$$

Hence we obtain the decomposition

$$\mathfrak{u}(1, n+1)_{\mathcal{I}} = (\mathbb{C} \oplus \mathfrak{u}(n)) \ltimes (\mathbb{C}^n \ltimes \mathbb{R}).$$

Denote by $\mathfrak{su}(1, n+1)_{\mathcal{I}}$ the subalgebra of $\mathfrak{su}(1, n+1)$ that preserves the subspace $\mathbb{R}X_1 \oplus \mathbb{R}X_2 \subset \mathbb{R}^{2,2n+2}$. Then

$$\mathfrak{su}(1, n+1)_{\mathcal{I}} = \{(a_1 + ia_2, B + iC, z_1 + iz_2, c) \in \mathfrak{u}(1, n+1)_{\mathcal{I}} | 2a_2 + \operatorname{tr}_{\mathbb{R}} C = 0\}$$

and

$$\mathfrak{u}(1, n+1)_{\mathcal{I}} = \mathfrak{su}(1, n+1)_{\mathcal{I}} \oplus \mathbb{R}J.$$

Note that

$$\mathfrak{u}(1,1)_{\mathcal{I}} = \left\{ \left(\begin{array}{cccc} a_1 & -a_2 & 0 & -c \\ a_2 & a_1 & c & 0 \\ 0 & 0 & -a_1 & -a_2 \\ 0 & 0 & a_2 & -a_1 \end{array} \right) \middle| \, a_1, a_2, c \in \mathbb{R} \right\}$$

If an indecomposable subalgebra $\mathfrak{h} \subset \mathfrak{u}(1, n+1)$ preserves a degenerate proper subspace $W \subset \mathbb{R}^{2,2n+2}$, then $\mathfrak{h}$ preserves the J-invariant 2-dimensional isotropic subspace $W_1 \subset \mathbb{R}^{2,2n+2}$, where $W_1 = (W \cap JW) \cap (W \cap JW)^{\perp}$ if $W \cap JW \neq \{0\}$ and $W_1 = (W \oplus JW) \cap (W \oplus JW)^{\perp}$ if $W \cap JW = \{0\}$. Therefore $\mathfrak{h}$ is conjugated to an indecomposable subalgebra of $\mathfrak{u}(1, n+1)_{\mathcal{I}}$.

The classification of indecomposable, non-irreducible holonomy algebras in $\mathfrak{u}(2, n+2)$ is given by the following theorem, for the second part of which we fix some notation: $0 \leq m \leq n$ is an integer, $\mathfrak{u} \subset \mathfrak{u}(m)$ is a subalgebra (as above, $\mathfrak{u} = \mathfrak{z} \oplus \mathfrak{u}'$), we fix the decomposition $\mathbb{C}^n = \mathbb{C}^m \oplus \mathbb{C}^{n-m}$, $\mathbb{R}^{n-m} \subset \mathbb{C}^{n-m}$ is a real form and $J_{n-m} \subset \mathfrak{u}(n-m) \subset \mathfrak{u}(n) \subset \mathfrak{u}(1, n+1)_{\mathcal{I}}$ is the complex structure on $\mathbb{C}^{n-m}$. Let $\varphi, \phi : \mathfrak{u} \to \mathbb{R}$ be linear maps with $\varphi|_{\mathfrak{u}'} = \phi|_{\mathfrak{u}'} = 0$.

Theorem 4.2. *1) A subalgebra $\mathfrak{h} \subset \mathfrak{u}(1,1)$ is the indecomposable, non-irreducible holonomy algebra of a pseudo-Kählerian manifold of signature $(2,2)$ if and only if $\mathfrak{h}$ is conjugated to $\mathfrak{u}(1,1)_{\mathcal{I}}$ or one of the following subalgebras of it:*

$$\mathfrak{h}^2_{n=0} = \left\{ \left(\begin{array}{cccc} a_1 & -a_2 & 0 & 0 \\ a_2 & a_1 & 0 & 0 \\ 0 & 0 & -a_1 & -a_2 \\ 0 & 0 & a_2 & -a_1 \end{array} \right) \middle| \, a_1, a_2 \in \mathbb{R} \right\};$$

$$\mathfrak{h}^{\gamma_1,\gamma_2}_{n=0} = \left\{ \left(\begin{array}{cccc} a\gamma_1 & -a\gamma_2 & 0 & -c \\ a\gamma_2 & a\gamma_1 & c & 0 \\ 0 & 0 & -a\gamma_1 & -a\gamma_2 \\ 0 & 0 & a\gamma_2 & -a\gamma_1 \end{array} \right) \middle| \, a, c \in \mathbb{R} \right\}, \text{ where } \gamma_1, \gamma_2 \in \mathbb{R};$$

2) Let $n \geq 1$. Then a subalgebra $\mathfrak{h} \subset \mathfrak{u}(1, n+1)$ is the indecomposable, non-irreducible holonomy algebra of a pseudo-Kählerian manifold of signature $(2, 2n+2)$ if and only if $\mathfrak{h}$ is conjugated to one of the following subalgebras of $\mathfrak{u}(1, n+1)_{\mathcal{I}}$:

$\mathfrak{h}^{m,\mathfrak{u}} = (\mathbb{R} \oplus \mathbb{R}(i + J_{n-m}) \oplus \mathfrak{u}) \ltimes ((\mathbb{C}^m \oplus \mathbb{R}^{n-m}) \ltimes \mathbb{R})$,

$\mathfrak{h}^{m,\mathfrak{u},\phi} = (\mathbb{R} \oplus \{\phi(A)(i + J_{n-m}) + A | A \in \mathfrak{u}\}) \ltimes ((\mathbb{C}^m \oplus \mathbb{R}^{n-m}) \ltimes \mathbb{R})$,

$\mathfrak{h}^{m,\mathfrak{u},\varphi,\phi} = \{\varphi(A) + \phi(A)(i + J_{n-m}) + A | A \in \mathfrak{u}\} \ltimes ((\mathbb{C}^m \oplus \mathbb{R}^{n-m}) \ltimes \mathbb{R})$,

$\mathfrak{h}^{m,\mathfrak{u},\varphi} = (\mathbb{R}(i + J_{n-m}) \oplus \{\varphi(A) + A | A \in \mathfrak{u}\}) \ltimes ((\mathbb{C}^m \oplus \mathbb{R}^{n-m}) \ltimes \mathbb{R})$,

$\mathfrak{h}^{m,\mathfrak{u},\lambda} = (\mathbb{R}(1 + \lambda(i + J_{n-m})) \oplus \mathfrak{u}) \ltimes ((\mathbb{C}^m \oplus \mathbb{R}^{n-m}) \ltimes \mathbb{R})$, *where* $\lambda \in \mathbb{R}$,

$\mathfrak{h}^{n,\mathfrak{u},\psi,k,l} = \{A + \psi(A) | A \in \mathfrak{u}\} \ltimes ((\mathbb{C}^k \oplus \mathbb{R}^{n-l}) \ltimes \mathbb{R})$,
where k and l are integers such that $0 < k \leq l \leq n$, we have the decomposition $\mathbb{C}^n = \mathbb{C}^k \oplus \mathbb{C}^{l-k} \oplus \mathbb{C}^{n-l}$, $\mathfrak{u} \subset \mathfrak{u}(k)$ is a subalgebra with $\dim \mathfrak{z}(\mathfrak{u}) \geq n + l - 2k$ and $\psi : \mathfrak{u} \to \mathbb{C}^{l-k} \oplus i\mathbb{R}^{n-l}$ is a surjective linear map with $\psi|_{\mathfrak{u}'} = 0$,

$\mathfrak{h}^{m,\mathfrak{u},\psi,k,l,r} = \{A + \psi(A) | A \in \mathfrak{u}\} \ltimes ((\mathbb{C}^k \oplus \mathbb{R}^{m-l} \oplus \mathbb{R}^{r-m}) \ltimes \mathbb{R})$,
where k, l, r and m are integers such that $0 < k \leq l \leq m \leq r \leq n$ and $m < n$, we have the decomposition $\mathbb{C}^n = \mathbb{C}^k \oplus \mathbb{C}^{l-k} \oplus \mathbb{C}^{m-l} \oplus \mathbb{C}^{r-m} \oplus \mathbb{C}^{n-r}$, $\mathfrak{u} \subset \mathfrak{u}(k)$ is a subalgebra with $\dim \mathfrak{z}(\mathfrak{u}) \geq n + m + l - 2k - r$ and $\psi : \mathfrak{u} \to \mathbb{C}^{l-k} \oplus i\mathbb{R}^{m-l} \oplus \mathbb{R}^{n-r}$ is a surjective linear map with $\psi|_{\mathfrak{u}'} = 0$.

