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Symplectic Geometry

Problem Set 5

1. Prove that a contact manifold of dimension $2n+1$ with n odd (i.e. of dimension $4m-1$) has a preferred orientation determined by the contact structure.

2. Consider the following three contact forms on $\mathbb{R}^3$:

- $\lambda_1 = dz - ydx$, where (x, y, z) are cartesian coordinates,
- $\lambda_2 = dz + xdy$, where (x, y, z) are cartesian coordinates,
- $\lambda_3 = dz + r^2 d\varphi$, where (r, φ) are polar coordinates in $\mathbb{R}^2$, and z is the third coordinate.

a) Picture these contact structures and their Reeb vector fields (these will be defined on Tuesday).

b) Prove that $(\mathbb{R}^3, \text{Ker } \lambda_i)$ are pairwise contactomorphic, i.e. there are diffeomorphisms $\Phi_{ij} : \mathbb{R}^3 \to \mathbb{R}^3$ and functions $\rho_{ij} : \mathbb{R}^3 \to \mathbb{R}$ such that $\Phi_{ij}^*(\lambda_i) = \rho_{ij}\lambda_j$.

c) Prove that for any $i \in \{1, 2, 3\}$ there is a contactomorphism of $(\mathbb{R}^3, \text{Ker } \lambda_i)$ with a bounded subset $B \subset (\mathbb{R}^3, \text{Ker } \lambda_i)$.

3. Let (M^{2n}, ω) be symplectic and let $W \subset M$ by a smooth hypersurface.

a) Prove that every point $x \in W$ has a neighborhood $U \subset M$ such that $W' = W \cap U$ is a hypersurface of contact type, i.e. there exists a vector field Y defined on a neighborhood $U' \subseteq U$ of W' such that Y is transverse to W' and $L_Y \omega = \omega$.

b) In fact, if U is sufficiently small, the normal bundle of $W' = W \cap U$ is trivial, and one can find such a vector field Y giving the normal bundle either of the two possible orientations.

Please turn!

4. Let (M^{2n}, ω) be symplectic and let $H : M \to \mathbb{R}$ be a function. Suppose $W := H^{-1}(0) \subset M$ is a smooth **closed** oriented hypersurface of contact type, i.e. there is a vector field Y defined near W and transverse to W such that $L_Y\omega = \omega$. As we have seen in class, this means that $\alpha := (\iota(Y)\omega)|_W$ is a contact form on W.

a) Assuming $n > 1$, prove that there is no *closed* 1-form β on W such that $\beta(X_H) > 0$ at all points of W. *Hint: You may want to use Stokes' Theorem.*

b) Use this to prove that, if $n > 1$, any other vector field Z also transverse to W and satisfying $L_Z\omega = \omega$ defines the same normal orientation of W as Y.

c) What happens for $n = 1$?

5. Consider a closed coisotropic submanifold $W \subset (M, \omega)$ of a symplectic manifold, and suppose that $W = H^{-1}(c)$ for a smooth function $H : M \to \mathbb{R}^k$ which has $c \in \mathbb{R}^k$ as a regular value. We denote by $H_1, \ldots, H_k$ the components of H.

a) Prove that the subbundle $\ker \omega|_W \subset TW$ is spanned pointwise by the Hamiltonian vector fields $X_{H_1}, \ldots, X_{H_k}$.

b) Prove that $[X_{H_i}, X_{H_j}]|_W = 0$ for all $1 \le i, j \le k$. By Frobenius' Theorem, this implies that $\ker \omega|_W$ is integrable, i.e. tangent to a foliation.

c) Suppose that the Hamiltonian flows $\varphi^{X_{H_i}}$ are 1-periodic, so that together these flows combine to a free action of the torus T^k on W. Prove that the quotient space W/T^k inherits a symplectic structure from M.

d) Prove that for $W = S^{2n-1} \subset (\mathbb{R}^{2n}, \omega_{\text{st}})$, viewed as a level set of the function $H(z) = \frac{1}{2}\|z\|^2$, part **c)** applies with $k = 1$, and the quotient space is $\mathbb{C}P^{n-1}$.