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Symplectic Homology

Problem Set 1

1. (Transversality) Suppose M is a d-dimensional manifold without boundary.

a) Prove that the transverse intersection of two submanifolds is again a submanifold, whose codimension is the sum of the codimensions of the two original submanifolds.

b) Prove that if the first submanifold is oriented and the second one is cooriented (i.e. it has an oriented normal bundle), then the intersection comes with a preferred orientation.

What would be the analogue of these statements for manifolds with boundary?

2. (Morse homology over $\mathbb{Z}$) Let a closed manifold M, a Morse function $f : M \to \mathbb{R}$ and a metric g on M be given. Then each critical point p of f has stable and unstable manifolds $W^s(p)$ and $W^u(p)$ with respect to the gradient flow of f, and we assume that (f, g) is a Morse-Smale pair. We *choose orientations* for the unstable manifolds $W^u(p)$, which *automatically give us coorientations* for the stable manifolds $W^s(p)$ (why?).

a) Explain how these choices, together with the standard orientation of $\mathbb{R}$, give rise to orientations of the moduli spaces of trajectories

$$\mathcal{L}(q, p) = (W^u(q) \cap W^s(p))/\mathbb{R}.$$

b) What does this mean when $\operatorname{ind} q - \operatorname{ind} p = 1$ (so that $\dim \mathcal{L}(q, p) = 0$)?

c) Prove that the conventions can be set up so that if $\operatorname{ind} q - \operatorname{ind} p = 2$ (which implies that $\mathcal{L}(q, p)$ is 1-dimensional), the boundary of $\mathcal{L}(q, p)$ equals

$$\coprod_{\operatorname{ind} p < \operatorname{ind} r < \operatorname{ind} q} \mathcal{L}(q, r) \times \mathcal{L}(r, p)$$

as oriented manifolds.

d) Conclude that the Morse complex of f can be defined over $\mathbb{Z}$.

Bitte wenden!

3. (Perfect Morse function) Consider the function $f : \mathbb{C}P^n \to \mathbb{R}$ given in homogeneous coordinates $[z_0 : \ldots : z_n]$ on $\mathbb{C}P^n$ as

$$f([z_0 : \ldots : z_n]) := \frac{1}{\|z\|^2} \sum_{k=0}^{n} k \cdot |z_k|^2.$$

Prove that f is a Morse function, and describe its Morse complex with respect to the standard metric on $\mathbb{C}P^n$. What happens when you restrict f to $\mathbb{R}P^n$?

4. (Energy and Asymptotics) Let $f : M \to \mathbb{R}$ be a Morse function on a closed manifold M, and let g be *any* metric on M.

a) Prove that the energy

$$E(\gamma) := \int_{\mathbb{R}} |\dot{\gamma}(t)|_g^2 \, dt$$

is uniformly bounded for all gradient flow lines, i.e. solutions $\gamma : \mathbb{R} \to M$ of $\dot{\gamma}(t) = \operatorname{grad} f(\gamma(t))$.

b) Deduce that there are critical points $x_\pm \in \operatorname{Crit}(f)$ of f such that

$$\lim_{t\to\infty} \gamma(t) = x_+ \quad \text{and} \quad \lim_{t\to-\infty} \gamma(t) = x_-.$$

c) Prove that in fact there are constants $C_\pm, \delta_\pm > 0$ such that for all $t \in \mathbb{R}$ we have

$$d(x_+, \gamma(t)) \leq C_+ e^{-\delta_+ t} \text{and} \quad d(x_-, \gamma(t)) \leq C_- e^{\delta_- t}.$$