Universität Hamburg
Janko Latschev Winter 2015/16

Symplectic Homology

Problem Set 3

In the following exercises, J_0 denotes the standard complex structure on $\mathbb{R}^{2n} \cong \mathbb{C}^n$.

1. a) Let $S : [0,1] \to Mat(2n, \mathbb{R})$ be a path of symmetric matrices, and let $R : [0,1] \to Mat(2n, \mathbb{R})$ be the solution of the initial value problem

$$\frac{d}{dt} R(t) = J_0 S(t) R(t), \qquad R(0) = \mathrm{id}\,.$$

Prove that R is a path of symplectic matrices.

b) Prove conversely that if $R : [0,1] \to Sp(2n)$ is a continuously differentiable path, then the path $S : [0,1] \to Mat(2n, \mathbb{R})$ defined as

$$S(t) = -J_0 R'(t) R(t)^{-1}$$

consists of symmetric matrices.

c) Compute the path $A_k(t) = e^{J_0 S_k t}$ for each of the following matrices

$$S_1 = \begin{pmatrix} \pi & 0 \\ 0 & \pi \end{pmatrix}, \qquad S_2 = \begin{pmatrix} -\pi & 0 \\ 0 & \pi \end{pmatrix}, \qquad S_3 = \begin{pmatrix} -\pi & 0 \\ 0 & -\pi \end{pmatrix},$$

and compute the Conley-Zehnder indices of these paths.

2. Prove the following properties of the Conley-Zehnder index that were claimed in the lecture:

a) If A and B are homotopic through paths in $SP(2n)$, then $\mu_{CZ}(A) = \mu_{CZ}(B)$.

b) If $A \in SP(2n)$ and $\Phi : [0,1] \to Sp(2n)$ is a loop based at id, then

$$\mu_{CZ}(\Phi A) = \mu_{CZ}(A) + 2\mu(\Phi),$$

where $\mu(\Phi)$ is the usual Maslov index of that loop.

c) If $S \in Mat(2n, \mathbb{R})$ is a nondegenerate symmetric matrix with $\|S\| < 2\pi$, and $A(t) = e^{J_0 S t}$, then $\mu_{CZ}(A) = \frac{1}{2}\,\mathrm{sign}(S)$, where $\mathrm{sign}(S)$ is the difference between the number of positive and the number of negative eigenvalues.

Bitte wenden!

d) If $A \in SP(2n)$ and $B \in Sp(2n)$ then

$$\mu_{CZ}(BAB^{-1}) = \mu_{CZ}(A).$$

e) If $A \in SP(2n)$ and for $t > 0$ the matrices $A(t)$ have no eigenvalues on S^1, then $\mu_{CZ}(A) = 0$.

f) If $A \in SP(2n)$ then

$$\mu_{CZ}(A^{-1}) = \mu_{CZ}(A^T) = -\mu_{CZ}(A).$$

3. Consider the paths of matrices

$$A(t) = \begin{pmatrix} 1 + 4\pi^2 t^2 & 2\pi t \\ 2\pi t & 1 \end{pmatrix} \qquad \text{and} \qquad B(t) = \begin{pmatrix} 1 - 4\pi^2 t^2 & -2\pi t \\ 2\pi t & 1 \end{pmatrix}.$$

a) Prove that A and B are in $SP(2)$.

b) Compute the Conley-Zehnder indices for both paths.