Note that $\mathfrak{h}_{n=0}^{\gamma_1=1,\gamma_2=0} = \mathfrak{su}(1,1)_{\mathcal{I}}$. As an example for the Lie algebras in the theorem, an element in the Lie algebra $\mathfrak{h}^{m,\mathfrak{u},\varphi,\phi}$ is given by

$$\begin{pmatrix} \varphi(A) & -\phi(A) & -z_1^t & -z_1'^t & -z_2^t & 0 & 0 & -c \\ \phi(A) & \varphi(A) & z_2^t & 0 & -z_1^t & -z_1'^t & c & 0 \\ 0 & 0 & B & 0 & -C & 0 & z_1 & -z_2 \\ 0 & 0 & 0 & 0 & 0 & -\phi(A)\mathbb{I}_{n-m} & z_1' & 0 \\ 0 & 0 & C & 0 & B & 0 & z_2 & z_1 \\ 0 & 0 & 0 & \phi(A)\mathbb{I}_{n-m} & 0 & 0 & 0 & z_1' \\ 0 & 0 & 0 & 0 & 0 & 0 & -\varphi(A) & -\phi(A) \\ 0 & 0 & 0 & 0 & 0 & 0 & \phi(A) & -\varphi(A) \end{pmatrix}$$

with $c \in \mathbb{R}$, $z_1, z_2 \in \mathbb{R}^m$, $z_1' \in \mathbb{R}^{n-m}$, $A = \left(\begin{smallmatrix} B & -C \\ C & B \end{smallmatrix}\right) \in \mathfrak{u}$.

As a corollary, we get the classification of indecomposable, non-irreducible holonomy algebras contained in $\mathfrak{su}(1, n+1)$, i.e. of the holonomy algebras of special pseudo-Kählerian manifolds (these manifolds are pseudo-Kählerian and Ricci-flat).

Corollary 4.3. *Let (M, h, J) be a indecomposable, non-irreducible pseudo-Kählerian manifold of signature $(2, n+2)$ which is Ricci-flat. Then its holonomy algebra is given by $\mathfrak{h}_{n=0}^{\gamma_1=0,\gamma_2=0}$ or $\mathfrak{su}(1,1)_{\mathcal{I}} = \mathfrak{h}_{n=0}^{\gamma_1=1,\gamma_2=0}$ in the case $n = 0$, and by*

$\mathfrak{h}^{m,\mathfrak{u},\phi}$, $\mathfrak{h}^{m,\mathfrak{u},\varphi,\phi}$ with $\mathfrak{u} \subset \mathfrak{su}(n)$ and $\phi(A) = -\frac{1}{n-m+2} tr_{\mathbb{C}} A$, and

$\mathfrak{h}^{n,\mathfrak{u},\psi,k,l}$, $\mathfrak{h}^{m,\mathfrak{u},\psi,k,l,r}$ with $\mathfrak{u} \subset \mathfrak{su}(k)$.

for $n > 0$.

As in the first case, we consider all subalgebras of $\mathfrak{u}(1,1)_{\mathcal{I}}$ and show which of these subalgebras are indecomposable Berger subalgebras. We realise the Lie algebras of Part 1) of Theorem 4.2 as the holonomy algebras of pseudo-Kählerian metrics on $\mathbb{R}^{2,2}$, see Section 5.2.

By analogy to the case of Lorentzian manifolds, the proof of Part 2) of Theorem 4.2 consists of the following 3 steps:

Step 1) Classification of indecomposable, non-irreducible subalgebras of $\mathfrak{su}(1, n+1)$.

Step 2) Classification of indecomposable, non-irreducible Berger subalgebras of $\mathfrak{u}(1, n+1)$.

Step 3) Construction of a pseudo-Riemannian manifold with the holonomy algebra $\mathfrak{h}$ for each indecomposable, non-irreducible Berger subalgebra $\mathfrak{h} \subset \mathfrak{u}(1, n+1)$.

Step 1) First we classify all connected subgroups of $SU(1, n+1)$ that act indecomposably and non-irreducibly on $\mathbb{R}^{2,2n+2}$, that is equivalent to the classification of indecomposable, non-irreducible subalgebras of $\mathfrak{su}(1, n+1)$. Any such subgroup preserves a 2-dimensional isotropic J-invariant subspace of $\mathbb{R}^{2,2n+2}$. We use a generalization of the method from [47] (see Section 3.2).

We denote by $\mathbb{C}^{1,n+1}$ the $(n+2)$-dimensional complex vector space given by $(\mathbb{R}^{2,2n+2}, J, \eta)$. Let g be the pseudo-Hermitian metric on $\mathbb{C}^{1,n+1}$ of signature $(1, n+1)$ corresponding to η. If a subgroup $G \subset U(1, n+1)$ acts indecomposably on $\mathbb{R}^{2,2n+2}$, then G acts indecomposably on $\mathbb{C}^{1,n+1}$, i.e. does not preserve any proper g-non-degenerate complex vector subspace.

We consider the boundary $\partial\mathbf{H}_{\mathbb{C}}^{n+1}$ of the complex hyperbolic space $\mathbf{H}_{\mathbb{C}}^{n+1}$. The boundary $\partial\mathbf{H}_{\mathbb{C}}^{n+1}$ consists of complex isotropic lines of $\mathbb{C}^{1,n+1}$. We identify $\partial\mathbf{H}_{\mathbb{C}}^{n+1}$ with the $(2n+1)$-dimensional sphere S^{2n+1}. Consider the complex isotropic line $\mathcal{I} = \mathbb{C}X_1 =$

$\mathbb{R}X_1 \oplus \mathbb{R}X_2 \subset \mathbb{C}^{1,n+1}$ and denote by $U(1,n+1)_{\mathcal{I}} \subset U(1,n+1)$ the connected Lie subgroup that preserves the line $\mathcal{I}$. The Lie algebra of the Lie group $U(1,n+1)_{\mathcal{I}}$ is $\mathfrak{u}(1,n+1)_{\mathcal{I}}$. Any connected subgroup $G \subset U(1,n+1)$ that acts on $\mathbb{C}^{1,n+1}$ indecomposably and non-irreducibly is conjugated to a subgroup of $U(1,n+1)_{\mathcal{I}}$. From the above decomposition of the Lie algebra $\mathfrak{u}(1,n+1)_{\mathcal{I}}$ we obtain the decomposition

$$U(1,n+1)_{\mathcal{I}} = (\mathbb{C}^* \times U(n)) \ltimes (\mathbb{C}^n \ltimes \mathbb{R}).$$

We identify the set $\partial \mathbf{H}^{n+1}_{\mathbb{C}} \backslash \{\mathcal{I}\} = S^{2n+1} \backslash \{\text{point}\}$ with the Heisenberg space $\mathcal{H}_n = \mathbb{C}^n \oplus \mathbb{R}$. Any element $f \in U(1,n+1)_{\mathcal{I}}$ induces a transformation $\Gamma(f)$ of $\mathcal{H}_n$, moreover, $\Gamma(f) \in \operatorname{Sim} \mathcal{H}_n$, where $\operatorname{Sim} \mathcal{H}_n$ is the group of the Heisenberg similarity transformations of $\mathcal{H}_n$. For the Lie group $\operatorname{Sim} \mathcal{H}_n$ we have the decomposition

$$\operatorname{Sim} \mathcal{H}_n = (\mathbb{R}_+ \times U(n)) \ltimes (\mathbb{C}^n \ltimes \mathbb{R}).$$

