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Winter 2019/20

Symplectic Geometry

Problem Set 11

1. **a)** Find a sequence $\{u_k\}$ of holomorphic maps

$$u_k : \mathbb{C}P^1 \to \mathbb{C}P^1 \times \mathbb{C}P^1$$

with the following properties:

- the compositions of each u_k with each of the two projections to $\mathbb{C}P^1$ have degree 1 (i.e. they are bijective),
- the images of all the maps u_k contain a fixed point $([w_0 : w_1], [w_0' : w_1']) \in \mathbb{C}P^1 \times \mathbb{C}P^1$, and
- the images converge in the Hausdorff sense to a subset of $\mathbb{C}P^1 \times \mathbb{C}P^1$ of the form $\mathbb{C}P^1 \times \{p\} \cup \{q\} \times \mathbb{C}P^1$.

b) Now find Möbius transformations $\varphi_k : \mathbb{C}P^1 \to \mathbb{C}P^1$ and $\psi_k : \mathbb{C}P^1 \to \mathbb{C}P^1$ so that the maps $v_k = u_k \circ \varphi_k$ converge to a holomorphic parametrization of $\mathbb{C}P^1 \times \{p\}$ and the maps $\overline{v}_k = u_k \circ \psi_k$ converge to a holomorphic parametrization of $\{q\} \times \mathbb{C}P^1$, both in C^∞_{loc} of the complement of suitable points in $\mathbb{C}P^1$.

2. Prove the following statement due to Hofer: Let (X, d) be a metric space, and $f : X \to [0, \infty)$ a continuous function. Fix $x \in X$ and $\delta > 0$, and assume that the closed ball $\overline{B(x, 2\delta)}$ is complete. Then there exists $y \in X$ and a positive number $0 < \epsilon \leq \delta$ such that

$$d(x, y) < 2\delta, \quad \sup_{B(y,\epsilon)} f \leq 2f(y) \quad \text{and} \quad \epsilon f(y) \geq \delta f(x).$$

Remark: In the lecture, we found one bubble in a sequence of J-holomorphic curves whose domain is a closed surface by rescaling at the sequence of maxima of the norm of the derivative. To find all bubbles, or more generally in other situations where the domain curve is not compact, one instead starts with sequences $\{x_k\}$ of points and $\delta_k \to 0$ with $\delta_k \|du_k(x_k)\| \to \infty$ and then uses the above result for each $f_k(x) = \|du_k(x)\|$ to find new centers y_k and numbers $0 < \epsilon_k < \delta_k$ for which the rescaled maps will have uniform gradient bounds.