Conformal holonomy of Lorentzian manifolds

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Conformal structures

Let (M, c) be a conformal manifold

c = equivalence class of semi-Riemannian metrics of signature (p,q)

$\tilde g \sim g \Leftrightarrow \exists\ \varphi \in C^\infty(M):\ \tilde g = e^{2\varphi} g$

Question: $\exists$? metrics in c with special properties?

E.g.
- flat metrics $\rightsquigarrow$ conformally flat metrics
- Einstein metrics $\rightsquigarrow$ conformally Einstein
- with constant scalar curvature.

$\tilde g = \sigma^{-2} g \Rightarrow$

$$\widetilde W = \sigma^{-2} W = 0 \iff \exists \text{ a flat metric in } c.$$

$$\widetilde P = P + \frac{1}{\sigma} H_\varphi - \frac{1}{\sigma^2} \|\mathrm{grad}\varphi\|^2 \cdot g \qquad (*)$$

$$\sigma^{-2}\tilde S = S - \frac{2(n-1)}{\sigma}\Delta\sigma - \frac{n(n-1)}{2\sigma^2}\|\mathrm{grad}\sigma\|^2$$

$(*)$ $\left.\begin{array}{l} P = \frac{1}{n-2}\left(Ric - \frac{S}{2(n-1)} g\right) \\ H_\sigma = \nabla d\sigma = g(\nabla \mathrm{grad}\sigma, .) \end{array}\right\} \in \Gamma(\odot^2 T^*M)$ Schouten tensor, Hessian of σ.

Conformal Einstein manifolds

$$(M,g) \text{ Einstein} \quad \Longleftrightarrow \quad P \;=\; f\cdot g$$

(M,g) conformally Einstein

$\Longleftrightarrow\ \exists\ \sigma \in C^{\infty}_{\neq 0}(M)$: $\tilde{g} = \sigma^{-2} g$ Einstein

$$\Longleftrightarrow \left\{ \begin{array}{rcl} \tilde{P} & = & P + \frac{1}{\sigma} H_\sigma - \frac{1}{2\sigma^2} \|\mathrm{grad}\sigma\|^2 \cdot g \\ // & & \\ f\cdot\tilde{g} & = & f\sigma^{-2}\cdot g \end{array} \right.$$

$$\Longleftrightarrow \boxed{0 = \ H_\sigma + \sigma\cdot P + \tau\cdot g} \qquad (*)$$

Recalling that $H_\sigma = \nabla d\sigma$ gives

$$\Longleftrightarrow \left\{ \begin{array}{rcl} 0 & = & d\sigma - \mu \\ 0 & = & \nabla\mu + \sigma P + \tau g. \end{array} \right.$$

Derivating and tracing $(*) \Rightarrow$

$$0 \;=\; d\tau - P(\mu^\sharp, .)$$

which gives a closed system.

Tractor bundle and tractor connection

$\mathcal{T} := \mathbb{R} \oplus TM \oplus \mathbb{R} \longrightarrow M$ vector bundle,
metric g on $M \rightsquigarrow \nabla^g$ on $TM \rightsquigarrow \nabla^g$ on $\mathcal{T}$ by:

$$\nabla^g_X \begin{pmatrix} \sigma \\ Y \\ \tau \end{pmatrix} := \begin{pmatrix} d\sigma(X) - g(X,Y) \\ \nabla^g_X Y + \sigma P(X)^\sharp + \tau X \\ d\tau(X) - P(X,Y) \end{pmatrix}$$

$\Rightarrow \nabla^g(\sigma, X, \tau) = 0 \iff \sigma^{-2} g$ is Einstein.

Indef. metric on $\mathcal{T}$: $h = \begin{pmatrix} 0 & 0 & 1 \\ 0 & g & 0 \\ 1 & 0 & 0 \end{pmatrix}$, ∇^g-parallel.

