Universität Hamburg
Janko Latschev
Pavel Hájek

Winter 2019/20

# Symplectic Geometry

## Problem Set 9

**1.** Let $(E, J) \to (\Sigma, j)$ be a complex vector bundle over a Riemann surface. A connection on $E$ is an $\mathbb{R}$-linear map

$$\nabla : \Gamma(E) \to \Gamma(\mathrm{Hom}_{\mathbb{R}}(T\Sigma, E))$$

such that for all $f : \Sigma \to \mathbb{R}$ and $s \in \Gamma(E)$ we have

$$\nabla(f \cdot s) = df \cdot s + f \cdot \nabla s.$$

Prove:

**a)** The difference $A = \nabla - \nabla'$ of two connections is a $C^\infty(\Sigma)$-linear map $A : \Gamma(E) \to \Gamma(\mathrm{Hom}_{\mathbb{R}}(T\Sigma, E))$.

**b)** For every connection $\nabla$ on $E$ the operator

$$D := \nabla + J \circ \nabla \circ j \qquad (\text{ so } Ds = \nabla s + J \circ (\nabla s) \circ j\ )$$

is a real linear Cauchy-Riemann operator on $(E, J)$.

**c)** If $\nabla$ is a complex connection, meaning that $\nabla J = 0$ (which is equivalent to $\nabla(Js) = J(\nabla s)$ for all $s \in \Gamma(E)$), then the operator $D$ from part **b)** is a complex linear Cauchy Riemann operator.

**d)** The difference $B = D - D'$ of two real linear Cauchy-Riemann operators is a $C^\infty(\Sigma)$-linear map $B : \Gamma(E) \to \Gamma(\overline{\mathrm{Hom}}_{\mathbb{C}}(T\Sigma, E))$.

**2.** *(holomorphic line bundles on surfaces)*
Prove the following assertions:

**a)** The cotangent bundle $K_\Sigma = T^*\Sigma$ of a Riemann surface $(\Sigma, j)$ is a holomorphic line bundle. It is called the *canonical bundle* of $\Sigma$.

**Please turn!**

**b)** Define $U \subseteq \mathbb{C}P^1 \times \mathbb{C}^2$ as the subset

$$U := \{ ([z], w) \,|\, \exists\, \lambda \in \mathbb{C} \,:\, w = \lambda z, \}.$$

Then with the obvious projection $\pi : U \to \mathbb{C}P^1$, $\pi([z], w) = [z]$ this is a holomorphic line bundle over $\mathbb{C}P^1$, called the *universal line bundle* over $\mathbb{C}P^1$.

**c)** Let $\mathcal{U}_i := \{ [z_0 : z_1] \,|\, z_i \neq 0 \} \subseteq \mathbb{C}P^1$ be the two open subsets giving the standard covering of $\mathbb{C}P^1$ by charts For every $k \in \mathbb{Z}$ we can define a holomorphic line bundle $E_k \to \mathbb{C}P^1$ by gluing the trivial bundles $E^0 = \mathcal{U}_0 \times \mathbb{C}$ and $E^1 = \mathcal{U}_1 \times \mathbb{C}$ via the transition map

$$\begin{aligned} \psi_k : E^0|_{\mathcal{U}_0 \cap \mathcal{U}_1} &\to E^1|_{\mathcal{U}_0 \cap \mathcal{U}_1} \\ ([z_0 : z_1], v) &\mapsto \left( [z_0 : z_1], \left( \frac{z_0}{z_1} \right)^k \cdot v \right). \end{aligned}$$

Then the bundle $E_k \to \mathbb{C}P^1$ admits nonzero holomorphic sections $s : \mathbb{C}P^1 \to E_k$ if and only if $k \geq 0$, in which case the dimension of the $\mathbb{C}$-vector space of holomorphic sections is $k + 1$.

**d)** To which $E_k$ do the canonical bundle $K_{\mathbb{C}P^1}$ and the universal bundle $U$ correspond?

**3.** Let $A : X \to Y$ be a bounded linear operator between Banach spaces with a bounded right inverse $B : Y \to X$. Prove that there is an $\epsilon > 0$ such that every bounded linear operator $C : X \to Y$ with $\|A - C\| < \epsilon$ also has a bounded right inverse.

**4.** Let $X$,$Y$ and $Z$ be Banach spaces, $D : X \to Z$ a Fredholm map (so it has finite-dimensional kernel, closed image and finite-dimension cokernel) and $A : Y \to Z$ a bounded linear map. Prove that if

$$L := D + A : X \oplus Y \to Z$$

is surjective, then the projection $\Pi : \ker L \to Y$, $\Pi(x, y) = y$ is Fredholm with $\ker \Pi \cong \ker D$ and $\operatorname{coker} \Pi \cong \operatorname{coker} D$.

**5.** Let $X$, $Y$ and $Z$ be Banach spaces, $A : X \to Y$ a bounded linear map and $K : X \to Z$ a compact linear map. Moreover, we assume that there are constants $C_1, C_2 > 0$ such that for all $x \in X$ we have

$$\|x\|_X \leq C_1 \|Ax\|_Y + C_2 \|Kx\|_Z.$$

Prove:

**a)** The kernel of $A$ is a finite-dimensional linear subspace of $X$.
*Hint: This is equivalent to the fact that the unit ball in this subspace is compact.*

**b)** The image of $A$ is closed in $Y$.