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Jan Henrik Sylvester Summer term 2016

# Functional analysis exercises
# Solutions – Sheet 10

**Problem 37.** Let $K$ be the operator from Exercise 12 or 36. Show:

1. For $T = \mathbb{1} - K$, $\ker T$ is closed.
2. $\ker T$ is finite dimensional.
3. The image of $T = \mathbb{1} - K$ is closed and has finite codimension.

*Solution.* 1. The kernel of any continuous linear operator ist closed.

2. In $\ker T$, $\overline{B_1(0)} = \overline{K(B_1(0))}$ is comact by Exercise 36, hence $\ker T$ must be finite dimensional.
3. Let $y_n \in \mathrm{IM}(T)$ and $y_0 = \lim_{n\to\infty} y_n$ and $x_n \in X$ with $Tx_n = y_n$. If $x_n$ is bounded, we have
$$x_n - Kx_n = y_n \to y_0$$
and
$$Kx_{n_j} \to x_0.$$
Hence, the sequence
$$x_{n_j} = y_0 + Kx_{n_j} \to y_0 + Kx_0$$
converges. We have
$$(\mathbb{1} - K)x_0 = x_0 - Kx_0 = y_0.$$
If the sequence of $x_n$ is not bounded, we try to modify it (by addition of elements from the kernel) to be bounded. This is possible unless the norm in $X/\ker T$ of the sequence $[x_n]$ is not bounded. This leads to a contradiction (compare to the proof of Theorem 4.2.2.9 from the lecture).
To obtain the finite codimension, use the Banach's closed range theorem and the proposition from the previous parts of the exercise for the dual operator (compare to the text after the proof of Theorem 4.2.2.9 in the lecture notes). □

**Problem 38.** For $m > \frac{1}{p}$, let $i_{m,p} : w^{m,p} \to \ell^p$ be the embedding $x \mapsto x$. Show that $i_{m,p}$ is well defined and $i_{m,p}(B_1^{w^{m,p}}(0)) \subseteq \ell^p$ is relatively compact.

*Solution.* $i_{m,p}(B_1(0))$ is relatively compact, iff the set is bounded and for every $\varepsilon > 0$ there is an $N \in \mathbb{N}$ with
$$\sigma_\ell^N(i_{m,p}(B_1(0))) \subseteq B_\varepsilon(0),$$
where $\sigma_\ell$ is the left shift.

To boundedness: We have
$$\|i_{m,p}(x)\|_p^p = \sum_{n=1}^{\infty} |x_n|^p \le \sum_{n=1}^{\infty} |n^m \cdot x_n|^p \le 1,$$

and hence $i_{m,p}(B_1^{m,p}(0)) \subseteq B_1^{\ell^p}(0)$.

To the shift property: For $\varepsilon > 0$ choose $N \in \mathbb{N}$ large enough, such that $\sum_{n=N+1}^{\infty} n^{-mp} < \varepsilon$. We get

$$\left\|\sigma_\ell^N(i_{m,p}(x))\right\|_p^p = \sum_{n=N+1}^{\infty} |x_n|^p = \sum_{n=N+1}^{\infty} n^{-mp} \cdot |n^m x_n|^p .$$

Since

$$\sum_{n=1}^{\infty} n^{pm} \cdot |x_n|^p \leq 1,$$

we have $n^{pm} \cdot |x_n|^p \leq 1$ for all $n \in \mathbb{N}$ and hence

$$\left\|\sigma_\ell^N(i_{m,p}(x))\right\|_p^p \leq \sum_{n=N+1}^{\infty} n^{-mp} < \varepsilon.$$

Thus, the set is compact. □

**Problem 39.** Let $O(n)$ be the group of orthogonal matrices in $\mathbb{R}^{n\times n}$. For $f \in L^p(\mathbb{R}^n;\mathbb{R})$, set

$$A.f(x) = f(A^{-1}x).$$

Show that the set $\{A.f \mid A \in O(n)\}$ is compact in $L^p(\mathbb{R}^n)$.

*Solution.* We use the Fréchet-Kolmogorov theorem. First we show that the set $\mathcal{O}_f = \{A.f \mid A \in O(n)\}$ is closed and bounded. Boundedness is due to the transformation theorem:

$$\|A.f\|_p^p = \int_{\mathbb{R}^n} |A.f(x)|^p \,\mathrm{d}x = \int_{\mathbb{R}^n} \left|f(A^{-1}x)\right| \,\mathrm{d}x = \int_{\mathbb{R}^n} |f(x)|^p \,\mathrm{d}x = \|f\|_p^p .$$

Let $A_n.f$ be an $L^p$ convergent sequence with limit $h \in L^p$. Since $O(n)$ is compact, the sequence of $A_n$ has a convergent subsequence with limit $A \in O(n)$. We have

$$\|A_n.f - A.f\|_p^p = \int_{\mathbb{R}^n} \left|f(A_n^{-1}Ax) - f(x)\right|^p \,\mathrm{d}x$$

and this converges to 0 for $n \to \infty$ (see the proof of the Fréchet-Kolmogorov theorem). We obtain $h = A.f$ and the sequence $A_n.f$ converges to $A.f$. Thus, $\mathcal{O}_f$ is closed.