Conformal invariance:
$\varphi \in C^\infty_{\neq 0}(M)$ defines bundle isomorphism of $\mathcal{T}$:

$$\Theta_\varphi \begin{pmatrix} \sigma \\ X \\ \tau \end{pmatrix} := \begin{pmatrix} \sigma \\ X + \sigma \cdot \varphi^{-1} \text{grad } \varphi \\ \tau - \varphi^{-1} d\varphi(X) - \frac{1}{2} ||\text{grad } \varphi||^2 \cdot \varphi^{-2} \cdot \sigma \end{pmatrix}$$

$\Rightarrow \Theta_\varphi \in SO(\mathcal{T}_p, h_p)$ and if $\tilde{g} = \varphi^2 g$, then

$$\boxed{\nabla^{\tilde{g}}_X (\Theta_\varphi(\sigma, Y, \tau)) = \Theta_\varphi \left(\nabla^g_X(\sigma, Y, \tau) \right).}$$

I.e. $(\mathcal{T}, \nabla, h)$ defined for (M, c)!

∇-parallel sections $\leftrightarrow$ Einstein metrics in c:

$$\nabla^g \begin{pmatrix} \sigma \\ Y \\ \tau \end{pmatrix} = 0 \;\Rightarrow\; \begin{cases} Y = \text{grad } \sigma, \\ H_\sigma + \sigma P + \tau g = 0, \\ \text{i.e. } \; \sigma^{-2} g \text{ Einstein.} \end{cases}$$

$$g \in c \text{ Einstein} \;\Rightarrow\; \nabla^g \begin{pmatrix} 1 \\ 0 \\ -S/2n(n-1) \end{pmatrix} = 0.$$

(σ, Y, τ) recurrent $\Rightarrow$ (σ, Y, τ) parallel and σ cannot vanish on open sets.

Tractor curvature:

$$\mathcal{F}(X,Y) \;=\; \begin{pmatrix} 0 & 0 & 0 \\ C(X,Y)^\sharp & W(X,Y) & 0 \\ 0 & -C(X,Y,Z) & 0 \end{pmatrix}.$$

$$C(X,Y,Z) := (\nabla_X P)(Y,Z) - (\nabla_Y P)(X,Z)$$

Conformal holonomy:

$$\begin{aligned} Hol_p(M,c) \; &:= \; Hol_p(\mathcal{T}, \nabla) \;\subset\; SO(\mathcal{T}_p, h_p) \\ \mathfrak{hol}_p(M,c) \; &:= \; LA(Hol_p(\mathcal{T}, \nabla)) \;\subset\; \mathfrak{so}(\mathcal{T}_p, h_p) \end{aligned}$$

Holonomy of vector bundles

Let $\mathcal{V}$ be a vector bundle with connection ∇.

$$\begin{aligned} Hol_p(\mathcal{V},\nabla) &:= \{\mathcal{P}_\gamma \mid \gamma(0)=\gamma(1)=p\} \\ \mathfrak{hol}_p(\mathcal{V},\nabla) &:= LA\,(Hol_p(M,\nabla)) \end{aligned}$$

$$\{Hol_p(\mathcal{V},\nabla) - \text{invariant subspaces of } \mathcal{V}_p\}$$
$$\updownarrow$$
$$\{\nabla - \text{invariant sub-bundles of } \mathcal{V}\}$$

$$\{\nabla\varphi = 0\} \leftrightarrow \{\mathfrak{hol}_p(\mathcal{V},\nabla)v = 0\}.$$

Ambrose-Singer holonomy theorem:
M connected $\Rightarrow \mathfrak{hol}_p(M,\nabla)$ is generated by curvature endomorphisms:

$$\mathfrak{hol}_p(\mathcal{V},\nabla) = \left\{ \begin{array}{l} \mathcal{P}_\gamma^{-1} \circ \mathcal{F}(X,Y) \circ \mathcal{P}_\gamma : \mathcal{V}_p \to \mathcal{V}_p \\ X, Y \in TM,\ \gamma(1) = p \end{array} \right\}$$

Holonomy of torsion free connections

$\mathcal{V} = TM$, ∇ torsion free.
Bianchi-identity for the curvature,
Ambrose-Singer $\rightsquigarrow$
algebraic constraints on $\mathfrak{hol}_p(M, \nabla) \subset \mathfrak{gl}(T_pM)$:

$\mathfrak{g} \subset \mathfrak{gl}(E)$ be a Lie algebra, E a vector space.

$$\mathcal{K}(\mathfrak{g}) := \ker\Big(\lambda : \Lambda^2 E^* \otimes \mathfrak{g} \to \Lambda^3 E^* \otimes E\Big)$$

$\mathfrak{g} \subset \mathfrak{gl}(E)$ is called *Berger algebra* $\iff$

$$\mathfrak{g} \stackrel{!}{=} \langle \{ R(x,y) \mid x, y \in E, R \in \mathcal{K}(\mathfrak{g}) \} \rangle$$

$\Rightarrow \mathfrak{hol}_p(M, \nabla) \subset \mathfrak{gl}(T_pM)$ is a Berger algebra.

