Analysis on (almost) complex manifolds

Almost complex manifolds

Suppose (M, J) is an almost complex manifold. For $p \in M$ we consider the complexified tangent space $T_pM \otimes \mathbb{C}$, which is a complex vector space of dimension $\dim M$ and a real vector space of dimension $2\dim M$. The real linear map J extends to a complex linear map J on this space, and since $J^2 = -\mathbf{Id}$ this map has eigenvalues $\pm i$. We denote the corresponding eigenspaces as[1]

$$T_{1,0} := \{w \in T_pM \otimes \mathbb{C} \mid Jw = iw\} \subset T_pM \otimes \mathbb{C}$$
$$T_{0,1} := \{w \in T_pM \otimes \mathbb{C} \mid Jw = -iw\} \subset T_pM \otimes \mathbb{C}$$

These are naturally complex subspaces of the complex vector space $T_pM \otimes \mathbb{C}$, where the complex structure is given by multiplication by i.

Exercise 1. Prove that the map $T_pM \to T_pM \otimes \mathbb{C}$ given by $v \mapsto v - iJv$ is an isomorphism of the complex vector spaces $(T_pM, J) \cong (T_{1,0}, i)$. Similarly $v \mapsto v + iJv$ is an isomorphism between the complex vector spaces (T_pM, J) and $(T_{0,1}, -i) =: \overline{T_{0,1}}$.

We have $T_pM \otimes \mathbb{C} \cong T_{1,0} \oplus T_{0,1}$ and denote the projections by

$$\pi_{1,0} : T_pM \otimes \mathbb{C} \to T_{1,0} \quad \text{and} \qquad \pi_{0,1} : T_pM \otimes \mathbb{C} \to T_{0,1}.$$

An analogous discussion applies to the complexified cotangent space $T_p^*M \otimes \mathbb{C}$. We have $T_p^*M \otimes \mathbb{C} \cong T_p^{1,0} \oplus T_p^{0,1}$ where

$$T_p^{1,0} := \{\phi \in T_p^*M \otimes \mathbb{C} \mid \phi \circ J = i\phi\} \subset T_p^*M \otimes \mathbb{C}$$
$$T_p^{0,1} := \{\phi \in T_p^*M \otimes \mathbb{C} \mid \phi \circ J = -i\phi\} \subset T_p^*M \otimes \mathbb{C}.$$

The corresponding projections are denoted by

$$\pi^{1,0} : T_p^*M \otimes \mathbb{C} \to T_p^{1,0} \quad \text{and} \qquad \pi^{0,1} : T_p^*M \otimes \mathbb{C} \to T_p^{0,1}.$$

As above, we have isomorphisms of complex vector spaces

$$T_p^{1,0} \cong (T_p^*M, J) \cong \overline{T_p^{0,1}}.$$

where the composition of the isomorphisms is literally complex conjugation, sending a given element $\phi - i\phi \circ J \in T_p^{1,0}$ to $\phi + i\phi \circ J \in \overline{T_p^{0,1}} = (T_p^{0,1}, -i)$.

[1] To avoid cumbersome notation, we leave off the base point here.

As this splitting of $T_p^*M \otimes \mathbb{C}$ can be done at each point, we get a global splitting of the complexified cotangent bundle as the direct sum of two subbundles

$$T^*M \otimes \mathbb{C} \cong T^{1,0} \oplus T^{0,1}.$$

The exterior powers of the complexified cotangent bundle split accordingly as

$$\Lambda^k(T^*M) \otimes \mathbb{C} \cong \bigoplus_{p+q=k} \underbrace{\Lambda^p T^{1,0} \wedge \Lambda^q T^{0,1}}_{=:\Lambda^{p,q}T^*M}.$$

