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Symplectic Geometry

Problem Set 6

1. Let (W, ξ) be a coorientable contact manifold, i.e. one which admits global contact forms. Prove that a contact vector field Y on W will be the Reeb vector field for some contact form α defining ξ if and only if Y is everywhere transverse to ξ.

2. Let $W \subseteq (M, \omega)$ be a regular level set of the function $H : M \to \mathbb{R}$. Assume that W is also a hypersurface of contact type, so there exists a vector field Y defined near W satisfying $Y \pitchfork W$ and $L_Y \omega = \omega$. We have seen that $\alpha = (\iota_Y \omega)|_W$ is a contact form on W. Prove the assertion made in class that the Reeb vector field of α and the restriction of the Hamiltonian vector field X_H to W are proportional.

3. *(Legendrian submanifolds)* Let (W, ξ) be a contact manifold of dimension $2n+1$.

a) Suppose $S \subseteq W$ is a submanifold everywhere tangent to ξ. Prove that $\dim S \leq n$.
Note: If $\dim S = n$*, then* S *is called a Legendrian submanifold of* W*.*

b) Prove that every Legendrian submanifold $S \subseteq W$ has a neighborhood which is contactomorphic to a neighborhood of the zero section in $(J^1S, \ker(dz - \lambda_{\text{can}})$ (which is obviously an example of a Legendrian submanifold).
Hint: Try to immitate the proof for Lagrangian submanifolds of symplectic manifolds. At some point Gray's stability theorem might be helpful.

4. *(Legendrian knots)* We consider the standard contact structure $\xi = \ker(dz - y dx)$ on $\mathbb{R}^3$.

a) Prove that every smooth curve $\gamma : [0,1] \to \mathbb{R}^2$, $\gamma(t) = (x(t), y(t))$ admits a unique lift $\widetilde{\gamma} : [0,1] \to \mathbb{R}^3$ which starts at $\widetilde{\gamma}(0) = (x(0), y(0), 0)$ and is tangent to ξ.

b) Which closed curves in the plane lift to closed curves in $\mathbb{R}^3$?

c) Compute the lift of $\gamma(t) = (\sin 2\pi t, \sin 4\pi t)$ explicitly and sketch its image in $\mathbb{R}^3$.