The elements $\lambda \in \mathbb{R}_+$, $A \in U(n)$ and $(z,u) \in \mathbb{C}^n \ltimes \mathbb{R}$ act on $\mathcal{H}_n = \mathbb{C}^n \oplus \mathbb{R}$ in the following way:

$$\begin{aligned} \lambda :& (z,u) \mapsto (\lambda z, \lambda^2 u) \quad \text{(real Heisenberg dilation about the origin)},\\ A :& (z,u) \mapsto (Az, u) \quad \text{(Heisenberg rotation about the vertical axis)},\\ (w,v) :& (z,u) \mapsto (w+z, v+u+2\operatorname{Im} g(w,z)) \quad \text{(Heisenberg translations)}. \end{aligned}$$

We show that $\Gamma : U(1,n+1)_{\mathcal{I}} \to \operatorname{Sim} \mathcal{H}_n$ is a surjective Lie group homomorphism with the kernel $\mathbb{T}$, where $\mathbb{T}$ is the 1-dimensional subgroup generated by the complex structure $J \in U(1,n+1)_{\mathcal{I}}$. In particular, $\mathbb{T}$ is the center of $U(1,n+1)_{\mathcal{I}}$. Let $SU(1,n+1)_{\mathcal{I}} = U(1,n+1)_{\mathcal{I}} \cap SU(1,n+1)$. Then $U(1,n+1)_{\mathcal{I}} = SU(1,n+1)_{\mathcal{I}} \cdot \mathbb{T}$ and the restriction

$$\Gamma|_{SU(1,n+1)_{\mathcal{I}}} : SU(1,n+1)_{\mathcal{I}} \to \operatorname{Sim} \mathcal{H}_n$$

is a Lie group isomorphism.

We consider the natural projection $\pi : \operatorname{Sim} \mathcal{H}_n \to \operatorname{Sim} \mathbb{C}^n$, where

$$\operatorname{Sim} \mathbb{C}^n = (\mathbb{R}_+ \times U(n)) \ltimes \mathbb{C}^n$$

is the group of similarity transformations of $\mathbb{C}^n$. The homomorphism π is surjective and its kernel is 1-dimensional.

We prove that *if a subgroup $G \subset U(1,n+1)_{\mathcal{I}}$ acts indecomposably on $\mathbb{C}^{1,n+1}$, then*

(1) *the subgroup $\pi(\Gamma(G)) \subset \operatorname{Sim} \mathbb{C}^n$ does not preserve any proper complex affine subspace of $\mathbb{C}^n$;*

(2) *if $\pi(\Gamma(G)) \subset \operatorname{Sim} \mathbb{C}^n$ preserves a proper non-complex affine subspace $L \subset \mathbb{C}^n$, then the minimal complex affine subspace of $\mathbb{C}^n$ containing L is $\mathbb{C}^n$.*

This is the key statement for our classification.

Since we are interested in connected Lie groups, it is enough to classify the corresponding Lie algebras. The classification is done in the following way:

• First we describe non-complex vector subspaces $L \subset \mathbb{C}^n$ with $\operatorname{span}_{\mathbb{C}} L = \mathbb{C}^n$ (it is enough to consider only vector subspaces, since we do the classification up to conjugacy). Any such non-complex vector subspace has the form $L = \mathbb{C}^m \oplus \mathbb{R}^{n-m}$, where $0 \leq m \leq n$. Here we have 3 types of subspaces: 1) $m = 0$ (L is a real form of $\mathbb{C}^n$); 2) $0 < m < n$; 3) $m = n$ ($L = \mathbb{C}^n$).

• We describe the Lie algebras $\mathfrak{f}$ of the connected Lie subgroups $F \subset \operatorname{Sim}\mathbb{C}^n$ preserving L. Without loss of generality, we can assume that each Lie group F does not preserve any proper affine subspace of L. This means that F acts irreducibly on L. By a theorem of D.V. Alekseevsky [2, 3], F acts transitively on L. In Theorem 3.4 we divided transitive similarity transformation groups of Euclidean spaces into 4 types. Here we unify type 2 and 3. The group F is contained in $(\mathbb{R}_+ \times SO(L) \times SO(L^{\perp_\eta})) \ltimes L$, where $\mathbb{R}_+$ is the group of real dilations of $\mathbb{C}^n$ about the origin and L is the group of all translations in $\mathbb{C}^n$ by vectors of L. In general situation we know only the projection of F on $\operatorname{Sim} L = (\mathbb{R}_+ \times SO(L)) \ltimes L$, but in our case the projection of F on $SO(L) \times SO(L^{\perp_\eta})$ is also contained in $U(n)$ and we know the full information about F. On this step we obtain 9 types of Lie algebras.
• Then we describe subalgebras $\mathfrak{a} \subset \mathrm{LA}(\operatorname{Sim}\mathcal{H}_n)$ with $\pi(\mathfrak{a}) = \mathfrak{f}$. For each $\mathfrak{f}$ we have 2 possibilities: $\mathfrak{a} = \mathfrak{f} + \ker\pi$ or $\mathfrak{a} = \{x + \zeta(x) | x \in \mathfrak{f}\}$, where $\zeta : \mathfrak{f} \to \ker\pi$ is a linear map. Using the isomorphism $(\Gamma|_{\mathfrak{su}(1,n+1)_{\mathcal{I}}})^{-1}$ we obtain a list of subalgebras $\mathfrak{h} \subset \mathfrak{su}(1, n+1)_{\mathcal{I}}$. This gives us 12 types of Lie algebras.
• Finally we check which of the obtained subalgebras of $\mathfrak{su}(1, n+1)_{\mathcal{I}} \subset \mathfrak{so}(2, 2n+2)$ are indecomposable. It turns out that some of the types contain Lie algebras that are not indecomposable. Giving new definitions to these types we obtain 12 types of indecomposable Lie algebras. Unifying some of the types we obtain 8 types of indecomposable subalgebras of $\mathfrak{su}(1, n+1)_{\mathcal{I}} \subset \mathfrak{so}(2, 2n+2)$.

Example 4.4. Let $\mathfrak{g} \subset \mathfrak{so}(n)$ be a subalgebra with non-trivial center and $\zeta : \mathfrak{g} \to \mathbb{R}$ be a non-zero linear map with $\zeta|_{\mathfrak{g}'} = 0$. Then the subalgebra $\{(0, B, z_1, \zeta(B)) | B \in \mathfrak{g},\ z_1 \in \mathbb{R}^n\} \subset \mathfrak{su}(1, n+1)_{\mathcal{I}} \subset \mathfrak{so}(2, 2n+2)$ is indecomposable.

Note that the Lie algebra of the above example was not considered in [56].

Step 2) In this step we classify indecomposable Berger subalgebras of $\mathfrak{u}(1, n+1)_{\mathcal{I}}$.

