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Differential Topology

Problem Set 8

1. Prove that if M is oriented and S_1 and S_2 are oriented closed submanifolds of complementary dimension, then the intersection numbers (defined using the coorientations induced from the orientations) satisfy

$$S_2 \bullet S_1 = (-1)^{\dim S_1 \cdot \dim S_2} S_1 \bullet S_2.$$

Use a similar argument to prove that the Euler characteristic of an odd dimensional closed manifold vanishes.

2. Let M be a compact manifold and $f : M \to M$ a smooth map. The Lefschetz number $L(f)$ of f is defined to be the intersection number (defined in $\mathbb{Z}$ if M is oriented or in $\mathbb{Z}_2$ otherwise)

$$L(f) := \Delta \bullet \operatorname{graph} f$$

of the diagonal $\Delta \subset M \times M$ and $\operatorname{graph} f = \{(x, f(x)) \in M \times M \mid x \in M\}$.

a) Prove that if $L(f)$ is not zero, then f has a fixed point.

b) If f is homotopic to the identity, then $L(f) = \chi(M)$.

c) Show that $\operatorname{graph} f \pitchfork \Delta$ if and only if at every fixed point $x \in M$ for f the differential $f_{*,x} : T_xM \to T_xM$ does not have 1 as an eigenvalue.

d) Prove that in this case the contribution of the fixed point $x \in M$ of f to the Lefschetz number $L(f)$ is $\operatorname{sgn}\det(f_{*,x} - \mathbb{1})$, where $\mathbb{1} : T_xM \to T_xM$ is the identity map.

3. In class we discussed examples of phase portraits of vector fields on the plane near an isolated zero of index $+1$ or -1. Draw phase portraits for other integer degrees - enough to convince yourself that all integers can occur. Can you also give explicit formulas?

4. In the lecture we defined the Grassmannian manifolds $G_{k,n}$ and the canonical bundles $\Gamma_{k,n} \to G_{k,n}$. Observe that $G_{1,n+1} \cong \mathbb{R}P^n$.

a) Prove directly that the universal bundle $\Gamma_{1,n+1} \to G_{1,n+1} \cong \mathbb{R}P^n$ is not a trivial bundle of rank 1.

b) Prove that $\Gamma_{1,n}$ is isomorphic to the normal bundle of the standard embedding $\mathbb{R}P^n \subseteq \mathbb{R}P^{n+1}$ induced from the inclusion $\mathbb{R}^{n+1} \times \{0\} \subseteq \mathbb{R}^{n+2}$.