Sectorial operators generate analytic semigroups

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Ergänzungen

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Definition. Let $A : X \to X$ be a linear operator with $D(A) \subset X$. It is called an operator of type $(\Phi, M)$ for $\Phi \in (0, \frac{\pi}{2})$, $M > 0$ if

1. $A$ is closed and $D(A)$ is dense in $X$.
2. The resolvent set $P(A)$ of $A$ contains the set

$$S_{\Phi} = \left\{ z \in \mathbb{C} \mid z \neq 0, \ 1 - \frac{1}{2} \pi < \arg(z) < \frac{3}{2} \pi + \Phi \right\}$$

and for all $\lambda \in S_{\Phi}$ we have an estimate of the resolvent as

$$\|R(\lambda, A)\| \leq \frac{M}{|\lambda|}. \quad (1)$$

The operator $A$ is called sectorial, if there exists a $\tau \in \mathbb{R}$ with $A - \tau \mathbb{1}$ is of type $(\Phi, M)$. 

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Statement of the Theorem

If $A$ is sectorial, then $-A$ generates an analytic semigroup $\{T(t)\}_{t \geq 0}$. Moreover we have:

1. For $t > 0$ the operators $AT(t)$, $\frac{d}{dt} T(t)$ are bounded linear with
   
   $$\frac{d}{dt} T(t)x = -AT(t)x, \quad \forall x \in X, \quad t > 0.$$ 

2. $T$ has an analytic continuation to a sector

   $$S = \left\{ z \in \mathbb{C} \mid |\arg z| < \varphi_1 \right\}$$

   which contains the positive real half axis. For $t, s \in S$ we have $T(t + s) = T(t) \circ T(s)$.

3. There exists an $a \in \mathbb{R}$, such that for all $t \in S$ we have an estimate of the following form

   $$\| T(t) \| \leq Ce^{-at} \quad \text{and} \quad \| AT(t) \| \leq \frac{C}{|t|} e^{-at}.$$
Proof – Preliminary

Wlog: $\tau = 0$.

Curve $\Gamma$

The left (red) curve indicates the boundary of the set which by assumption contains the resolvent set of $-A$. The blue curve represents $\Gamma$. If we multiply $t$ with $\lambda \in \Gamma$, then $\text{Re}(\lambda t) < 0$ and $\lim_{|\lambda| \to \infty} \text{Re}(\lambda t) = 0$. 

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Proof – Construction of $T(t)$

We define

$$T(t)x = \frac{1}{2\pi i} \int_{\Gamma} R(\lambda, -A) \exp(\lambda t)x \, d\lambda.$$ 

We write $\Gamma_n = \Gamma \cup B_n(0)$ and $\Gamma_{c,n} = \Gamma \setminus \Gamma_n$. Then

$$T(t)x = \frac{1}{2\pi i} \int_{\Gamma_n} R(\lambda, -A) \exp(\lambda t)x \, d\lambda + \frac{1}{2\pi i} \int_{\Gamma_{c,n}} R(\lambda, -A) \exp(\lambda t)x \, d\lambda.$$ 

We estimate

$$\left\| \frac{1}{2\pi i} \int_{\Gamma_{c,n}} R(\lambda, -A) \exp(\lambda t)x \, d\lambda \right\| \leq \frac{1}{2\pi} \int_{\Gamma_{c,n}} \frac{M}{|\lambda|} \exp(\text{Re}(\lambda t)) \, d\lambda \leq \frac{MC(n)}{2\pi n}.$$
Proof – Convergence of $T(t)$

Therefore

$$\lim_{n \to \infty} \frac{1}{2\pi i} \int_{\Gamma_n} R(\lambda, -A) \exp(\lambda t)x \, d\lambda$$

exists and

$$T(t)x = \frac{1}{2\pi i} \int_{\Gamma} R(\lambda, -A) \exp(\lambda t)x \, d\lambda.$$

is defined.
The dark blue curve indicates a new curve $\Gamma_0$. To show that using $\Gamma_0$ we get the same integral, we choose $f \in X^*$ and $x \in X$. Define curves $\Gamma_{0,n}$ and $\Gamma_{0,c,n}$ similar to the above construction.
Proof – Semigroup Property – Independence of curve