First we get a list of candidates for the indecomposable subalgebras of $\mathfrak{u}(1, n+1)_{\mathcal{I}} \subset \mathfrak{so}(2, 2n+2)$. For each $\mathfrak{f} \subset \mathrm{LA}(\operatorname{Sim}\mathcal{H}_n)$ as above and for each $\mathfrak{h} \subset \mathfrak{su}(1, n+1)_{\mathcal{I}}$ with $\pi(\Gamma(\mathfrak{h})) = \mathfrak{f}$ we consider the Lie algebras $\mathfrak{h}^J = \mathfrak{h} \oplus \mathbb{R}J$ and $\mathfrak{h}^\xi = \{x + \xi(x) | x \in \mathfrak{h}\}$, where $\xi : \mathfrak{h} \to \mathbb{R}$ is a non-zero linear map. As we claimed above, any indecomposable subalgebra of $\mathfrak{u}(1, n+1)_{\mathcal{I}} \subset \mathfrak{so}(2, 2n+2)$ is of the form $\mathfrak{h}$, $\mathfrak{h}^J$ or $\mathfrak{h}^\xi$. These subalgebras are candidates for the indecomposable subalgebras of $\mathfrak{u}(1, n+1)_{\mathcal{I}} \subset \mathfrak{so}(2, 2n+2)$. We associate with each of these subalgebras an integer $0 \leq m \leq n$. If $m > 0$, then the subalgebras of the form $\mathfrak{h}$, $\mathfrak{h}^J$ and $\mathfrak{h}^\xi \subset \mathfrak{u}(1, n+1)_{\mathcal{I}}$ are indecomposable. We have inclusions $\mathfrak{u}(m) \subset \mathfrak{u}(n) \subset \mathfrak{u}(1, n+1)_{\mathcal{I}}$ and projection maps $\mathrm{pr}_{\mathfrak{u}(m)} : \mathfrak{u}(1, n+1)_{\mathcal{I}} \to \mathfrak{u}(m)$, $\mathrm{pr}_{\mathfrak{u}(n)} : \mathfrak{u}(1, n+1)_{\mathcal{I}} \to \mathfrak{u}(n)$.

For any integer $0 \leq m \leq n$ and subalgebra $\mathfrak{u} \subset \mathfrak{u}(m) \oplus ((\mathfrak{so}(n-m) \oplus \mathfrak{so}(n-m)) \cap \mathfrak{u}(n-m))$ we consider a subalgebra $\mathfrak{h}_0^{m,\mathfrak{u}} \subset \mathfrak{u}(1, n+1)_{\mathcal{I}}$ and describe the space $\mathcal{K}(\mathfrak{h}_0^{m,\mathfrak{u}})$. The Lie algebras of the form $\mathfrak{h}_0^{m,\mathfrak{u}}$ contain all candidates for the indecomposable subalgebras of $\mathfrak{u}(1, n+1)_{\mathcal{I}}$. For any subalgebra $\mathfrak{h} \subset \mathfrak{h}_0^{m,\mathfrak{u}}$ the space $\mathcal{K}(\mathfrak{h})$ can be found from the following condition

$$R \in \mathcal{K}(\mathfrak{h}) \text{ if and only if } R \in \mathcal{K}(\mathfrak{h}_0^{m,\mathfrak{u}}) \text{ and } R(\mathbb{R}^{2,2n+2}, \mathbb{R}^{2,2n+2}) \subset \mathfrak{h}.$$

Using this, we easily find all indecomposable, non-irreducible Berger subalgebras of $\mathfrak{u}(1, n+1)_{\mathcal{I}}$.

Step 3) As the last step of the classification, we construct metrics on $\mathbb{R}^{2n+4}$ that realise all Berger algebras obtained above as holonomy algebras. Idea of constructions of the metrics is similar to the one in Section 4.1. The coefficients of the metrics are polynomial functions, hence the corresponding Levi-Civita connections are analytic and in each case the holonomy algebra at the point $0 \in \mathbb{R}^{2n+4}$ is generated by the operators images of the curvature tensor and of all its derivatives. We explicitly compute for each metric the components of the curvature tensor and its derivatives. Then using the induction, we find the holonomy algebra for each of the metrics.

4.3. Examples of 4-dimensional Lie groups with left-invariant pseudo-Kählerian metrics.

Let G be a Lie group endowed with a left-invariant metric g and let $\mathfrak{g}$ be the Lie algebra of G. We will consider $\mathfrak{g}$ as the Lie algebra of left-invariant vector fields on G and as the tangent space at the identity $e \in G$. Let $X, Y, Z \in \mathfrak{g}$. Since X, Y, Z and g are left-invariant, from the Koszul formulae it follows that the Levi-Civita connection on (G, g) is given by

$$2g(\nabla_X Y, Z) = g([X, Y], Z) + g([Z, X], Y) + g(X, [Z, Y]), \tag{21}$$

where $X, Y, Z \in \mathfrak{g}$. In particular, we see that the vector field $\nabla_X Y$ is also left-invariant. Hence ∇_X can be considered as the linear operator $\nabla_X : \mathfrak{g} \to \mathfrak{g}$. Obviously, $\nabla_X \in \mathfrak{so}(\mathfrak{g}, g)$. For the curvature tensor $\mathcal{R}$ of (G, g) at the point $e \in G$ we have

$$\mathcal{R}(X, Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}, \tag{22}$$

where $X, Y \in \mathfrak{g}$. The holonomy algebra $\mathfrak{h}_e$ at the point $e \in G$ is given by

$$\mathfrak{h}_e = \mathfrak{m}_0 + [\mathfrak{m}_1, \mathfrak{m}_0] + [\mathfrak{m}_1, [\mathfrak{m}_1, \mathfrak{m}_0]] + \cdots, \tag{23}$$

where

$$\mathfrak{m}_0 = \operatorname{span}\{\mathcal{R}(X, Y) | X, Y \in \mathfrak{g}\} \quad \text{and} \quad \mathfrak{m}_1 = \operatorname{span}\{\nabla_X | X \in \mathfrak{g}\}.$$

Now we consider 4-dimensional Lie algebras with the basis X_1, X_2, Z_1, Z_2 and with the metric that has the Gram matrix $\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$ with respect to this basis. Define the following Lie algebras by giving their non-zero brackets:

$\mathfrak{g}_1$: $[X_1, Z_1] = X_1 + Z_2$, $[X_1, Z_2] = -X_2 - Z_1$, $[X_2, Z_1] = X_2 + Z_1$, $[X_2, Z_2] = X_1 + Z_2$;

$\mathfrak{g}_2$: $[X_1, Z_2] = X_1$, $[X_2, Z_1] = -X_1$, $[Z_1, Z_2] = X_1 + Z_1$;

Example 4.5. The holonomy algebras of the Levi-Civita connections on the simply connected Lie groups corresponding to the Lie algebras $\mathfrak{g}_1$ and $\mathfrak{g}_2$ are $\mathfrak{h}^2_{n=0}$ and $\mathfrak{h}^{\gamma_1=0,\gamma_2=1}_{n=0}$, respectively.

5. Holonomy in neutral signature

The Berger list in Theorem 2.4 shows that neutral signature (n, n) has the largest variety of irreducible, non-symmetric holonomy groups of all signatures. Apart from $SO(n, n)$ we have the unitary groups $U(p, p)$ and $SU(p, p)$ and the symplectic groups $Sp(q, q)$ and $Sp(1) \cdot Sp(q, q)$, but also

$$SO(r, \mathbb{C}),\ Sp(p, \mathbb{R}) \cdot Sl(2, \mathbb{R}),\ \text{and}\ Sp(p, \mathbb{C}) \cdot Sl(2, \mathbb{C}),$$

and finally the exceptional groups

$$G_2^{\mathbb{C}} \subset SO(7,7),\ Spin(4,3) \subset SO(4,4), \text{ and } Spin(7)^{\mathbb{C}} \subset SO(8,8).$$

Nevertheless, for a complete classification of holonomy groups of neutral signature one also has to consider indecomposable, non-irreducible ones. In [12] some partial results for the holonomy algebras of pseudo-Riemannian manifolds of signature (n,n) were obtained. In particular, a complete classification for $n = 2$ can be given.

