Recent developments in pseudo-Riemannian holonomy theory

Thomas Leistner

Humboldt University Berlin

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Outline

1. Holonomy
   - The holonomy group of a linear connection
   - Classification problem and Berger algebras
   - Holonomy and geometric structure
   - Riemannian holonomy
2. Lorentzian holonomy
   - Preliminaries
   - Classification
   - Proof of the Classification
   - Applications
3. Other signatures
   - Signature $(2, n+2)$
   - Neutral signature (n, n)
   - Signature $(2, 2)$
4. Open problems

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$$Gl(T_pM)$$

and its Lie algebra $\mathfrak{hol}_p(M, \nabla)$.

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- S^n the round sphere: $Hol_p(S^n) = SO(n)$.

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M connected $\Longrightarrow$ $\mathfrak{hol}_p(M, \nabla)$ is spanned by

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Classification of Berger algebras:

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- Berger '55: $\mathfrak{g} \subset \mathfrak{so}(p,q)$,

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- Berger '55: $\mathfrak{g} \subset \mathfrak{so}(p,q)$,
- Schwachhöfer/Merkulov '99: $\mathfrak{g} \subset \mathfrak{gl}(n, \mathbb{R})$.

$\rightsquigarrow$ Classification of irreducible holonomy algebras of torsion free connections.

Holonomy and geometric structure

$$\left\{ \begin{array}{l} F \in \otimes^r_s T_pM := \otimes^r T_pM \otimes \otimes^s T_p^*M \\ Hol_p(M,\nabla)\cdot F = F \end{array} \right\} \overset{\mathcal{P}_\gamma}{\simeq} \left\{ \begin{array}{l} \varphi \in \Gamma(\otimes^r_s TM) \;: \\ \nabla\varphi = 0 \end{array} \right\}$$

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$\mathcal{P}_\gamma(\mathcal{V}) \subset \mathcal{V} \iff \nabla_X : \mathcal{V} \to \mathcal{V}$, in particular $\mathcal{V}$ is integrable.

Decomposition of a semi-Riemannian manifold (M, g)

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$$T_pM = \oplus_{i=0}^{k} V_k, \text{ with } V_0 \text{ trivial and } V_i \underbrace{\text{indecomposable}}_{\text{non-degenerate and only degenerate inv. subspaces}} \text{ for } i > 0$$

Then $$(M, g) \simeq (M_1, g_1) \times \ldots \times (M_k, g_k)$$

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- (M_i, g_i) *complete and 1-connected,*
- (M_i, g_i) *flat or with indecomposable holonomy representation,*
- $Hol_p(M,g) \simeq Hol_{p_1}(M_1, g_1) \times \ldots \times Hol_{p_k}(M_k, g_k)$.

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$$Hol_p(M, g) \overset{O(n)}{\sim} \quad \Big| \; SO(n) \; \Big| \; U(\tfrac{n}{2}) \Big| SU(\tfrac{n}{2}) \Big| \quad Sp(\tfrac{n}{4}) \quad \Big| Sp(1) \cdot Sp(\tfrac{n}{4}) \Big| G_2 \Big| Spin(7)$$

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Holonomy of Riemannian manifolds (M, g)

Positive definite metric $\Longrightarrow$ indecomposable $=$ irreducible
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$dim\{\nabla\varphi = 0\}$ $\uparrow$ *par.spinor*	*0*	*0*	*2*	*q+1*	*0*	*1*	*1*

Wu–Decomposition for a Lorentz manifold (M, g)

Let (M, g) be a complete, 1-connected Lorentzian manifold.

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Classify holonomy for these!

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The orthogonal part is reductive:

$$\mathfrak{g} := pr_{\mathfrak{so}(n)}\mathfrak{h} = \underbrace{\mathfrak{z}}_{centre} \oplus \underbrace{\mathfrak{g}'}_{= [\mathfrak{g},\mathfrak{g}] \text{ semisimple}} \qquad (\textit{Levi} - \textit{decomposition})$$

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Note: $\exists$ holonomy groups of uncoupled type III and IV which are non-closed, first examples in Berard-Bergery/Ikemakhen '96

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An indecomposable subalgebra $\mathfrak{h} \subset \mathfrak{so}(1, n+1)_L$ is characterised by: $\mathfrak{g} = pr_{\mathfrak{so}(n)}(\mathfrak{h})$, $pr_{\mathbb{R}}(\mathfrak{h})$, and possibly by $\varphi : \mathfrak{z}(\mathfrak{g}) \to \mathbb{R}$, or $\psi : \mathfrak{z}(\mathfrak{g}) \to \mathbb{R}^k$.