We integrate

\[ g(\lambda, t) = f \left( R(\lambda, -A) \exp(\lambda t) \right) \]

along \( \Gamma \) and \( \Gamma_0 \) and take the difference. Denote the brown arcs by \( A_1 \) and \( A_2 \), such that \( \Gamma_n + A_1 - \Gamma_0, n - A_2 \) is a closed curve \( \gamma \).

\[
\left| \int_{\Gamma} g(\lambda, t) \, d\lambda - \int_{\Gamma_0} g(\lambda, t) \, d\lambda \right| \leq \left| \int_{\gamma} g(\lambda, t) \, d\lambda \right| + \\
+ \int_{|\Gamma_{c,n}| + |\Gamma_{0,c,n}| + |A_1| + |A_2|} g(\lambda, t) \, d\lambda
\]
By Cauchy’s theorem we have

\[ \int_{\gamma} g(\lambda, t) \, d\lambda = 0. \]

Then for \( N = \Gamma_{c,n} \) or \( N = \Gamma_{0,c,n} \) we have

\[ \left| \int_{N} g(\lambda, t) \, d\lambda \right| \leq \frac{M}{n} \| f \|_X \int_{N} \exp(\Re \lambda t) \, d\lambda \to 0 \text{ as } n \to \infty. \]

For \( A_i \)

\[ \left| \int_{A_i} g(\lambda, t) \, d\lambda \right| \leq \frac{M}{n} \| f \|_X \int_{A_i} \exp(\Re \lambda t) \, d\lambda \to 0 \text{ as } n \to \infty. \]
Proof – 1.) Growth condition

We use the estimate from above for $R\left(\frac{\lambda}{|t|}, -A\right)$, i.e.

$$\| R\left(\frac{\lambda}{|t|}, -A\right) \| \leq \frac{M|t|}{|\lambda|}$$

to obtain with $\Gamma' = |t|\Gamma$, a change of coordinates $\lambda' = \lambda|t|$ and $t = |t|\xi$, $|\xi| = 1$ and Cauchy’s Theorem (replacing $\Gamma'$ by $\Gamma$):

$$\| T(t) \| \leq \frac{M}{2\pi} \int_{\Gamma} |\exp(\lambda' \xi)| \frac{d\lambda'}{\lambda'} = \frac{MC(\xi)}{2\pi}.$$
Proof – 2.) Growth condition

Since $A$ is closed, we can write $A$ into the integral over $\Gamma$ (we had such an argument several times before). Therefore we have an expression for $AT(t)$ given by

$$AT(t) = \frac{1}{2\pi i} \int_{\Gamma} \exp(\lambda' \xi) AR\left(\frac{\lambda'}{|t|}, -A\right) \frac{d\lambda'}{|t|}.$$ 

We get

$$AT(t) = \frac{1}{2\pi i} \int_{\Gamma} \exp(\lambda' \xi)(-1 + \frac{\lambda'}{|t|}) R\left(\frac{\lambda'}{|t|}\right) \frac{d\lambda'}{|t|}.$$ 

The term with the $1$ gives a constant, the second term yields an estimate of the form

$$\frac{M}{2\pi} \int_{\Gamma} \left| \exp(\lambda' t) \right| \frac{\lambda'}{|t|} \frac{|t|}{\lambda'} \frac{d\lambda'}{|t|}.$$ 

This yields an estimate of the form $\leq \frac{C}{|t|}$. 

