Universität Hamburg
Janko Latschev

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Differential Topology

Problem Set 11

1. **a)** Compute (e.g. inductively using the Mayer-Vietoris sequence) the de Rham cohomology of the complement of k points in $\mathbb{R}^n$ for any $k \geq 1$.

b) A closed oriented surface Σ_g of genus g can be obtained from S^2 by removing $2g$ disks and gluing the resulting boundary circles to each other in pairs. Alternatively, one can also just remove the centers of the disks and glue cylinders $[-1,1] \times S^1$ to a pair of the resulting annuli (disk $\setminus$ center point). Can you use this to compute the de Rham cohomology of Σ_g?

2. **a)** Compute the de Rham cohomology groups of $X := \mathbb{R}^3 \setminus A$, where A is the union of two coordinate lines.

b) Compute the de Rham cohomology groups of $Y := \mathbb{R}^4 \setminus B$, where B is the union of two coordinate planes intersecting transversely, i.e. only in $\{0\}$.

c) Are X and Y homotopy equivalent?

3. Prove that every simply connected manifold M satisfies $H^1(M) \cong 0$.

4. Use Poincaré duality in the noncompact case to compute $H_c^p(\mathbb{R}^n \setminus \{0\})$ for all $p \geq 0$, and find explicit differential forms whose cohomology classes make up a basis for $H_c^*(\mathbb{R}^n \setminus \{0\})$.
Hint: If you find the second part hard, consider at least the special case $n = 2$.

5. **a)** Prove the 5-Lemma stated in the lecture:
If the commuting diagram

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "f_A"'] & B \arrow[r] \arrow[d, "f_B"'] & C \arrow[r] \arrow[d, "f_C"'] & D \arrow[r] \arrow[d, "f_D"'] & E \arrow[d, "f_E"'] \\
A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & D' \arrow[r] & E'
\end{tikzcd}$$

has exact rows and f_A, f_B, f_D and f_E are isomorphisms, then f_C is also an isomorphism.

b) Can you weaken the assumptions and still obtain the same conclusion?