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Symplectic Geometry

Problem Set 10

1. Let $B \subseteq \mathbb{C}$ denote the unit ball, and consider a smooth family $J : B \to \mathrm{End}(\mathbb{R}^{2n}) = \mathrm{End}_{\mathbb{R}}(\mathbb{C}^n)$ of almost complex structures on $\mathbb{R}^{2n}$ parametrized by $z \in B$. Also fix $A \in L^\infty(B, \mathrm{End}_{\mathbb{R}}(\mathbb{C}^n))$ and $p > 2$. Now adapt the proof of the Carleman similarity principle to show that for every solution $u \in W^{1,p}(B, \mathbb{C}^n)$ of

$$\frac{1}{2}(\partial_s u(z) + J(u)\partial_t u(z)) + A(z)u(z) = 0, \qquad u(0) = 0 \tag{1}$$

there exists $\epsilon > 0$ and maps $\Phi \in W^{1,p}(B_\epsilon, \mathrm{End}_{\mathbb{R}}(\mathbb{C}^n))$ and $f \in C^\infty(B_\epsilon, \mathbb{C}^n)$ such that

$$\Phi(0) = \mathrm{id}_{\mathbb{C}^n}, \quad \overline{\partial} f = 0 \quad \text{and } u(z) = \Phi(z) f(z).$$

Hint: Use a smooth family $\Psi : B \to \mathrm{End}_{\mathbb{R}}(\mathbb{C}^n)$ with $\Psi(z)J(z) = i\Psi(z)$ and consider the equation satisfied by the map $v \in W^{1,p}(B, \mathbb{C}^n)$ which is defined via $u(z) = \psi(z)v(z)$.

2. Use the result of the previous exercise to prove the following statements:

a) For every nonconstant solution $u \in W^{1,p}(B, \mathbb{C}^n)$ of (1) there exists some $\delta > 0$ such that $u(z) \neq 0$ for all $z \in B_\delta \setminus \{0\}$.

b) For every nonconstant solution $u \in C^\infty(B, \mathbb{C}^n)$ of (1) with $A = 0$ (i.e. a J-holomorphic disk) there exists some $\delta > 0$ such that $du(z) \neq 0$ for all $z \in B_\delta \setminus \{0\}$.
Hint: Derive the equation with respect to s to obtain an equation of the same type for $v = \partial_s u$.

Please turn!

3. *This exercise discusses facts that are useful in the discussion of compactness of moduli spaces of J-holomorphic spheres.*

We consider the group $\mathrm{PSL}(2,\mathbb{C})$ of conformal automorphisms of the sphere $S^2 = \mathbb{C} \cup \{\infty\}$. It's elements are Möbius transformations, which can be written as

$$\varphi(z) = \frac{az+b}{cz+d} \qquad \text{where } a,b,c,d \in \mathbb{C},\, ad-bc = 1.$$

Prove the following facts:

a) Every Möbius transformation is uniquely determined by its values at any three distinct points $z_1, z_2, z_3 \in S^2$, and any triple of distinct points can be mapped to any other triple of distinct points.

b) With respect to the Fubini-Study metric $g_{FS} = \frac{1}{1+|z|^2} g_{st}$ on $S^2 = \mathbb{C} \cup \{\infty\}$ the norm of the differential of a Möbius transformation φ satisfies

$$\|d\varphi(z)\| = \sqrt{2}|\varphi'(z)|\frac{1+|z|^2}{1+|\varphi(z)|^2} = \sqrt{2}\frac{1+|z|^2}{|az-b|^2+|cz+d|^2}.$$

c) If φ_k is a sequence of Möbius transformations such that

$$\sup_k \sup_{z \in S^2} \|d\varphi_k(z)\| < \infty,$$

then there exists a subsequence φ_{k_n} which converges on all of S^2 uniformly with all derivatives to some $\varphi \in \mathrm{PSL}(2,\mathbb{C})$.

d) If instead φ_k is a sequence of Möbius transformations which does not admit such a uniformly convergent subsequence, then there are points $x, y \in S^2$ and a subsequence φ_{k_n} which converges on any compact subset of $S^2 \setminus \{x\}$ to the constant map $\varphi(z) = y$.