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Differential Topology

Problem Set 5

1. Use Sard's theorem or Sard's theorem for families to prove precise versions of each the following statements:

a) Most pairs of lines in $\mathbb{R}^n$ with $n \geq 3$ do not intersect.

b) If $f : \mathbb{R} \to \mathbb{R}$ is a C^1 function, then most horizontal lines in $\mathbb{R}^3$ are not tangent to its graph.

2. Suppose $f_1 : M_1 \to N$ and $f_2 : M_2 \to N$ are smooth maps between manifolds. Recall that we defined transversality for such maps as

$$f_1 \pitchfork f_2 :\Longleftrightarrow \quad \begin{array}{c} \text{for all } x \in M_1, y \in M_2 \text{ with } f_1(x) = f_2(y) \text{ we have} \\ (f_1)_* T_x M_1 + (f_2)_+ (T_y M_2) = T_z N \quad \text{where } z = f_1(x) = f_2(y). \end{array}$$

Prove that $f_1 \pitchfork f_2$ in this sense if and only if the map

$$F = f_1 \times f_2 : M_1 \times M_2 \to N \times N \quad , \quad F(x,y) := (f_1(x), f_2(y))$$

is transverse to the diagonal

$$\Delta := \{(z,z) \in N \times N \mid z \in N\}.$$

So in fact transversality for maps is not more general than transversality of one map to a submanifold of the target, and all statements about the latter have reformulations for the former.

3. **a)** Prove that Brouwer's Theorem is false for the open ball.

b) Find a map of the solid torus in $\mathbb{R}^3$ to itself with no fixed points. Where does our proof of Brouwer's theorem fail in this situation?

c) Use Brouwer's theorem to prove the following result of Frobenius: If $A \in \mathrm{Mat}(n, \mathbb{R})$ is a real $n \times n$ matrix with all entries nonnegative, then it must have a real eigenvalue $\lambda \geq 0$. *Hint: You can assume w.l.o.g. that* $\det A \neq 0$ *(why?). Under this additional assumption, can you use* A *to define a map* $S^{n-1} \to S^{n-1}$ *which preserves the intersection of* S^{n-1} *with the closed first quadrant?*