Complex valued differential forms of degree k are smooth sections of the bundle $\Lambda^k(T^*M) \otimes \mathbb{C}$, so they can be split according to type as well. More precisely, we define $\Omega^{p,q}(M)$ to be the space of smooth sections of $\Lambda^{p,q}(T^*M)$, and obtain

$$\Omega^k(M, \mathbb{C}) = \bigoplus_{p+q=k} \Omega^{p,q}(M).$$

With respect to this splitting, we have projections

$$\pi^{p,q} : \Omega^k(M, \mathbb{C}) \to \Omega^{p,q}(M).$$

We would now like to understand the behavior of the exterior differential $d : \Omega^k(M, \mathbb{C}) \to \Omega^{k+1}(M, \mathbb{C})$ with respect to the above splitting. Since d is a derivation, it is essential to understand it on functions and 1-forms. For functions $C^\infty(M, \mathbb{C}) = \Omega^0(M, \mathbb{C})$ we of course have

$$d : \Omega^0(M, \mathbb{C}) \to \Omega^1(M, \mathbb{C}) \cong \Omega^{1,0}(M) \oplus \Omega^{0,1}(M),$$

and we *define* the maps

$$\begin{aligned} \partial &: \Omega^0(M, \mathbb{C}) \to \Omega^{1,0}(M) \quad , \quad \partial := \pi^{1,0} \circ d \quad \text{and} \\ \overline{\partial} &: \Omega^0(M, \mathbb{C}) \to \Omega^{0,1}(M) \quad , \quad \overline{\partial} := \pi^{0,1} \circ d. \end{aligned}$$

Over a contractible open subset $U \subset M$ we can trivialize the bundles $\Lambda^{1,0}T^*M$ and $\Lambda^{0,1}T^*M$, and so we can locally find $n = \dim_{\mathbb{C}} M$ sections $\alpha_1, \ldots, \alpha_n$ of $\Omega^{1,0}(U)$. Their complex conjugates $\overline{\alpha}_1, \ldots, \overline{\alpha}_n$ will be sections of $\Omega^{0,1}(U)$ such that any complex valued 1-form $\phi \in \Omega^1(U, \mathbb{C})$ can be written as

$$\phi = \sum_{k=1}^n f_k \alpha_k + \sum_{\ell=1}^n g_\ell \overline{\alpha}_\ell$$

with smooth complex valued functions $f_k : U \to \mathbb{C}$ and $g_\ell : U \to \mathbb{C}$. More generally, any form $\eta \in \Omega^{p,q}(U)$ can be written as

$$\eta = \sum_{|K|=p, |L|=q} \eta_{K,L} \alpha_K \wedge \overline{\alpha}_L, \tag{1}$$

with the obvious multiindex notation. Now

$$\Omega^2(M, \mathbb{C}) \cong \Omega^{2,0}(M) \oplus \Omega^{1,1}(M) \oplus \Omega^{0,2}(M),$$

so the exterior differential of one of the forms α_k will generally have the three components

$$\pi^{2,0}(d\alpha_k) \in \Omega^{2,0}(M) \quad , \quad \pi^{1,1}(d\alpha_k) \in \Omega^{1,1}(M) \quad \text{and} \quad \pi^{0,2}(d\alpha_k) \in \Omega^{0,2}(M),$$

and similarly for their complex conjugates. Applying the derivation property $d(\eta_1 \wedge \eta_2) = (d\eta_1) \wedge \eta_2 + (-1)^{\deg \eta_1} \eta_1 \wedge (d\eta_2)$ to a general (p,q)-form η written as in (1) one finds that on a general almost complex manifold (M,J) the exterior differential d is a direct sum of maps

$$d : \Omega^{p,q}(M) \to \Omega^{p-1,q+2}(M) \oplus \Omega^{p,q+1}(M) \oplus \Omega^{p+1,q}(M) \oplus \Omega^{p+2,q-1}(M), \qquad (2)$$

and typically all four components can and will be nonzero.