Classification of *irreducible* Berger algebras:
- $\mathfrak{g} \subset \mathfrak{gl}(m)$ by Schwachhöfer/Merkulow '99
- $\mathfrak{g} \subset \mathfrak{so}(r, s)$ by M. Berger '55 ("Berger list")

(M, g) Riemannian, complete, 1-connected $\Rightarrow$

$$Hol_p(M, g) := \begin{cases} SO(n) \\ U(n), SU(n) \\ Sp(n), Sp(1) \cdot Sp(n) \\ Spin(7), G_2 \end{cases}$$

or isotropy group of a symmetric space.

Holonomy groups of semi-Riemannian mfs.

DeRham/Wu-decomposition theorem

$$Hol(M,g) = H_1 \times \ldots \times H_k \iff (M,g) \overset{\text{isom.}}{\simeq} (M_1,g_1) \times \ldots \times (M_k,g_k),$$

$H_i = Hol(M_i,g_i)$ trivial or $\underbrace{\text{indecomposable}}_{\text{no } \textit{non-degenerate} \text{ invariant subspace}}$.

Riemannian: indecomposable = irreducible $\rightsquigarrow$ Classification of Riemannian holonomy groups (for (M,g) complete, simply connected).

Lorentzian: indecomposable $\neq$ irreducible

- $\nexists$ proper irreducible subgroups of $SO_0(1,n)$ [DiScala/Olmos 01].
- indecomposable, non-irreducible $\Rightarrow$

$$Hol(M,g) \subset (\mathbb{R} \times SO(n)) \ltimes \mathbb{R}^n$$

$\exists$ Classification [Berard-Bergery/Ikemakhen '93, Galaev '05, - '03]

Algebraic constaints on conformal holonomy

Bianchi-identity for W and $C \Rightarrow$

$$\mathcal{F}(X_1,X_2)\begin{pmatrix} s_3 \\ X_3 \\ t_3 \end{pmatrix}+\mathcal{F}(X_2,X_3)\begin{pmatrix} s_1 \\ X_1 \\ t_1 \end{pmatrix}+\mathcal{F}(X_3,X_1)\begin{pmatrix} s_2 \\ X_2 \\ t_2 \end{pmatrix}$$
$$= \begin{pmatrix} 0 \\ s_1 \cdot C(X_2,X_3)^\sharp + s_2 \cdot C(X_3,X_1)^\sharp + s_3 \cdot C(X_1,X_2)^\sharp \\ 0 \end{pmatrix}.$$

Proposition.
If g is the metric of a C-space, i.e. $C = 0$, then $\mathfrak{hol}(M,[g]) \subset \mathfrak{so}(p+1,q+1)$ is a Berger algebra.

Proof: $\mathcal{P}_\gamma = \begin{pmatrix} \mathcal{P}_\gamma^- \\ \mathcal{P}_\gamma^0 \\ \mathcal{P}_\gamma^+ \end{pmatrix} \in End(\mathcal{T}_{\gamma(0)}, \mathcal{T}_{\gamma(1)})$ and
$\left(\mathcal{P}_\gamma^{\mathcal{T}}\right)^{-1} \circ \mathcal{F}\left(\mathcal{P}_\gamma^0(\ .\ ), \mathcal{P}_\gamma^0(\ .\ )\right) \circ \mathcal{P}_\gamma \in \mathcal{K}\left(\mathfrak{hol}_p(M,c)\right)$.

Special case: Conformal class contains a locally symmetric metric.