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In fact, there are polynomial metrics for any possible holonomy algebra.

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Theorem (— '02)

If $\mathfrak{h} \subset \mathfrak{so}(n)(1, n+1)_L$ is an indecomposable Berger algebra, then $\mathfrak{g} := proj_{\mathfrak{so}(n)}(\mathfrak{h})$ is a weak-Berger algebra.

Weak Berger algebras II

Decomposition Property for (weak) Berger algebras

Let $\mathfrak{g} \subset \mathfrak{so}(n)$ *be a (weak) Berger algebra, and* $\mathbb{R}^n$ *decomposed as follows:*

$$\begin{array}{lcccccccl} & \mathbb{R}^n & = & E_0 & \oplus & E_1 & \oplus \ldots \oplus & E_k, & E_0 \text{ trivial, } E_i \text{ irreducible.} \\ \text{Then} & \mathfrak{g} & = & & & \mathfrak{g}_1 & \oplus \ldots \oplus & \mathfrak{g}_k, & \mathfrak{g}_i \text{ ideals, such that} \end{array}$$

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$\Longrightarrow$ in order to classify $\mathfrak{g} = pr_{\mathfrak{so}(n)}\mathfrak{hol}(M,h)$ we need to classify irreducible weak Berger algebras.

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$$\begin{array}{lcccccccl} & \mathbb{R}^n & = & E_0 & \oplus & E_1 & \oplus \ldots \oplus & E_k, & E_0 \text{ trivial, } E_i \text{ irreducible.} \\ \text{Then} & \mathfrak{g} & = & & & \mathfrak{g}_1 & \oplus \ldots \oplus & \mathfrak{g}_k, & \mathfrak{g}_i \text{ ideals, such that} \end{array}$$

$\mathfrak{g}_i$ *acts irreducibly on* E_i *and trivial on* E_j*, and is a (weak) Berger algebra.*

$\Longrightarrow$ in order to classify $\mathfrak{g} = pr_{\mathfrak{so}(n)}\mathfrak{hol}(M, h)$ we need to classify irreducible weak Berger algebras.

Method: Representation theory for (complex) semisimple Lie algebras.

Weak Berger algebras II

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Corollary

Lorentzian holonomy groups of uncoupled type I and II are closed.

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Parallel spinors on a Lorentzian spin manifold (M, g)

Let $\left(\Sigma, \nabla^{\Sigma}\right)$ be the spinor bundle over (M, g).

Assume: $\exists \varphi \in \Gamma(\Sigma)$ with $\nabla^{\Sigma}\varphi = 0$ a parallel spinor field.

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$$\begin{array}{lcccc} g(X_\varphi, X_\varphi) < 0 & : & (M,g) & = & (\mathbb{R}, -dt^2) \\ g(X_\varphi, X_\varphi) = 0 & : & (M,g) & = & (\overline{M}, \overline{g}) \end{array} \times \begin{array}{l}\text{Riemannian mf.} \\ \text{with parallel spinor}\end{array}$$

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Theorem (— '03)

(M^{n+2}, g) indecomposable Lorentzian spin with parallel spinor. Then $Hol_p(M, g) = G \ltimes \mathbb{R}^n$ where G is a product of the following groups:

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This generalizes the result for $n \leq 9$ in [Bryant '99].

Lorentzian Einstein manifolds

Theorem (Galaev/— '07)

The holonomy of an indecomposable non-irreducible Lorentzian Einstein manifold is uncoupled, i.e.

$$Hol_p^0(M,g) = \begin{cases} (\mathbb{R}^+ \times G) \ltimes \mathbb{R}^n, & \text{or} \\ G \ltimes \mathbb{R}^n \end{cases}$$

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Corollary

A Lorentzian Einstein manifold with parallel light-like vector field is Ricci-flat.