For a point p in a pseudo-Riemannian manifold of signature (n,n) we always fix a basis in T_pM such that the metric h at p is of the form

$$\begin{pmatrix} 0 & \mathbb{I}_n \\ \mathbb{I}_n & 0 \end{pmatrix}, \tag{24}$$

where $\mathbb{I}_n$ is the identity on $\mathbb{R}^n$. If V is a degenerate subspace which is invariant under the holonomy, again we can form the totally isotropic subspace $V \cap V^\perp$, but in contrary to the previous cases this subspace can have any dimension from 1 up to n. In the case in which this dimension is n we have that $V^\perp = V$, i.e. the orthogonal complement gives no further information, but this case contains a very special situation which will be described first.

5.1. Para-Kähler structures.

Recall that we have defined a holonomy representation to be *indecomposable* if any proper invariant subspace is degenerate. Neutral signature (n,n) is the only case where this property does not prevent the holonomy representation from *decomposing*, i.e. to split into two invariant subspaces which are complementary. This is the case if we have two invariant totally isotropic and complementary subspaces V^+ and V^- of dimension n, i.e.

$$T_pM = V^+ \oplus V^-, \tag{25}$$

with V^+ and V^- totally isotropic and holonomy invariant. This property is equivalent to the existence of a *para Kähler structure* on (M,h) (see [35] and [34]).

Definition 5.1. A *para-Kähler manifold* (M,J,h) is a pseudo-Riemannian manifold (M,h) equipped with a non-trivial section J in the endomorphism bundle such that:

1. $\nabla J = 0$ where ∇ is the Levi-Civita connection of h,
2. $J^*h = -h$, and
3. $J^2 = Id$ and the eigen distributions $V^\pm := \ker(Id \mp J)$ have the same rank.

The second condition forces the metric to be of neutral signature (n,n), the third condition ensures that the eigen distributions are totally isotropic of dimension n. That J is parallel is equivalent to the fact that $V^\pm$ are holonomy invariant. It can be shown that the local coefficients of the metric of a para-Kähler metric can be expressed as second derivatives of a *para-Kähler potential* Ω (for a proof see [81], [12], or [35] in terms of para-Kähler manifolds). Regarding the holonomy groups one gets the following result:

Theorem 5.2 (Bérard-Bergery, Ikemakhen, [12]). *Let (M,h) be a pseudo-Riemannian manifold of signature (n,n) and let $H := Hol_p(M,h)$ be its holonomy group at a point $p \in M$. If H leaves invariant two totally isotropic complementary subspaces of dimension n, then*

$$H \subset G := \left\{ \begin{pmatrix} U & 0 \\ 0 & (U^t)^{-1} \end{pmatrix} \mid U \in Gl(n,\mathbb{R}) \right\}. \tag{26}$$

Moreover, there exists co-ordinates $(x_1, \dots x_n, y_1, \dots, y_n)$ *around* p *sending* p *to* $0 \in \mathbb{R}^{2n}$*, and a smooth function* Ω *on a neighborhood of* $0 \in \mathbb{R}^{2n}$ *such that:*

1. $h = \sum_{i,j=1}^{n} D_{ij} dx_i dy_j$ *with* $D_{ij} = \frac{\partial^2 \Omega}{\partial x_i \partial y_j}$.

2. *The Taylor series of* Ω *in* 0 *starts with* $x_1 y_1 + \dots + x_n y_n$ *and continues with terms which are at least quadratic in the* x_i*'s and quadratic in the* y_j*'s.*

3. H *is the smallest connected subgroup of* G *which contains the element*

$$D := \begin{pmatrix} D & 0 \\ 0 & (D^t)^{-1} \end{pmatrix} \in G.$$

Here Ω is the para-Kähler potential. In [5] it is shown that cones over para-Sasakian manifolds are examples of manifolds with the property (25).

5.2. Neutral metrics in dimension four. Now we consider the case where the dimension of the manifold M is four, i.e. the signature is $(2,2)$. Let H be the indecomposable, non-irreducible holonomy group of (M,h). In this case the invariant totally isotropic subspace is a null-line or a totally isotropic plane. The first case is contained in the second: If $(e_1, \dots, e_4)$ is a basis of T_pM such that the metric has the form (24) with e_1 spanning the invariant null-line, then the plane spanned by e_1 and e_2 is invariant as well because Ae_2 is orthogonal to e_1 for A in H. Hence, in both case H leaves invariant a totally isotropic plane $\mathcal{I}$, i.e. the Lie algebra $\mathfrak{h}$ of H is contained in

$$\mathfrak{so}(2,2)_{\mathcal{I}} = \left\{ \begin{pmatrix} U & aJ \\ 0 & -U^t \end{pmatrix} \mid U \in \mathfrak{gl}(2,\mathbb{R}), a \in \mathbb{R} \right\}, \quad \text{with } J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$\mathfrak{so}(2,2)_{\mathcal{I}}$ is a semi-direct sum, $\mathfrak{so}(2,2)_{\mathcal{I}} = \mathfrak{gl}(2,\mathbb{R}) \ltimes \mathcal{A}$, where $\mathcal{A}$ is the ideal spanned by $\begin{pmatrix} 0 & J \\ 0 & 0 \end{pmatrix}$, with the commutator

$$[(U,a),(V,b)] = ([U,V], b \cdot \mathrm{trace}(U) - a \cdot \mathrm{trace}(V)).$$

The one can prove:

Theorem 5.3 (Bérard-Bergery, Ikemakhen, [12]). *Let* $\mathfrak{h}$ *be an indecomposable subalgebra of* $\mathfrak{so}(2,2)_{\mathcal{I}} \subset \mathfrak{so}(2,2)$*. Then either* $\mathfrak{h}$ *contains an ideal which is conjugated in* $SO_0(2,2)$ *to* $\mathcal{A}$*, or it is conjugated to one of the three exceptions*

$$\mathfrak{h}_1 = \mathfrak{so}(2) \oplus \mathbb{R} = \left\{ \begin{pmatrix} r\mathbb{I}_2 + sJ & 0 \\ 0 & -r\mathbb{I}_2 + sJ \end{pmatrix} \mid r, s \in \mathbb{R} \right\},$$

$\mathfrak{h}_2 = \mathbb{R}\begin{pmatrix} L & J \\ 0 & L \end{pmatrix}$, *or* $\mathfrak{h}_3 = \mathbb{R}\begin{pmatrix} J & J \\ 0 & J \end{pmatrix}$, *with* J *as above and* $L = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

Note that the other cases of subalgebras contained in the Lie algebra $\mathfrak{g}$ of G defined in Theorem 5.2 contain an ideal which is conjugated to $\mathcal{A}$, namely the ideals spanned by matrices $U \in Gl(2,\mathbb{R})$ which are nonzero only at one entry. Note also that $\mathfrak{h}_1 = \mathfrak{h}^2_{n=0}$ defined in Theorem 4.2. One can show that the algebras $\mathfrak{h}_2$ and $\mathfrak{h}_3$ are not Berger algebras and therefore cannot be holonomy algebras.

Corollary 5.4. *Let $\mathfrak{h}$ be an indecomposable Berger algebra in $\mathfrak{so}(2,2)_{\mathcal{I}} \subset \mathfrak{so}(2,2)$. Then $\mathfrak{h}$ is conjugated to a subalgebra which is either contained in $\mathfrak{g}$ or contains the ideal $\mathcal{A}$.*

Based on this in [12] a complete classification of indecomposable, non-irreducible holonomy groups of pseudo-Riemannian manifolds of signature $(2,2)$ is obtained.

