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Symplectic Geometry

Problem Set 4

1. Give examples of closed submanifolds of $T^4 = \mathbb{R}^4/\mathbb{Z}^4$ which are isotropic or coisotropic or Lagrangian or symplectic with respect to the standard symplectic structure $\omega_{\mathrm{st}} = dx_1 \wedge dy_1 + dx_2 \wedge dy_2$ on T^4! Can you find some that are not tori?

2. Let (M, ω) be a symplectic manifold and $S \subset M$ a closed oriented hypersurface.

a) Prove that $L := TS^{\perp_\omega}$ is a 1-dimensional subbundle of TS which inherits an orientation from S.

b) Prove that if $S = H^{-1}(c)$ for a regular value $c \in \mathbb{R}$ of a function $H : M \to \mathbb{R}$, then the restriction of X_H to S is a section of L.

Any one-dimensional subbundle of the tangent bundle of a manifold S is integrable, i.e. it is tangent to a family of 1-dimensional submanifolds of S. In the situation above, this family consists of the flow lines of X_H as in **b)**. It is called *the characteristic foliation of the hypersurface* $S \subset (M, \omega)$.

c) Describe the subbundle L and the characteristic foliation for

$$S_{a,b} = \{(z_1, z_2) \mid \frac{|z_1|^2}{a^2} + \frac{|z_2|^2}{b^2} = 1\} \subset \mathbb{C}^2 \cong (\mathbb{R}^4, \omega_{\mathrm{st}}),$$

where $a, b > 0$ (Consider the three cases: $a = b$, $\frac{a}{b} \in \mathbb{Q} \setminus \{1\}$ and $\frac{a}{b} \notin \mathbb{Q}$).

d) Conclude that there is no symplectomorphism $\varphi : (\mathbb{R}^4, \omega_{\mathrm{st}}) \to (\mathbb{R}^4, \omega_{\mathrm{st}})$ which maps the standard sphere $S^{2n-1} = S_{1,1}$ onto $S_{a,b}$ for $(a, b) \neq (1, 1)$.

3. *(Lagrangian surgery)*

a) Show that if L_1 and L_2 are two Lagrangian submanifolds passing through $p = (x, y) \in (\mathbb{R}^{2n}, \omega_{\mathrm{st}})$ such that $T_pL_0 \cap T_pL_1 = \{0\}$, there exists a symplectomorphism $\varphi : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ such that for a sufficiently small $\epsilon > 0$ one has $\varphi(L_0 \cup L_1) \cap B(p, \epsilon) = (\mathbb{R}^n \times \{y\} \cup \{x\} \times \mathbb{R}^n) \cap B(p, \epsilon)$.

Please turn!

b) Construct a Lagrangian submanifold $L \subset (\mathbb{R}^{2n}, \omega_{\mathrm{st}})$ diffeomorphic to $\mathbb{R} \times S^{n-1}$, such that

$$L \cap (\mathbb{R}^{2n} \setminus B^{2n}(0,1)) = (\mathbb{R}^n \times \{0\} \cup \{0\} \times \mathbb{R}^n) \cap (\mathbb{R}^{2n} \setminus B^{2n}(0,1)).$$

Together with Darboux' theorem, **a)** *and* **b)** *show that one can form the connected sum of two Lagrangian submanifolds which intersect transversely at one point such that the result is a new Lagrangian submanifold.*

c)* Formulate and prove a similar result for Lagrangian submanifolds that intersect cleanly, i.e. such that $L_1 \cap L_2$ is a submanifold and for each point $x \in L_1 \cap L_2$ one has $T_x L_1 \cap T_x L_2 = T_x(L_1 \cap L_2)$.

4. We consider the two Lagrangian submanifolds of $(\mathbb{R}^{2n}, \omega_{\mathrm{st}})$ that were discussed in the lecture.

a) Verify directly that for $n \geq 1$ and any fixed $0 < \epsilon < 1$ the map

$$\begin{aligned} \varphi_0 : S^1 \times S^{n-1} &\to \mathbb{C}^n \cong \mathbb{R}^{2n} \\ (e^{it}, x) &\mapsto (1 + \epsilon e^{it}) \cdot x \end{aligned}$$

is a Lagrangian embedding, and compute the Maslov index of the loop $\gamma_0 : \mathbb{R}/\mathbb{Z} \to L$, given by $\gamma_0(t) = \varphi_0(e^{2\pi i t}, (1, 0, \ldots, 0))$.

b) For $n \geq 2$, consider the Lagrangian submanifold $L \subset \mathbb{R}^{2n}$ given as the image of the immersion

$$\begin{aligned} \varphi_1 : S^1 \times S^{n-1} &\to \mathbb{C}^n \cong \mathbb{R}^{2n} \\ (\lambda, x) &\mapsto \lambda \cdot x, \end{aligned}$$

and compute the Maslov index of the loop $\gamma_1 : \mathbb{R}/\mathbb{Z} \to L$, given by $\gamma_1(t) = \varphi_1(e^{i\pi t}, (\cos(\pi t), \sin(\pi t), 0, \ldots, 0))$.