Einstein metrics with $S \neq 0$

(M,g) Einstein with $S \neq 0$. Construct:

$$\left(\overline{M} := \mathbb{R} \times M \times \mathbb{R}^+,\ \bar{g} := \frac{n(n-1)}{S}\left(dt^2 - ds^2\right) + t^2 g\right)$$

$\Rightarrow \frac{\partial}{\partial s}$ parallel w.r.t. LC-connection of $\bar{g}$, and

$$\begin{array}{ccc} Hol_{(1,p,1)}(\overline{M},\bar{g}) & = & Hol_p(M,[g]) \\ || & & \\ Hol_{(1,p,1)} \underbrace{(\hat{M},\hat{g})} & & \end{array}$$

cone over (M,g):
$\left(\hat{M} := \mathbb{R}^+ \times M, \hat{g} = \frac{n(n-1)}{S} dt^2 + t^2 g\right)$,

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g Riemannian $\Rightarrow$ The cone is:

$$\left.\begin{array}{lcl} \text{- Riemannian} & \Longleftrightarrow & S > 0 \\ \text{- Lorentzian} & \Longleftrightarrow & S < 0 \end{array}\right\} \rightsquigarrow \text{holonomy known.}$$

Proof: construct bundle isom. $\mathcal{T} \simeq T\overline{M}|_{\{1\}\times M \times \{1\}}$:

$$\begin{array}{ccccc} (0,X,0) & \mapsto & X & \in & TM \subset T\overline{M}|_{\{1\}\times M\times\{1\}} \\ (1,0,\frac{S}{2n(n-1)}) & \mapsto & \frac{S}{n(n-1)}\frac{\partial}{\partial t} & \in & T\overline{M}|_{\{1\}\times M\times\{1\}} \\ (1,0,-\frac{S}{2n(n-1)}) & \mapsto & \frac{S}{n(n-1)}\frac{\partial}{\partial s} & \in & T\overline{M}|_{\{1\}\times M\times\{1\}} \end{array}$$

Einstein metrics with $S = 0$ (Ricci flat)

(M, g) Ricci flat. Construct:

$$\left(\overline{M} := \mathbb{R} \times M \times \mathbb{R}^+ \; , \; \bar{g} := 2dxdz + z^2 \cdot g \right)$$

$\Rightarrow \frac{\partial}{\partial x}$ parallel and

$$\begin{array}{ccc} Hol_{(1,p,1)}(\overline{M}, \overline{g}) & \overset{(1)}{=} & Hol_p(M, [g]) \\ (2) \| & & \\ Hol_{(p)}(M, g) \ltimes \underbrace{\mathbb{R}^{n-k}}_{\substack{\text{if dim } M = n \text{ and} \\ k = \#\text{parallel vector fields on } (M,g)}} & & \end{array}$$

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g Riemannian or Lorentzian $\rightsquigarrow$ $Hol(\overline{g})$ known.

Proof: (1) Bundle isomorphism $\mathcal{T} \simeq T\overline{M}|_{\{1\} \times M \times \{1\}}$:

$$\begin{array}{lcccl} (0, Y, 0) & \mapsto & Y & \in & TM \subset T\overline{M}|_{\{1\}\times M\times\{1\}} \\ (1, 0, 0) & \mapsto & \frac{\partial}{\partial x} & \in & T\overline{M}|_{\{1\}\times M\times\{1\}} \\ (0, 0, 1) & \mapsto & \frac{\partial}{\partial z} & \in & T\overline{M}|_{\{1\}\times M\times\{1\}} \end{array}$$

(2) $Hol_{\{1\}\times M\times\{1\}}(\overline{M}, \overline{g})$ generated by paths running in $\{1\} \times M \times \{1\}$.

Conformal holonomy in Riemannian signature

g Riemannian metric $\rightsquigarrow$
3 cases for $Hol_p(M,[g]) \subset SO(1,n+1)$:

1. irreducible $\Rightarrow Hol_p(M,[g]) = SO(1,n+1)$

2. with degenerate invariant subspace $V \iff$
   $V \cap V^\perp$ light-like, invariant line $\iff$
   light-like parallel section of $\mathcal{T} \iff$
   g conformally Ricci flat.

3. with non-degenerate invariant subspace:
   $\iff$ locally $\exists$ product of Einstein metrics in the conformal class, $g_1 \times g_2 \in [g]$ with
   $$n_2(n-n_2-1)S_1 = -n_1(n_1-1)S_2,$$
   and
   $Hol(M,[g]) = Hol(M_1,[g_1]) \times Hol(M_2,[g_2])$.
   [Leitner '04, Armstrong '05]

Conformal holonomy in Lorentzian signature

g Lorentzian metric $\rightsquigarrow$
3 cases for $Hol_p(M,[g]) \subset SO(2,n)$:

1. with one-dimensional invariant subspace $\iff$ conformally Einstein.