Theorem 5.5. *Let $H \subset SO(2,2)$ be the indecomposable, non-irreducible acting holonomy group of a 4-dimensional pseudo-Riemannian manifold of signature $(2,2)$ and $\mathfrak{h}$ its Lie algebra. Then H leaves invariant a totally isotropic plane $\mathcal{I}$ and it holds one of the following:*

(A) H leaves invariant another totally isotropic plane complementary to $\mathcal{I}$, in which case $\mathfrak{h} \subset \mathfrak{g}$ and $\mathfrak{h}$ is conjugated in $SO_0(2,2)$ to one of the following: $\mathfrak{gl}(2,\mathbb{R})$, $\mathfrak{sl}(2,\mathbb{R})$, the strictly upper triangular, the upper triangular matrices,

$$\begin{aligned} \mathfrak{k}_\lambda &= \left\{ \begin{pmatrix} a & b \\ 0 & \lambda a \end{pmatrix} \mid a,b \in \mathbb{R} \right\} \text{ for } \lambda \in \mathbb{R}, \text{ or} \\ \mathfrak{h}_1 &= \left\{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \mid a,b \in \mathbb{R} \right\}, \end{aligned}$$

under use of the identification $\mathfrak{g} \simeq \mathfrak{gl}(2,\mathbb{R})$.

(B) H does not leave invariant a complementary totally isotropic plane. In this case the Lie algebra $\mathfrak{h}$ of H is conjugated to a semi-direct sum

$$\mathfrak{h}' \ltimes \mathcal{A},$$

where $\mathfrak{h}'$ is conjugated to one of the subalgebras of $\mathfrak{g}$ listed in (A), to $\mathbb{R} \cdot \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, or to $\mathfrak{u}_\mu := \mathbb{R} \cdot \begin{pmatrix} \mu & 1 \\ -1 & \mu \end{pmatrix}$.

Remark 5.6. From this Theorem we can also recover the classification of unitary holonomy algebras in signature $(2,2)$ given in Theorem 4.2. First we get the obvious subalgebras of $\mathfrak{u}(1,1)$:

$$\mathcal{A} = \mathfrak{h}_{n=0}^{\gamma_1=0,\gamma_2=0}, \quad \mathfrak{h}_1 = \mathfrak{h}_{n=0}^{2}, \quad \mathfrak{h}_1 \ltimes \mathcal{A} = \mathfrak{u}(1,1)_{\mathcal{I}}, \text{ and } \mathfrak{u}_\mu \ltimes \mathcal{A} = \mathfrak{h}_{n=0}^{\mu\gamma_2,\gamma_2\neq 0}.$$

For the remaining subalgebra $\mathbb{R} \cdot \begin{pmatrix} \mathbb{I}_2 & 0 \\ 0 & -\mathbb{I}_2 \end{pmatrix} \oplus \mathcal{A} = \mathfrak{h}_{n=0}^{\gamma_1=1,\gamma_2=0} = \mathfrak{su}(1,1)_{\mathcal{I}}$ one has to pay attention to the fact that in order to classify subalgebras of $\mathfrak{u}(1,1)$ one needs to classify them up to conjugation in $U(1,1)$. In fact, $\mathfrak{h}_{n=0}^{\gamma_1=1}$ is conjugated to $\left\{ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \mid a,b \in \mathbb{R} \right\} = \mathfrak{k}_1$ of (A), which is obviously not a subalgebra of $\mathfrak{u}(1,1)$ because the conjugation lies in $O(2,2)$ but not in $U(1,1)$.

From Theorem 5.5 we get a conclusion about the closedness of holonomy groups.

Corollary 5.7. *The holonomy group of a 4-dimensional pseudo-Riemannian manifold of neutral signature $(2,2)$ is closed.*

Finally we want to address the question, which of the algebras obtained can be realised as holonomy algebras. First we recall results in [12], where it is shown that all the possible holonomy groups that leave invariant a pair of complementary totally isotropic planes listed in (A) of Theorem 5.5 can be realised as holonomy groups of metrics which are not locally symmetric. As seen in Theorem 5.2, for these manifolds the metric can be written in terms of the para-Kähler potential Ω, which is given by

$$\Omega = x_1y_1 + x_2y_2 + f(x_1, x_2, y_1, y_2),$$

where f is a smooth function. In the generic case $\mathfrak{h} = \mathfrak{gl}(2,\mathbb{R})$, the Taylor series of f in 0 is at least quadratic in the x_i's and quadratic in the y_i's, $f = x_1x_2y_1y_2$ is an example. In [12] the other algebras are realised by specifying f:

$\mathfrak{h}$	f
$\left\{\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}\right\}_{a,b,c\in\mathbb{R}}$	$f = y_1^2 f_1(x_1, x_2, y_1) + x_2^2 f_2(x_2, y_1, y_2)$, where the Taylor series of f_1 and f_2 in 0 start with polynomials of degree 2 in (x_1, x_2) and (y_1, y_2) respectively, e.g. $f x_2 y_1(x_1y_1 + x_2y_2)$.
$\mathfrak{k}_{\lambda\neq 0}$	$f = x_1 \int_0^{y_1} x_2y_1\hat{f}(x_2,t)dt + y_2 \int_0^{x_2} ((1 + x_2y_1\hat{f}((t, y_1))^2 - 1)dt$, with $\hat{f}$ non-zero
$\mathfrak{k}_0$	$f = x_1x_2y_1^2\hat{f}(x_1, x_2, y_1)$, with $\hat{f}$ non-zero,
$\mathbb{R}\cdot\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$	$f = x_2^2y_1^2\hat{f}(x_2, y_1)$, with $\hat{f}$ non-zero.

A metric with holonomy $\mathfrak{h}_1 \subset \mathfrak{u}(1,1)_{\mathcal{I}}$ is given in terms of complex co-ordinates. For the existence of a metric with holonomy $\mathfrak{sl}(2,\mathbb{R})$ as listed in (A) of Theorem 5.5, in [12] is argued on general grounds referring to [23] and [24]. [12] leaves open all the cases in (B) of Theorem 5.5.

We will solve this problem partially, first by giving metrics with holonomy $\mathfrak{sl}(2,\mathbb{R}) \ltimes \mathcal{A}$. This metric has the form

$$g = 2dx_1dy_1 + 2dx_2dy_2 + f_1dy_1^2 + f_2dy_2^2 + 2f_3dy_1dy_2 \tag{27}$$

with the functions $f_1 = x_1^2$, $f_2 = x_2^2$, $f_3 = -2x_1x_2$.

To all the subalgebras of $\mathfrak{u}(1,1)_{\mathcal{I}}$ obtained in Theorem 4.2 our method of constructing metrics as sketched in Section 4.2 applies and ensures that all of them can be realised as holonomy algebras. The metrics again have the form (27) with the following functions:

$\mathfrak{h}$	f_1, f_2, f_3
$\mathfrak{u}(1,1)_{\mathcal{I}}$	$f_1 = -2x_2y_1 - x_1y_1^2, \quad f_2 = -f_1 \quad f_3 = 2x_1y_1 - x_2y_1^2$
$\mathfrak{h}^2_{n=0}$	$f_1 = x_1^2 - x_2^2, \quad f_2 = -f_1, \quad f_3 = 2x_1x_2$
$\mathfrak{h}^{\gamma_1,\gamma_2}_{n=0}$ $(\gamma_1^2 + \gamma_2^2 \neq 0)$	$f_1 = -2\gamma_1x_2y_1 - 2\gamma_2x_1y_1, \quad f_2 = -f_1, \quad f_3 = 2\gamma_1x_1y_1 - 2\gamma_2x_2y_1$
$\mathfrak{h}^{\gamma_1=0,\gamma_2=0}_{n=0}$	$f_1 = y_2^2, \quad f_2 = f_3 = 0$

These metrics disprove a claim in [49], that the Lie algebras

$$\mathfrak{sl}(2,\mathbb{R}),\quad \mathfrak{sl}(2,\mathbb{R})\ltimes\mathcal{A},\quad \mathfrak{u}_\mu\ltimes\mathcal{A} \text{ for } \mu\neq 0, \text{ and } \mathfrak{h}_1\ltimes\mathcal{A}=\mathfrak{u}(1,1)_{\mathcal{I}}$$

— in [49] denoted by A_{21}, A_{29}, A_{12}, and A_{24} — cannot be realised as holonomy algebras in $\mathfrak{so}(2,2)$. The only case we have to leave undecided, and which is left undecided in [49] as A_{13}, is the one of

$$\mathfrak{h}=\mathbb{R}\cdot\begin{pmatrix}1&1\\0&1\end{pmatrix}\ltimes\mathcal{A}. \tag{28}$$

However, very recently L. Bérard-Bergery and T. Krantz developed a general construction to build connections on vector bundles which give new holonomies [13]. In particular, by a construction on the cotangent bundle of a surface they can show that all the algebras listed in Theorem 5.5, even the last one in Equation (28), are holonomy algebras of a metric of signature $(2,2)$.

