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Symplectic Geometry

Problem Set 2

1. In a finite dimensional real vector space V, any euclidean inner product g determines an open *ellipsoid* via

$$E_g = \{v \in V \,:\, g(v,v) < 1\}.$$

a) Let g be a euclidean inner product on $\mathbb{R}^2$, and let $\omega_{\mathrm{st}} = e^* \wedge f^*$ be the standard symplectic form. Prove that one can find a new symplectic basis $\{e', f'\}$ such that $e' \perp_g f'$ and $\|e'\|_g = \|f'\|_g$.

b) Prove that for any ellipsoid $E \subset (\mathbb{R}^{2n}, \omega_{\mathrm{st}})$ there exists a symplectic linear map $\varphi : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ such that $\varphi(E)$ is a *standard symplectic ellipsoid*, meaning it is of the form

$$E(r_1, \ldots, r_n) := \{(z_1, \ldots, z_n) \in \mathbb{C}^n \cong \mathbb{R}^{2n} \,:\, \sum_j \frac{|z_j|^2}{r_j^2} < 1\}.$$

Here the numbers $0 < r_1 \leq r_2 \leq \cdots \leq r_n$ are uniquely determined by E.

2. Prove that given two Lagrangian subspaces $L_0, L_1 \subset (V, \omega)$ of a symplectic vector space such that $L_0 \cap L_1 = \{0\}$ (i.e. L_0 and L_1 are transverse), there exists a symplectic basis $e_1, f_1, \ldots e_n, f_n$ such that $L_0 = \mathrm{span}(e_1, \ldots, e_n)$ and $L_1 = \mathrm{span}(f_1, \ldots, f_n)$.

3. We denote by $\mathcal{L}(n)$ the space of Lagrangian subspaces of $(\mathbb{R}^{2n}, \omega_{\mathrm{st}})$

a) Prove that the loop $\Psi : \mathbb{R}/\mathbb{Z} \to \mathrm{Sp}(4, \mathbb{R})$ defined in the lecture as

$$\Psi(t) := e^{\pi i t} \begin{pmatrix} \cos(\pi t) & -\sin(\pi t) \\ \sin(\pi t) & \cos(\pi t) \end{pmatrix} \in U(2) \subset \mathrm{Sp}(4, \mathbb{R})$$

has Maslov index 1.

Please turn!

b) Prove that with $\Lambda_0(t) = e^{\pi i t} \cdot \mathbb{R} \in \mathcal{L}(1)$ and $\Lambda(t) := \Lambda_0(t) \oplus \Lambda_0(t) \in \mathcal{L}(2)$ we have

$$\Lambda(t) = \Psi(t) \cdot (\mathbb{R}^2 \oplus \{0\}).$$

As discussed in the lecture, this proves that $\mu(\Lambda_0) = 1$.

c) Prove that the Maslov index for Lagrangian subspaces is characterized uniquely by the (homotopy), (product), (direct sum) and (zero) axioms.

d) Prove that the Maslov index for Lagrangian loops has the concatenation property: If Λ_1 and Λ_2 are two loops in $\mathcal{L}(n)$ with $\Lambda_1(0) = \Lambda_2(0)$, then

$$\mu(\Lambda_1 \star \Lambda_2) = \mu(\Lambda_1) + \mu(\Lambda_2).$$

e) The space $\mathcal{L}^{\mathrm{or}}(n)$ of oriented Lagrangian subspaces of $(\mathbb{R}^{2n}, \omega_{\mathrm{st}})$ is a double cover (two-sheeted covering space) of $\mathcal{L}(n)$. Prove that if $p : \mathcal{L}^{\mathrm{or}}(n) \to \mathcal{L}(n)$ is the covering projection map, then $p_*(\pi_1(\mathcal{L}^{\mathrm{or}}(n)) = 2\mathbb{Z} \subset \mathbb{Z} \cong \pi_1(\mathcal{L}(n))$. In other words: the Maslov index of a loop of oriented Lagrangian subspaces is even.

4. This exercise assumes some familiarity with techniques in differential topology. Still, it is included to give an idea of an alternative approach to the Maslov index for loops of symplectic matrices. Throughout, we consider $(\mathbb{R}^{2n}, \omega_{\mathrm{st}})$ with coordinates $(x_1, \ldots, x_n, y_1, \ldots y_n)$.

a) Prove that a matrix $\Psi = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{GL}(2n, \mathbb{R})$ is symplectic if and only if

$$C^T A = A^T C, \quad D^T B = B^T D \quad \text{and} \quad A^T D - C^T B = \mathbb{1}.$$

Now we define the (noncompact) *Maslov cycle* $\Delta \subseteq \mathrm{Sp}(2n, \mathbb{R})$ by

$$\begin{aligned} \Delta &:= \{\Psi = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}(2n, \mathbb{R}) \mid \det(B) = 0\} \\ &= \Delta_1 \sqcup \Delta_2 \sqcup \cdots \sqcup \Delta_n, \end{aligned}$$

where $\Delta_i = \{\Psi \mid \mathrm{rank}(B) = n - i\}$ has closure $\overline{\Delta_i} = \cup_{j \geq i} \Delta_j$.

b) For $n = 1$, describe $\Delta = \Delta_1 \subseteq \mathrm{Sp}(2, \mathbb{R})$ explicitly, and prove that its complement consists of two regions, both diffeomorphic to $\mathbb{R}^3$.

c) Prove that every loop in $\mathrm{Sp}(2, \mathbb{R})$ is homotopic to a new loop Ψ such that $\Psi \cap \Delta$ consists of finitely many transverse intersections. We call such a loop *regular*, and the corresponding transverse intersection points $\Psi \cap \Delta$ *regular crossings*.

See next page!

d) For a loop Ψ in $\mathrm{Sp}(2,\mathbb{R})$, we call a regular crossing $\Psi(t)$ positive if $-d(t)b'(t) > 0$, and negative otherwise. Prove that for a regular loop Ψ in $\mathrm{Sp}(2,\mathbb{R})$ its Maslov index $\mu(\Psi)$ is given by

$$\mu(\Psi) = \frac{1}{2}(\#(\text{positive crossings}) - \#(\text{negative crossings})),$$

where $\#$ denotes the count.

Hint: Check that this gives the right answer on the standard loops

$$\Psi_k(t) = \begin{pmatrix} \cos 2k\pi t & -\sin 2k\pi t \\ \sin 2k\pi t & \cos 2k\pi t \end{pmatrix}$$

and then use homotopy invariance of the right hand side.

For more information, including a discussion of the higher dimensional case and the Lagrangian version, see the book "Introduction to symplectic topology" by McDuff and Salamon or the article by Robbin and Salamon, "The Maslov index for paths", Topology 32 (1993), 827–844.