2. with non-degenerate invariant subspace $\iff$ $\exists$ product of Einstein metrics in $[g]$ as above ... [Leitner '04, or generalise Armstrong '05]

3. with 2-dimensional, totally isotropic, invariant subspace: ['05]
   This is the case $\iff$ g is conformally equivalent to a Lorentzian metric with

   • light-like recurrent vector field, $\nabla X \sim X$, and

   • totally isotropic Ricci-tensor, i.e.
   $$g(Ric(U), Ric(V)) = 0 \ \forall\ U, V.$$

Lorentzian mfd's with recurrent, light-like vector field

(M^{n+2}, g) Lorentzian with recurrent vector field X, i.e. $\nabla_Y X = \theta(Y) X \ \forall Y$.

$\Rightarrow \quad \mathbb{R} \cdot X_p \subset X_p^{\perp} \subset T_p M$
is a $Hol_p(M,g)$-invariant flag in $T_p M$ and

$\mathbb{R} \cdot X \subset X^{\perp} \subset TM$ are ∇-invariant distributions.

$\Rightarrow \quad Hol_p(M,g) \subset (\mathbb{R}^* \times SO(n)) \ltimes \mathbb{R}^n$

(M, g) totally Ricci-isotropic
$\iff Y \lrcorner Ric = 0 \ \forall Y \in X^{\perp}$
$\iff \ Ric^{\sharp} : TM \to \mathbb{R} \cdot X$ with $X^{\perp} \subset \ker Ric^{\sharp}$.

In particular: $S = 0$, i.e. $(n-2) Ric = P$.

$Hol(M,[g])$ has 2-dimensional totally isotropic subspace $\iff$ g admits light-like recurrent vf. and totally isotripic Ricci tensor.

($\Leftarrow$) $\mathcal{H} := \mathbb{R} \oplus \mathbb{R} \cdot X \subset \mathcal{T}$ is invariant by the tractor connection: $\nabla_U : \Gamma(\mathcal{H}) \to \Gamma(\mathcal{H})$ $\forall U \in TM$, because

$$\nabla_U \begin{pmatrix} \sigma \\ f \cdot X \\ 0 \end{pmatrix} = \begin{pmatrix} U(\sigma) - h(f \cdot X, U) & & \\ \nabla_U f \cdot X + \sigma P^\sharp(U) & \sim & X \\ P(Y,U) & = & 0 \end{pmatrix}.$$

($\Rightarrow$) If $\mathcal{H}$ is the invariant distribution, then

$$pr_{TM}\left(\underbrace{\mathcal{H} \cap (TM \oplus \mathbb{R})}_{\{(0,X,\rho)\in\mathcal{H}\}}\right)$$

will give the line bundle in TM which is invariant under ∇^g for appropriate choice of $g \in c$.

Example: Plane waves

Plane wave metric on $\mathbb{R}^{n+2} = \{x, y_1, \ldots, y_n, z)$:

$$g = 2dx\ dz + \left(\sum_{i,j=1}^{n} a_{ij}(z)\ y_i y_j \right) dz^2 + \sum_{i=1}^{n} dy_i^2,$$

a_{ij} are functions only of z.
$X := \frac{\partial}{\partial x}$ is parallel, $Ric = (\sum_{i=1}^{n} a_{ii}) dz^2$,

$Hol(M, g) = \mathbb{R}^n \subset (\mathbb{R}^* \times SO(n)) \ltimes \mathbb{R}^n$.
Tractor derivative:

$$\begin{aligned} \nabla_U \begin{pmatrix} \sigma \\ \tau \cdot X \\ 0 \end{pmatrix} &= \begin{pmatrix} U(\sigma) - \tau h(U, X) \\ \left(U(\tau) + \frac{a}{n-2}\ dz(U) \right) \cdot X \\ 0 \end{pmatrix} \\ &= 0 \end{aligned}$$

$\iff \quad \sigma = \sigma(z)$ and $\tau = \tau(z)$ satisfying

$$\begin{aligned} \sigma' &= \tau \\ \tau' &= \frac{a}{n-2}\ \sigma. \end{aligned}$$

$\Rightarrow$ 2 solutions, i.e. locally conformally Ricci flat in two ways, and $Hol(M, [g]) = \mathbb{R}^{2n+1}$.