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Differential Topology

Problem Set 9

1. Prove that the manifolds $G_{k,N}$ and $G_{N-k,N}$ are diffeomorphic.

2. Let $E \longrightarrow B$ be a vector bundle of rank k over a manifold of dimension n.

a) Prove that if $k > n$, then there exists a bundle $E' \longrightarrow B$ of rank $k-1$ such that

$$E \cong E' \oplus \underline{\mathbb{R}}.$$

b) Show by example that in general E' is not uniquely determined by E.

c) Prove that if $k > n+1$, then E' is uniquely determined by E.

3. We consider real vector bundles $E \longrightarrow S^n$ for $n \geq 1$. Any such vector bundle is trivial over the upper and lower hemispheres

$$S^n_\pm := \{x \in S^n \mid \pm x_{n+1} \geq 0\} \subset S^n \subset \mathbb{R}^{n+1}.$$

In particular, it is completely described by the transition map

$$g : S^{n-1} = S^n_+ \cap S^n_- \to \mathrm{GL}(k, \mathbb{R})$$

between the two trivializations.

a) Using the fact that every vector bundle over a compact manifold admits a Riemannian metric, prove that the transition map can be arranged to take values in the subgroup $\mathrm{O}(k) \subset \mathrm{GL}(k, \mathbb{R})$, and even in $\mathrm{SO}(k)$ for $n \geq 2$.

b) Let $n \geq 2$. Prove that the transition maps g and g' for two bundles E and E' of the same rank are homotopic if and only if the bundles are isomorphic. In particular, for any $k \geq 1$ we have

$$\mathrm{Vect}_k(S^n) \cong \mathrm{Vect}_k^{\mathrm{or}}(S^n) \cong \pi_{n-1}(\mathrm{SO}(k)).$$

c) Describe a bijective map from $\mathrm{Vect}_2(S^2)$ to $\mathbb{Z}$.

d) Which integer does the tangent bundle of S^2 correspond to under your map? Give a pictorial proof of your answer.