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$$\mathfrak{h} \subset \mathfrak{so}(2, n+2)_{\mathcal{I}} = \left\{ \left(\begin{array}{cc|c|cc} \multicolumn{2}{c|}{\multirow{2}{*}{B}} & X^t & 0 & -c \\ & & Y^t & c & 0 \\ \hline 0 & 0 & A & -X & -Y \\ \hline 0 & 0 & 0 & \multicolumn{2}{c}{\multirow{2}{*}{-B^t}} \\ 0 & 0 & 0 & & \end{array} \right) \middle| \begin{array}{c} B \in \mathfrak{gl}(2, \mathbb{R}), \\ A \in \mathfrak{so}(n) \\ X, Y \in \mathbb{R}^n \\ c \in \mathbb{R} \end{array} \right\}.$$

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For $\mathfrak{g} \subset \mathfrak{so}(n)$ define the indecomposable subalgebra of $\mathfrak{so}(2, n+2)_{\mathcal{I}}$

$$\mathfrak{h}^{\mathfrak{g}} = \left\{ \left(\begin{array}{cc|c|cc} 0 & 0 & X^t & 0 & -c \\ 0 & 0 & Y^t & c & 0 \\ \hline 0 & 0 & A & -X & -Y \\ \hline 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right) \middle| \begin{array}{l} A \in \mathfrak{g} \\ X, Y \in \mathbb{R}^n \\ c \in \mathbb{R} \end{array} \right\} = (\mathfrak{g} \oplus \mathbb{R}) \ltimes \mathbb{R}^{2n} \subset \mathfrak{so}(2, n+2)_{\mathcal{I}}.$$

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This is in sharp contrast to the Lorentzian situation where $\mathfrak{g} = pr_{\mathfrak{so}(n)}(\mathfrak{h})$ had to be a Riemannian holonomy algebra!

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Let $\mathfrak{h} \subset \mathfrak{u}(1, n+1)$ indecomposable, non-irreducible. $\Longrightarrow$
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2. Classification of indecomposable Berger subalgebras of $\mathfrak{u}(1, n+1)_{\mathcal{I}}$.
3. For each of those $\mathfrak{h}$'s: Construction of a $(2, n+2)$-Kähler metric with holonomy algebra $\mathfrak{h}$.

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[Cortés et al '04, '05: Special para-Kähler manifolds]

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Theorem (Bérard-Bergery, Ikemakhen '97)

Let (M^{2n}, g) be a para-Kähler manifold, i.e. $Hol^0_p(M, g) \subset G$.
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2. *The Taylor series of Ω in 0 starts with $x_1 y_1 + \dots + x_n y_n$ and continues with terms which are at least quadratic in the x_i's and quadratic in the y_j's.*

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$\Rightarrow \forall p \in M\ \exists$ *co-ordinates* $(U, \varphi = (x_1, \ldots x_n, y_1, \ldots, y_n))$: $\varphi(p) = 0 \in \mathbb{R}^{2n}$, *and* $\Omega \in C^\infty(\varphi(U))$ *(para-Kähler potential):*

1. $g = \sum_{i,j=1}^{n} D_{ij} dx_i dy_j$ *with* $D_{ij} = \frac{\partial^2 \Omega \circ \varphi}{\partial x_i \partial y_j} : \varphi(U) \to \mathbb{R}$.
2. *The Taylor series of* Ω *in* 0 *starts with* $x_1 y_1 + \ldots + x_n y_n$ *and continues with terms which are at least quadratic in the* x_i*'s and quadratic in the* y_j*'s.*
3. $Hol_p(U, g)$ *is the smallest connected subgroup of* G *which contains*

$$\left\{ \begin{pmatrix} D_{ij}(q) & 0 \\ 0 & (D_{ij}(q)^t)^{-1} \end{pmatrix} \in G \mid q \in \varphi(U) \right\}.$$

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$$\mathfrak{so}(2,2)_{\mathcal{I}} = \left\{ \left(\begin{array}{cc} U & aJ \\ 0 & -U^t \end{array} \right) \mid U \in \mathfrak{gl}(2,\mathbb{R}), a \in \mathbb{R} \right\} = \mathfrak{gl}(2,\mathbb{R}) \ltimes \mathcal{A}$$

where $\mathcal{A}$ is the ideal spanned by $\left(\begin{array}{cc} 0 & J \\ 0 & 0 \end{array} \right)$ and $J = \left(\begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array} \right)$

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