

Intermezzo—A compact embedding theorem

Reiner Lauterbach

January 10, 2017

Compact embeddings

Definition

Let X and Y be Banach spaces, as sets we assume $X \subset Y$.

Compact embeddings

Definition

Let X and Y be Banach spaces, as sets we assume $X \subset Y$.

1. We say $X \subset Y$ is continuously embedded, if the identity $\mathbb{1}_X : X \rightarrow Y$ is continuous (or bounded).

Compact embeddings

Definition

Let X and Y be Banach spaces, as sets we assume $X \subset Y$.

1. We say $X \subset Y$ is continuously embedded, if the identity $\mathbb{1}_X : X \rightarrow Y$ is continuous (or bounded).
2. X is compactly embedded into Y , $X \subset\subset Y$ if $\mathbb{1}_X : X \rightarrow Y$ is compact (maps bounded sets onto relatively compact sets).

Compact embeddings

Definition

Let X and Y be Banach spaces, as sets we assume $X \subset Y$.

1. We say $X \subset Y$ is continuously embedded, if the identity $\mathbb{1}_X : X \rightarrow Y$ is continuous (or bounded).
2. X is compactly embedded into Y , $X \subset\subset Y$ if $\mathbb{1}_X : X \rightarrow Y$ is compact (maps bounded sets onto relatively compact sets).

Example

We consider the space $Y = C([0, 1]; \mathbb{R})$ with its usual norm and let $X \subset Y$ be the set of functions

$$X = \left\{ f \in Y \mid f \in C^1((0, 1); \mathbb{R}), \sup_{x \in (0, 1)} |f'(x)| < \infty \right\}.$$

with norm

$$\|f\|_X = \sup_{x \in [0, 1]} |f(x)| + \sup_{x \in (0, 1)} |f'(x)|.$$

Example continued

Example (Continuation)

Obviously X and Y are Banach spaces, $f \in X \Rightarrow f \in Y$. Moreover the mapping $\mathbb{1}_X : X \rightarrow Y$ is continuous since for $f \in X$ we have $\|f\|_Y \leq \|f\|_X$. This embedding is compact, since for any bounded set in X the set as a subset of Y is equicontinuous, hence precompact.

Example continued

Example (Continuation)

Obviously X and Y are Banach spaces, $f \in X \Rightarrow f \in Y$. Moreover the mapping $\mathbb{1}_X : X \rightarrow Y$ is continuous since for $f \in X$ we have $\|f\|_Y \leq \|f\|_X$. This embedding is compact, since for any bounded set in X the set as a subset of Y is equicontinuous, hence precompact.

For the proof of the embedding theorem, the interpolation inequality for Lebesgue-spaces is useful. It reads as follows.

Example continued

Example (Continuation)

Obviously X and Y are Banach spaces, $f \in X \Rightarrow f \in Y$. Moreover the mapping $\mathbb{1}_X : X \rightarrow Y$ is continuous since for $f \in X$ we have $\|f\|_Y \leq \|f\|_X$. This embedding is compact, since for any bounded set in X the set as a subset of Y is equicontinuous, hence precompact.

For the proof of the embedding theorem, the interpolation inequality for Lebesgue-spaces is useful. It reads as follows.

Lemma (Interpolation inequality)

Assume $1 \leq p \leq r \leq q \leq \infty$. Choose $\theta \in [0, 1]$ such that

$$\frac{1}{r} = \frac{\theta}{p} + \frac{1-\theta}{q}.$$

Suppose $u \in L^p(\Omega) \cap L^q(\Omega)$, then $u \in L^r(\Omega)$ and

$$\|u\|_{L^r(\Omega)} \leq \|u\|_{L^p(\Omega)}^\theta \|u\|_{L^q(\Omega)}^{1-\theta}.$$

Proof of the interpolation inequality

Proof. Observe that the function $U : x \mapsto \max\{|u(x)|^p, |u(x)|^q\}$ is integrable with

$$\int_{\Omega} U(x) dx \leq \|u\|_{L^p(\Omega)}^p + \|u\|_{L^q(\Omega)}^q.$$

Since $|u(x)|^r \leq U(x)$ it is clear that $u \in L^r(\Omega)$.

Proof of the interpolation inequality

Proof. Observe that the function $U : x \mapsto \max\{|u(x)|^p, |u(x)|^q\}$ is integrable with

$$\int_{\Omega} U(x) dx \leq \|u\|_{L^p(\Omega)}^p + \|u\|_{L^q(\Omega)}^q.$$

Since $|u(x)|^r \leq U(x)$ it is clear that $u \in L^r(\Omega)$. Then by definition we have

$$\frac{r\theta}{p} + \frac{r(1-\theta)}{q} = 1 \text{ and } r\theta + r(1-\theta) = r.$$

Moreover it is clear that $|u|^{r\theta} \in L^{\frac{p}{r\theta}}(\Omega)$ and $|u|^{r(1-\theta)} \in L^{\frac{q}{r(1-\theta)}}(\Omega)$.