To obtain compactness by Fréchet-Kolmogorov, we have to check two conditions. We start with

$$\lim_{t\to 0} \int_{\mathbb{R}^n} |A.f(x+t) - A.f(x)|^p \,\mathrm{d}x \to 0$$

uniformly in $A \in O(n)$. For $\varepsilon > 0$, there is a $\delta > 0$ with

$$\|t\| < \delta \implies \int_{\mathbb{R}^n} |f(x+t) - f(x)|^p \,\mathrm{d}x < \varepsilon.$$

Anyhow, we have

$$\int_{\mathbb{R}^n} |A.f(x+t) - A.f(x)|^p \,\mathrm{d}x = \int_{\mathbb{R}^n} \left|f(x + A^{-1}t) - f(x)\right|^p \,\mathrm{d}x$$

and $\left\|A^{-1}t\right\| = \|t\|$. Hence, we obtain $\int_{\mathbb{R}^n} |A.f(x+t) - A.f(x)|^p \, \mathrm{d}x < \varepsilon$ for $\|t\| < \delta$. The first condition is fulfilled.

Moreover, for every $\varepsilon > 0$ there must be an $M > 0$ with

$$\int_{\mathbb{R}^n \setminus B_M(0)} |A.f(x)|^p \, \mathrm{d}x < \varepsilon \; \forall \, A \in O(n).$$

Since $f \in L^p(\mathbb{R}^n;\mathbb{R})$, for $\varepsilon > 0$, there is $M > 0$ with

$$\int_{\mathbb{R}^n \setminus B_M(0)} |f(x)|^p \, \mathrm{d}x < \varepsilon.$$

Due to the invariance of balls by $A$, we get $A(B_M(0)) = B_M(0)$ and hence with the transformation theorem

$$\int_{\mathbb{R}^n \setminus B_M(0)} |A.f(x)|^p \, \mathrm{d}x = \int_{\mathbb{R}^n \setminus B_M(0)} |f(x)|^p \, \mathrm{d}x < \varepsilon.$$

Thus, $\mathcal{O}_f$ is compact. □

**Problem 40.** Let $X$ be a B-space over $\mathbb{K}$. Show:

1. $X$ is reflexive, iff $X'$ is reflexive.
2. If $X$ is reflexive, any closed subspace $Y \subseteq X$ is reflexive.

*Solution.*

1. We start with $X$ reflexive. We need to show that $J_{X'} : X' \to X'''$, $J_{X'}(x')(x'') = x''(x')$ is an isometric isomorphism. Since isometry is always given, we need to show surjectivity. Consider any $x''' \in X'''$. Then $x' : X \to \mathbb{K} : x \mapsto x'''(J_X(x))$ is a functional in $X'$ and we have $J_{X'}(x')(x'') = x''(x')$. Since $X$ is reflexive, there is an $x \in X$ with $x'' = J_X(x)$, thus we get $x''(x') = x'(x)$. We calculate $J_{X'}(x')(x'') = x'(x) = x'''(J_X(x)) = x'''(x'')$. Thus, $x'$ is the preimage of $x'''$ and $J_{X'}$ is surjective.

   Now let $X'$ be reflexive. Again, we only need to show surjectivity of $J_X : X \to X''$, $J_X(x)(x') = x'(x)$. Let $x''' \in X'''$ be a functional with $x'''(x'') = 0$ for all $x'' \in J_X(X)$. Since $X'$ is reflexive, there is an $x' \in X'$ with $x'''(x'') = x''(x')$ for all $x'' \in X''$. Particularly, we obtain $0 = x'''(J_X(x)) = J_X(x)(x') = x'(x)$ for all $x \in X$. Hence, $x'$ and $x'''$ are both zero. Then $X'' = J_X(X)$.

   Remark: To follow $X$ reflexive from $X'$ reflexive, one can also use the next part of the exercise to follow $X$ reflexive from $X''$ reflexive, which is a consequence of $X'$ reflexive.

2. Let $Y \subseteq X$ be closed. Again, we have to show surjectivity of the embedding. Consider any $y'' \in Y''$. Set $x'' \in X''$ by

   $$x''(x') = y''(x'\big|_Y).$$

   $x''$ is obviously continuous and since $X$ is reflexive, there is a $y \in X$ with $x'' = J_X(y)$. Assume that $y \notin Y$, then we have a functional $x' \in X'$ with $x'\big|_Y \equiv 0$ and $x'(y) = 1$. This leads to the contradiction

   $$1 = x'(y) = x''(x') = y''(x'\big|_Y) = 0.$$

Therefore $y \in Y$. Consider any extension $\hat{y}' \in X'$ of $y' \in Y'$. We have

$$y''(y') = x''(\hat{y}') = \hat{y}'(y) = y'(y).$$

Thus, $Y$ is reflexive. □

**Exercises are due on June 21, 2016.**

## Space of the week

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|---|---|
| Name: | Orlicz spaces $L^\varphi(\Omega; X)$, $(\Omega, \mathcal{A}, \mu)$ $\sigma$ finite measure space, $X$ B-algebra, $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ convex, $\lim_{x\to 0} \frac{\varphi(x)}{x} = 0$, $\lim_{x\to\infty} \frac{\varphi(x)}{x} = \infty$ |
| Definition: | $\left\{ f : \Omega \to X \;\middle|\; f \text{ measurable}, \int_\Omega \varphi(\|f(\omega)\|_X) \,\mathrm{d}\mu < \infty \right\}$ |
| Norm: | $\|f\|_{L^\varphi(\Omega;X)} = \sup \left\{ \|fg\|_1 \;\middle|\; \int_\Omega \varphi(\|g(\omega)\|_X) \,\mathrm{d}\mu \leq 1 \right\}$ |
| Dual space: | |
| Dual space of: | |
| Reflexive: | |
| Criterion for compactness: | |
| Criterion for weak convergence: | |
| Additional aspects: | generalization of $L^p$ spaces |