References

[1] D. V. Alekseevskiĭ. Riemannian spaces with unusual holonomy groups. *Funkcional. Anal. i Priložen*, 2(2):1–10, 1968.

[2] D. V. Alekseevskiĭ. Homogeneous Riemannian spaces of negative curvature. *Mat. Sb. (N.S.)*, 96(138):93–117, 168, 1975.

[3] D. V. Alekseevskij, È. B. Vinberg, and A. S. Solodovnikov. Geometry of spaces of constant curvature. In *Geometry, II*, volume 29 of *Encyclopaedia Math. Sci.*, pages 1–138. Springer, Berlin, 1993.

[4] D. V. Alekseevsky and V. Cortés. Classification of indefinite hyper-Kähler symmetric spaces. Asian J. Math. 5 (2001), 663–684.

[5] D. V. Alekseevsky, V. Cortés, A. Galaev and T. Leistner. Cones over pseudo-Riemannian manifolds and their holonomy. 0707.3063 at http://arxiv.org, 47 pages, submitted, 2007.

[6] C. Atindogbe and K. L. Duggal. Conformal screen on lightlike hypersurfaces. *Int. J. Pure Appl. Math.*, 11(4):421–442, 2004.

[7] H. Baum and O. Müller. Codazzi spinors and globally hyperbolic Lorentzian manifolds with special holonomy, 2005. ESI preprint 1757.

[8] A. Beauville. Variétés Kähleriennes dont la première classe de Chern est nulle. *J. Differential Geom.*, 18(4):755–782 (1984), 1983.

[9] A. Bejancu and K. L. Duggal. *Lightlike Submanifolds of Semi-Riemannian Manifolds and Applications*, volume 364 of *Mathematics and Its Applications*. Kluwer Academic Press, 1996.

[10] Y. Benoist and P. de la Harpe. Adhérence de Zariski des groupes de Coxeter. *Compos. Math.*, 140(5):1357–1366, 2004.

[11] L. Bérard-Bergery and A. Ikemakhen. On the holonomy of Lorentzian manifolds. In *Differential Geometry: Geometry in Mathematical Physics and Related Topics (Los Angeles, CA, 1990)*, volume 54 of *Proc. Sympos. Pure Math.*, pages 27–40. Amer. Math. Soc., Providence, RI, 1993.

[12] L. Bérard-Bergery and A. Ikemakhen. Sur l'holonomie des variétés pseudo-riemanniennes de signature (n,n). *Bull. Soc. Math. France*, 125(1):93–114, 1997.

[13] L. Bérard-Bergery and T. Krantz. Personal Communication, September 2007.

[14] M. Berger. Sur les groupes d'holonomie homogène des variétés a connexion affine et des variétés riemanniennes. *Bull. Soc. Math. France*, 83:279–330, 1955.

[15] M. Berger. Les espace symétriques non compacts. *Ann. Sci. École Norm. Sup.*, 74:85–177, 1957.

[16] A. L. Besse. *Einstein Manifolds.* Springer Verlag, Berlin-Heidelberg-New York, 1987.

[17] N. Bezvitnaya. Lightlike foliations on Lorentzian manifolds with weakly irreducible holonomy algebra, 2005. math.DG/0506101 at http://arxiv.org.

[18] A. Borel and A. Lichnerowicz. Groupes d'holonomie des variétés riemanniennes. *Acad. Sci. Paris*, 234:1835–1837, 1952.

[19] C. Boubel. *Sur l'holonomie des variétés pseudo-riemanniennes.* PhD thesis, Université Henri Poincaré, Nancy, 2000.

[20] C. Boubel and A. Zeghib. Isometric actions of Lie subgroups of the Moebius group. *Nonlinearity*, 17(5):1677–1688, 2004.

[21] H. W. Brinkmann. Einstein spaces which are mapped conformally on each other. *Math. Ann.*, 94:119–145, 1925.

[22] R. Brown and A. Gray. Riemannian manifolds with holonomy group *Spin*(9). In *Differential geometry (in honor of Kentaro Yano)*, pages 41 – 59. Kinokuniya, Tokyo, 1972.

[23] R. L. Bryant. Metrics with exceptional holonomy. *Annals of Mathematics*, 126(2):525–576, 1987.

[24] R. L. Bryant. Classical, exceptional, and exotic holonomies: a status report. In *Actes de la Table Ronde de Géométrie Différentielle (Luminy, 1992)*, volume 1 of *Sémin. Congr.*, pages 93–165. Soc. Math. France, Paris, 1996.

[25] R. L. Bryant. Recent advances in the theory of holonomy. *Séminair BOURBAKI*, 51(861):1–24, 1999.

[26] R. L. Bryant. Pseudo-Riemannian metrics with parallel spinor fields and vanishing Ricci tensor. *Global Analysis and Harmonic Analysis, Séminaires et Congrès*, 4:53–93, 2000.

[27] R. L. Bryant and S. M. Salamon. On the construction of some complete metrics with exceptional holonomy. *Duke Math. J.*, 58(3):829–850, 1989.

[28] M. Cahen and M. Parker. Sur des classes d'espace pseudo-Riemanniens symétrique. *Bulletin de la Société Mathéthematique de Belgique*, XXII:339 – 354, 1970.

[29] M. Cahen and M. Parker. *Pseudo-Riemannian Symmetric Spaces*, volume 24, No.229 of *Memoirs of the AMS.* American Mathematical Society, 1980.

[30] M. Cahen and N. Wallach. Lorentzian symmetric spaces. *Bull. Amer. Math. Soc.*, 79:585–591, 1970.

[31] E. Cartan. Les groupes de transformations continus, infinis, simples. *Ann. Ec. Norm.*, 26:93–161, 1909.

[32] E. Cartan. Les groupes projectifs continus réels qui ne laissant invariante aucune multiplicité plane. *Journ. Math. pures et appl.*, 10:149–186, 1914. or Œvres complètes, vol. 1, pp. 493-530.

[33] E. Cartan. Sur une classe remarquable d'espaces de Riemann. *Bull. Soc. Math. France*, 54:214–264, 1926.

[34] V. Cortés, M.-A. Lawn, and L. Schäfer. Affine hyperspheres associated to special para-Kähler manifolds. *Int. J. of Geom. Meth. in Mod. Phys.*, 3(5-6): 995–1009, 2006

[35] V. Cortés, C. Mayer, T. Mohaupt, and F. Saueressig. Special geometry of Euclidean supersymmetry. I. Vector multiplets. *J. High Energy Phys.*, no. 3:028, 73 pp. (electronic), 2004.

[36] G. de Rham. Sur la réducibilité d'un espace de Riemann. *Math. Helv.*, 26:328–344, 1952.

[37] A. Derdzinski and W. Roter. Walker's theorem without coordinates. *J. Math. Phys.*, 47:062504, 2006.

[38] A. J. Di Scala, T. Leistner, and T. Neukirchner. Geometric applications of irreducible representations of Lie groups, this volume.

[39] A. J. Di Scala and C. Olmos. The geometry of homogeneous submanifolds in hyperbolic space. *Mathematische Zeitschrift*, 237(1):199–209, 2001.

[40] K. L. Duggal and A. Giménez. Lightlike hypersurfaces of Lorentzian manifolds with distinguished screen. *J. Geom. Phys.*, 55(1):107–122, 2005.