Proof of the interpolation inequality

Proof. Observe that the function $U : x \mapsto \max\{|u(x)|^p, |u(x)|^q\}$ is integrable with

$$\int_{\Omega} U(x) dx \leq \|u\|_{L^p(\Omega)}^p + \|u\|_{L^q(\Omega)}^q.$$

Since $|u(x)|^r \leq U(x)$ it is clear that $u \in L^r(\Omega)$. Then by definition we have

$$\frac{r\theta}{p} + \frac{r(1-\theta)}{q} = 1 \text{ and } r\theta + r(1-\theta) = r.$$

Moreover it is clear that $|u|^{r\theta} \in L^{\frac{p}{r\theta}}(\Omega)$ and $|u|^{r(1-\theta)} \in L^{\frac{q}{r(1-\theta)}}(\Omega)$. Using Hölders inequality with the function spaces $L^{\frac{p}{r\theta}}(\Omega)$ and $L^{\frac{q}{r(1-\theta)}}(\Omega)$ we get

Proof of the interpolation inequality – Continuation

$$\begin{aligned}\int_{\Omega} |u|^r dx &= \int_{\Omega} |u|^{r\theta} |u|^{r(1-\theta)} dx \\ &\leq \| |u|^{r\theta} \|_{L^{\frac{p}{r\theta}}(\Omega)} \| |u|^{r(1-\theta)} \|_{L^{\frac{q}{r(1-\theta)}}(\Omega)} \\ &= \| u \|_{L^p(\Omega)}^{r\theta} \| u \|_{L^q(\Omega)}^{r(1-\theta)}\end{aligned}$$

Proof of the interpolation inequality – Continuation

$$\begin{aligned}\int_{\Omega} |u|^r dx &= \int_{\Omega} |u|^{r\theta} |u|^{r(1-\theta)} dx \\ &\leq \| |u|^{r\theta} \|_{L^{\frac{p}{r\theta}}(\Omega)} \| |u|^{r(1-\theta)} \|_{L^{\frac{q}{r(1-\theta)}}(\Omega)} \\ &= \| u \|_{L^p(\Omega)}^{r\theta} \| u \|_{L^q(\Omega)}^{r(1-\theta)}\end{aligned}$$

Taking left and right hand side to the $\frac{1}{r}$ -th power yields the result.

Compact Sobolev Embedding

Theorem (Rellich-Kondrachov: Compact Embedding)

Let $\Omega \subset \mathbb{R}^n$ be bounded and assume $\partial\Omega$ is of class C^1 . Let $1 \leq p < n$ then for

$$1 \leq q < p^* = \frac{np}{n-p}$$

we have

$$W^{1,p}(\Omega) \subset\subset L^q(\Omega).$$

Proof – First Step

1. We first prove a continuous embedding: we use the fact that for any open set U with $\bar{\Omega} \subset U \subset \mathbb{R}^n$ and any $u \in W^{1,p}(\Omega)$ we find a function $\bar{u} : U \rightarrow \mathbb{R}$ with $\text{supp } \bar{u} \subset U$ and $u \in W^{1,p}(\mathbb{R}^n)$.

Proof – First Step

1. We first prove a continuous embedding: we use the fact that for any open set U with $\bar{\Omega} \subset U \subset \mathbb{R}^n$ and any $u \in W^{1,p}(\Omega)$ we find a function $\bar{u} : U \rightarrow \mathbb{R}$ with $\text{supp } \bar{u} \subset U$ and $u \in W^{1,p}(\mathbb{R}^n)$.

Proof – First Step

1. We first prove a continuous embedding: we use the fact that for any open set U with $\bar{\Omega} \subset U \subset \mathbb{R}^n$ and any $u \in W^{1,p}(\Omega)$ we find a function $\bar{u} : U \rightarrow \mathbb{R}$ with $\text{supp } \bar{u} \subset U$ and $u \in W^{1,p}(\mathbb{R}^n)$. Hence there is an operator

$$E : W^{1,p}(\Omega) \rightarrow W^{1,p}(\mathbb{R}^n)$$

which is bounded.

Proof – First Step

1. We first prove a continuous embedding: we use the fact that for any open set U with $\bar{\Omega} \subset U \subset \mathbb{R}^n$ and any $u \in W^{1,p}(\Omega)$ we find a function $\bar{u} : U \rightarrow \mathbb{R}$ with $\text{supp } \bar{u} \subset U$ and $u \in W^{1,p}(\mathbb{R}^n)$. Hence there is an operator

$$E : W^{1,p}(\Omega) \rightarrow W^{1,p}(\mathbb{R}^n)$$

which is bounded.