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Proof – Strong Continuity

1. For $x \in D(A)$ we have

$$T(t)x - x = \frac{1}{2\pi i} \int_{\Gamma} \exp(\lambda t)[R(\lambda, -A) - \lambda^{-1}]x \, d\lambda$$

$$= -\frac{1}{2\pi i} \int_{\Gamma} \lambda^{-1}e^{\lambda t}AR(\lambda, -A)x \, d\lambda.$$

This gives for $t > 0$

$$\|T(t)x - x\| \leq C\|Ax\|t$$

(use $\mu = \lambda t$.) (Observe $R(\lambda, -A)(\lambda I + A) = I$ and hence $AR(\lambda, -A)\lambda^{-1} + R(\lambda, -A) = \lambda^{-1}I$ implying $R(\lambda, -A) - \lambda^{-1} = -AR(\lambda, -A)\lambda^{-1}$)

2. $D(A) \subset X$ is dense $T(t)$ bounded:

$$\|T(t)x - x\| \leq \|T(t)x - T(t)x_n\| + \|T(t)x_n - x_n\| + \|x_n - x\|.$$
Let $x \in D(A)$ and $t > 0$, then

$$\frac{d}{dt} AT(t) + AT(t) = \frac{1}{2\pi i} \int_{\Gamma} e^{\lambda t} (\lambda + A) R(\lambda, -A)x \, d\lambda = 0.$$ 

Then for $x \in D(A)$, $t > 0$ we have

$$\frac{1}{t} \int_{0}^{t} \frac{d}{ds} T(s)x \, ds = -\frac{1}{t} \int_{0}^{t} T(s)Ax \, ds.$$ 

The limit on the right hand side exists and yields $-Ax$. Therefore $-A$ is contained in the generator $\tilde{A}$.
Define for $\lambda$ sufficiently large (given by the growth rate $\omega$ of $T(t)$)

$$R(\lambda)x = \int_{0}^{\infty} e^{-\lambda t} T(t)x \, dt.$$ 

$R = R(\lambda)$ is bounded linear operator.
Consider for \( x \in X \)

\[
\frac{1}{h}(T(h)Rx - Rx) = \frac{1}{h} \left( \int_0^\infty e^{-\lambda t} T(t+h)x \, dt - \int_0^\infty e^{-\lambda t} T(t)x \, dt \right)
\]

\[
= \frac{e^{\lambda h}}{h} \int_0^\infty e^{-\lambda u} T(u)x \, du - \frac{1}{h} \int_0^\infty e^{-\lambda t} T(t)x \, dt
\]

\[
= \frac{e^{\lambda h} - 1}{h} \int_0^\infty e^{-\lambda u} T(u)x \, du - \frac{1}{h} \int_0^h e^{\lambda t} T(t)x \, dt
\]

\[
\rightarrow \lambda Rx - x \text{ for } h \to 0.
\]
This implies for \( x \in X \) that \( R(\lambda)x \in D(\bar{A}) \) and \( \bar{A}Rx - \lambda Rx = -x \) i.e. we have \( (\lambda \mathbb{1} - \bar{A})R(\lambda)x = x \) for all \( x \in X \). Then, trivially, for \( x \in D(\bar{A}) \) we have \( R(\lambda)x \in D(\bar{A}) \) and

\[
\bar{A}R(\lambda)x = \bar{A} \int_0^\infty e^{\lambda t} T(t) x \, dt = \int_0^\infty e^{\lambda t} T(t) \bar{A}x \, dt = R(\lambda)\bar{A}x.
\]

Therefore we have \( \bar{R}Ax - \lambda Rx = -x \) and hence

\[
R(\lambda)(\lambda \mathbb{1} - \bar{A})x = x \quad \text{for} \quad x \in D(\bar{A}).
\]

Therefore \( R(\lambda) = (\lambda \mathbb{1} - \bar{A})^{-1} \), Then \( P(A) \cap P(\bar{A}) \neq \emptyset \). For \( \lambda \in D(A) \cap D(\bar{A}) \) we have

\[
(\lambda \mathbb{1} - \bar{A})D(A) = (\lambda \mathbb{1} - A)D(A) = X.
\]

Since \( \lambda \mathbb{1} - \bar{A} \) is injective and \( D(A) \subset D\bar{A} \) we have

\[
D(A) = D(\bar{A}).
\]