[41] L. P. Eisenhardt. Fields of parallel vectors in Riemannian space. *Annals of Mathematics*, 39:316–321, 1938.

[42] J. M. Figueroa-O'Farrill. Breaking the M-waves. *Classical Quantum Gravity*, 17(15):2925–2947, 2000.

[43] A. S. Galaev and T. Leistner. Holonomy groups of Lorentzian manifolds: classification, examples, and applications. To appear in: Recent developments in pseudo-Riemannian Geometry, ESI Lectures in Mathematics and Physics, EMS Publishing House.

[44] A. S. Galaev. Remark on holonomy groups of pseudo-Riemannian manifolds of signature (2,n+2). math.DG/0406397 at http://arxiv.org.

[45] A. S. Galaev. Classification of connected holonomy groups of pseudo-kählerian manifolds of index 2. ESI 1764, 2004. math.DG/0405098 at http://arxiv.org.

[46] A. S. Galaev. The space of curvature tensors for holonomy algebras of Lorentzian manifolds. *Differential Geom. Appl.*, 22(1):1–18, 2005.

[47] A. S. Galaev. Isometry groups of Lobachevskian spaces, similarity transformation groups of Euclidean spaces and Lorentzian holonomy groups. *Rend. Circ. Mat. Palermo* (2) Suppl. No. 79, 87–97, 2006.

[48] A. S. Galaev. Metrics that realize all types of Lorentzian holonomy algebras, 2005. *Int. J. Geom. Methods Mod. Phys.* 3, no. 5-6, 1025–1045, 2006.

[49] R. Ghanam and G. Thompson. The holonomy Lie algebras of neutral metrics in dimension four. *J. Math. Phys.*, 42(5):2266–2284, 2001.

[50] R. Ghanam and G. Thompson. Two special metrics with R_{14}-type holonomy. *Classical Quantum Gravity*, 18(11):2007–2014, 2001.

[51] M. Goto. *Semisimple Lie Algebras*, volume 38 of *Lecture Notes in Pure and Applied Mathematics.* Marcel Dekker Inc., New York, Basel, 1978.

[52] G. S. Hall. Space-times and holonomy groups. In *Differential Geometry and Its Applications. Proc. Conf. Opava August 24-28, 1993*, pages 201–210. Silesian University, 1993.

[53] G. S. Hall and D. P. Lonie. Holonomy groups and spacetimes. *Classical Quantum Gravity*, 17(6):1369–1382, 2000.

[54] J. Hano and H. Ozeki. On the holonomy groups of linear connections. *Nagoya Math. Journal*, 10:97–100, 1965.

[55] A. Ikemakhen. Examples of indecomposable non-irreducible Lorentzian manifolds. *Ann. Sci. Math. Québec*, 20(1):53–66, 1996.

[56] A. Ikemakhen. Sur l'holonomie des variétés pseudo-riemanniennes de signature $(2, 2+n)$. *Publ. Mat.*, 43(1):55–84, 1999.

[57] N. Iwahori. On real irreducible representations of Lie algebras. *Nagoya Mathematical Journal*, pages 59–83, 1959.

[58] D. D. Joyce. Compact 8-manifolds with holonomy $Spin(7)$. *Invent. Math.*, 123(3):507–552, 1996.

[59] D. D. Joyce. Compact Riemannian 7-manifolds with holonomy G_2. I, II. *J. Differential Geom.*, 43(2):291–328, 329–375, 1996.

[60] I. Kath and M. Olbrich. On the structure of pseudo-Riemannian symmetric spaces. math.DG/0408249 at http://arxiv.org.

[61] I. Kath and M. Olbrich. New examples of indefinite hyper-Kähler symmetric spaces. *J. Geom. Phys.* 57(8), 1697–1711, 2007.

[62] I. Kath and M. Olbrich. The classification problem for pseudo-Riemannian symmetric spaces. To appear in: Recent developments in pseudo-Riemannian Geometry, ESI Lectures in Mathematics and Physics, EMS Publishing House.

[63] S. Kobayashi and T. Nagano. On filtered Lie algebras and geoemtric structures II. *J. Math. Mech.*, 14:513–521, 1965.

[64] C. LeBrun. Quaternionic-Kähler manifolds and conformal geometry. *Math. Ann.*, 284(3):353–376, 1989.

[65] C. LeBrun and S. Salamon. Strong rigidity of positive quaternion-Kähler manifolds. *Invent. Math.*, 118(1):109–132, 1994.

[66] T. Leistner. Berger algebras, weak-Berger algebras and Lorentzian holonomy, 2002. SFB 288-Preprint no. 567, `ftp://ftp-sfb288.math.tu-berlin.de/pub/Preprints/preprint567.ps.gz`.

[67] T. Leistner. Lorentzian manifolds with special holonomy and parallel spinors. *Rend. Circ. Mat. Palermo (2) Suppl.*, (69):131–159, 2002.

[68] T. Leistner. Towards a classification of Lorentzian holonomy groups, 2003. math.DG/0305139 at http://arxiv.org.

[69] T. Leistner. Towards a classification of Lorentzian holonomy groups. Part II: Semisimple, non-simple weak Berger algebras, 2003. math.DG/0309274 at http://arxiv.org.

[70] T. Leistner. *Holonomy and Parallel Spinors in Lorentzian Geometry.* Logos Verlag, 2004. Dissertation, Mathematisches Institut der Humboldt-Universität Berlin, 2003.

[71] T. Leistner. Conformal holonomy of C-spaces, Ricci-flat, and Lorentzian manifolds. *Differential Geom. Appl.*, 24(5):458–478, 2006.

[72] T. Leistner. Screen bundles of Lorentzian manifolds and some generalisations of pp-waves. *J. Geom Phys.*, 56(10):2117–2134, 2006.

[73] T. Leistner. On the classification of Lorentzian holonomy groups. *J. Differ. Geom.*, 76(3):423–484, 2007.

[74] S. Merkulov and L. J. Schwachhöfer. Classification of irreducible holonomies of torsion-free affine connections. *Ann. Math.*, 150:77–149, 1999.

[75] J. F. Schell. Classification of 4-dimensional Riemannian spaces. *J. of Math. Physics*, 2:202–206, 1960.

[76] R. Schimming. Riemannsche Räume mit ebenfrontiger und mit ebener Symmetrie. *Mathematische Nachrichten*, 59:128–162, 1974.

[77] L. J. Schwachhöfer. *On the Classification of Holonomy Representations.* 1999. Habilitationsschrift, Mathematisches Institut der Universität Leipzig.

[78] L. J. Schwachhöfer. Connections with irreducible holonomy representations. *Adv. Math.*, 160(1):1–80, 2001.

[79] R. Shaw. The subgroup structure of the homogeneous Lorentz group. *Quart. J. Math. Oxford*, 21:101–124, 1970.

[80] J. Simons. On the transitivity of holonomy systems. *Annals of Mathematics*, 76(2):213–234, September 1962.

[81] G. Thompson. Normal form of a metric admitting a parallel field of planes. *J. Math. Phys.*, 33(12):4008–4010, 1992.

[82] H. Wakakuwa. *Holonomy groups.* Publications of the Study Group of Geometry. Vol. 6. Okayama, Japan: Okayama University, Dept. of Mathematics, 168 p. , 1971.

[83] A. G. Walker. On parallel fields of partially null vector spaces. *Quart. J. Math., Oxford Ser.*, 20:135–145, 1949.

[84] A. G. Walker. Canonical form for a Riemannian space with a parallel field of null planes. *Quart. J. Math., Oxford Ser. (2)*, 1:69–79, 1950.

[85] A. G. Walker. Canonical forms. II. Parallel partially null planes. *Quart. J. Math., Oxford Ser. (2)*, 1:147–152, 1950.

[86] H. Wu. On the de Rham decomposition theorem. *Illinois J. Math.*, 8:291–311, 1964.

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