Especially we can consider $E : W^{1,p}(\Omega) \rightarrow W_0^{1,p}(U)$ be bounded and moreover by Theorem 4.4.5 we have an embedding of $W_0^{1,p}(U) \subset L^{p^*}(U)$. Hence we have a bounded embedding $W^{1,p}(\Omega)$ into $L^q(\Omega)$ for $1 \leq q < p^*$.

Proof – Continuation

2. Choose a bounded sequence $\{u_m\}_{m \in \mathbb{N}}$ in $W^{1,p}(\Omega)$, we need to show that it is relatively compact as a subset in $W_0^{1,p}(U) \subset L^q(U)$.

Proof – Continuation

2. Choose a bounded sequence $\{u_m\}_{m \in \mathbb{N}}$ in $W^{1,p}(\Omega)$, we need to show that it is relatively compact as a subset in $W_0^{1,p}(U) \subset L^q(U)$.
3. We consider the smoothed functions $J_\varepsilon(u_m)$ for $\varepsilon < \varepsilon_0$.
W.l.o.g. we may assume that $\text{supp } J_\varepsilon u_m \subset U$ for all $m, \varepsilon < \varepsilon_0$.

Proof – Continuation

- Choose a bounded sequence $\{u_m\}_{m \in \mathbb{N}}$ in $W^{1,p}(\Omega)$, we need to show that it is relatively compact as a subset in $W_0^{1,p}(U) \subset L^q(U)$.
- We consider the smoothed functions $J_\varepsilon(u_m)$ for $\varepsilon < \varepsilon_0$. W.l.o.g. we may assume that $\text{supp } J_\varepsilon u_m \subset U$ for all m , $\varepsilon < \varepsilon_0$.
- $J_\varepsilon u_m$ converges uniformly in m to u_m with respect to the space $L^q(U)$. We prove this first for u_m smooth, the full result follows by approximation of arbitrary functions (in $W^{1,p}(U)$ by smooth functions). As before we write (compare with the proof of Lemma 4.2.5)

$$\begin{aligned}(J_\varepsilon u)(x) &= \frac{1}{\varepsilon^n} \int_U \rho\left(\frac{x-y}{\varepsilon}\right) u(y) \, dy \\ &= \int_{B_1(0)} \rho(z) u(x - \varepsilon z) \, dz.\end{aligned}$$

Proof – Continuation

4. (Continued) We consider for a differentiable u_m

$$\begin{aligned}(J_\varepsilon u_m)(x) - u_m(x) &= \int_{B_1(0)} \rho(z) (u_m(x - \varepsilon z) - u_m(x)) dz \\ &= \int_{B_1(0)} \rho(z) \left(\int_0^1 \frac{d}{dt} u_m(x - \varepsilon tz) dt \right) dz \\ &= -\varepsilon \int_{B_1(0)} \rho(z) \left(\int_0^1 Du_m(x - t\varepsilon z) z dt \right) dz\end{aligned}$$

Proof – Continuation

4. (Continued) Therefore we can estimate

$$\begin{aligned} \int_U |(J_\varepsilon u_m)(x) - u_m(x)| dx &\leq \varepsilon \int_{B_1(0)} \rho(y) \int_0^1 \int_U |Du_m(x - ty)| dx dt \\ &\leq \varepsilon \int_U |Du_m(x)| dx. \end{aligned}$$

Proof – Continuation

4. (Continued) Therefore we can estimate

$$\begin{aligned} \int_U |(J_\varepsilon u_m)(x) - u_m(x)| dx &\leq \varepsilon \int_{B_1(0)} \rho(y) \int_0^1 \int_U |Du_m(x - ty)| dx dt \\ &\leq \varepsilon \int_U |Du_m(x)| dx. \end{aligned}$$

5. It follows that

$$\|J_\varepsilon u_m - u_m\|_{L^1(U)} \leq \varepsilon \|Du_m\|_{L^1(U)} \leq C\varepsilon \|Du_m\|_{L^p(U)}$$

where the last inequality follows from the Hölder inequality
This generalizes (by approximation) to arbitrary
 $W^{1,p}(U)$ -functions.

Proof – Continuation

4. (Continued) Therefore we can estimate

$$\begin{aligned} \int_U |(J_\varepsilon u_m)(x) - u_m(x)| dx &\leq \varepsilon \int_{B_1(0)} \rho(y) \int_0^1 \int_U |Du_m(x - ty)| dx dt \\ &\leq \varepsilon \int_U |Du_m(x)| dx. \end{aligned}$$

5. It follows that

$$\|J_\varepsilon u_m - u_m\|_{L^1(U)} \leq \varepsilon \|Du_m\|_{L^1(U)} \leq C\varepsilon \|Du_m\|_{L^p(U)}$$

where the last inequality follows from the Hölder inequality
This generalizes (by approximation) to arbitrary
 $W^{1,p}(U)$ -functions.

6. Therefore $J_\varepsilon u_m \rightarrow u_m$ uniformly in m with respect to the $L^1(U)$ -norm.

Proof – Continuation

7. Let $1 \leq q \leq p^*$ and choose $q < \tilde{p} < p^*$ and write

$$\frac{1}{q} = \theta + \frac{1 - \theta}{\tilde{p}}.$$

Then we have from the interpolation inequality

$$\|J_\varepsilon u_m - u_m\|_{L^q(U)} \leq \|J_\varepsilon u_m - u_m\|_{L^1(U)}^\theta \|J_\varepsilon u_m - u_m\|_{L^{\tilde{p}}(U)}^{1-\theta}.$$

By bounded embedding $W_0^{1,p}(U) \rightarrow L^{\tilde{p}}(U)$ we have

$$\|J_\varepsilon u_m - u_m\|_{L^q(U)} \leq C \|J_\varepsilon u_m - u_m\|_{L^1(U)}^\theta$$

and we have uniform (with respect to m) convergence of $J_\varepsilon u_m \rightarrow u_m$ in $L^q(U)$.

Proof – Continuation

8. Fix $0 < \varepsilon < \varepsilon_0$. We show $\{J_\varepsilon u_m\}_{m \in \mathbb{N}}$ is uniformly bounded. For $x \in U$ (or $x \in \mathbb{R}^n$) we have

$$\begin{aligned} |J_\varepsilon u_m(x)| &\leq \frac{1}{\varepsilon^n} \int_U \rho\left(\frac{x-y}{\varepsilon}\right) |u_m(y)| \, dy \\ &\leq \frac{C}{\varepsilon^n} \|\rho\|_{L^\infty(U)} \|u_m\|_{L(U)} \\ &\leq \frac{C}{\varepsilon^n} < \infty \end{aligned}$$

Proof – Continuation

8. Fix $0 < \varepsilon < \varepsilon_0$. We show $\{J_\varepsilon u_m\}_{m \in \mathbb{N}}$ is uniformly bounded. For $x \in U$ (or $x \in \mathbb{R}^n$) we have

$$\begin{aligned} |J_\varepsilon u_m(x)| &\leq \frac{1}{\varepsilon^n} \int_U \rho\left(\frac{x-y}{\varepsilon}\right) |u_m(y)| \, dy \\ &\leq \frac{C}{\varepsilon^n} \|\rho\|_{L^\infty(U)} \|u_m\|_{L^1(U)} \\ &\leq \frac{C}{\varepsilon^n} < \infty \end{aligned}$$

9. For $0 < \varepsilon < \varepsilon_0$ the family $\{u_m\}_{m \in \mathbb{N}}$ is equicontinuous:

$$\begin{aligned} |DJ_\varepsilon u_m(x)| &\leq \frac{C}{\varepsilon^n} \int_U \left| D\rho\left(\frac{x-y}{\varepsilon}\right) \right| |u_m(y)| \, dy \\ &\leq \frac{C}{\varepsilon^n} \|D\rho\|_{L^\infty(U)} \|u_m\|_{L^1(U)} \\ &\leq \frac{C}{\varepsilon^{n+1}}. \end{aligned}$$

Proof – Continuation

10. Find a convergent subsequence, fix $\delta > 0$. Then we find an $\varepsilon_1 \in (0, \varepsilon_0)$ with for all $(0 < \varepsilon < \varepsilon_1)$ we have

$$\forall m \in \mathbb{N} : \|J_\varepsilon u_m - u_m\|_{L^q(U)} < \frac{\delta}{3}.$$

By equicontinuity and the Ascoli-Arzelá theorem we find a subsequence $\{u_{m_j}\}_{j \in \mathbb{N}}$ such that $\{J_\varepsilon u_{m_j}\}_{j \in \mathbb{N}}$ is (with respect to uniform convergence) a Cauchy sequence, since U is bounded it follows that this is a Cauchy sequence in $L^q(U)$, i.e. there exists a $N \in \mathbb{N}$ such that $k, j > N$ implies

$$\|J_\varepsilon u_{m_k} - J_\varepsilon u_{m_j}\|_{L^q(U)} < \frac{\delta}{3}.$$

Then we have with $X = L^q(U)$, $0 < \varepsilon < \varepsilon_1$ and $j, k > N$

$$\begin{aligned} \|u_{m_k} - u_{m_j}\|_X &\leq \|u_{m_k} - J_\varepsilon u_{m_k}\|_X + \|J_\varepsilon u_{m_k} - J_\varepsilon u_{m_j}\|_X + \|u_{m_j} - J_\varepsilon u_{m_j}\|_X \\ &\|u_{m_k} - u_{m_j}\|_X < \delta. \end{aligned}$$