Complex manifolds

On a *complex* manifold (M,J), the situation simplifies drastically. Indeed, on a sufficiently small open subset $U \subset M$ we can now find *complex coordinates* $z_k = x_k + iy_k$ such that J becomes standard in these coordinates, meaning that

$$J(\partial_{x_k}) = \partial_{y_k} \quad \text{and} \quad J(\partial_{y_k}) = -\partial_{x_k},$$

so that

$$dx_k \circ J = -dy_k \quad \text{and} \quad dy_k \circ J = dx_k.$$

It follows that we can choose as our local basis of sections $\{\alpha_k\}$ for $T^{1,0}|_U$ the forms $dz_k = dx_k + idy_k (= dx_k - idx_k \circ J)$, with $\overline{\alpha}_k = d\overline{z}_k = dx_k - idy_k$. These forms are exact, hence also closed, and so for a general (p,q)-form

$$\eta = \sum_{K,L} \eta_{K,L} dz_K \wedge d\overline{z}_L$$

the exterior derivative simplifies to

$$d\eta = \underbrace{\sum_{K,L} (\partial \eta_{K,L}) dz_k \wedge d\overline{z}_L}_{\in \Omega^{p+1,q}(M)} + \underbrace{\sum_{K,L} (\overline{\partial} \eta_{k,L}) dz_k \wedge d\overline{z}_L}_{\in \Omega^{p,q+1}(M)} .$$

Since the projections $\pi^{r,s} : \Omega^{k+1}(M,\mathbb{C}) \to \Omega^{r,s}(M)$ are invariantly defined, we have proven that in contrast to general almost complex manifolds, on a complex manifold (M,J) only two out of the possible four terms in (2) are nontrivial, and the exterior derivative splits as

$$\begin{aligned} &d = \partial + \overline{\partial} \qquad \text{with} \\ &\partial := \pi^{p+1,q} \circ d : \Omega^{p,q}(M) \to \Omega^{p+1,q}(M), \\ &\overline{\partial} := \pi^{p,q+1} \circ d : \Omega^{p,q}(M) \to \Omega^{p,q+1}(M). \end{aligned}$$

In particular, the equation $d^2 = 0$ can be written as

$$0 = (\partial + \overline{\partial})(\partial + \overline{\partial}) = \partial^2 + (\partial\overline{\partial} + \overline{\partial}\partial) + \overline{\partial}^2.$$

Since each of the three summands on the right maps an element $\eta \in \Omega^{p,q}(M) \subset \Omega^k(M, \mathbb{C})$ into a different component of $\Omega^{k+2}(M, \mathbb{C})$, each of them has to vanish separately. In particular, for each fixed $p \geq 0$ we get a complex

$$0 \longrightarrow \Omega^{p,0}(M) \xrightarrow{\overline{\partial}} \Omega^{p,1}(M) \xrightarrow{\overline{\partial}} \Omega^{p,2}(M) \xrightarrow{\overline{\partial}} \dots$$

Its cohomology is called *Dolbeault cohomology*,

$$H^{p,q}(M) := \frac{\ker(\overline{\partial} : \Omega^{p,q}(M) \to \Omega^{p,q+1}(M))}{\operatorname{im}(\overline{\partial} : \Omega^{p,q-1}(M) \to \Omega^{p,q}(M))}.$$

Exercise 2. Prove the following assertions:

(a) The vector fields

$$\partial_k := \frac{\partial}{\partial z_k} := \frac{1}{2}(\partial_{x_k} - i\partial_{y_k})$$

give a global (complex) basis of sections for the bundle $T_{1,0}(\mathbb{C}^n) \subset T_{\mathbb{R}}\mathbb{C}^n \otimes \mathbb{C}$, and the vector fields

$$\overline{\partial}_k := \frac{\partial}{\partial \overline{z}_k} := \frac{1}{2}(\partial_{x_k} + i\partial_{y_k})$$

form a global (complex) basis of sections for the bundle $T_{0,1}(\mathbb{C}^n) \subset T_{\mathbb{R}}\mathbb{C}^n \otimes \mathbb{C}$. These vector basis are dual to the basis dz_k and $d\overline{z}_k$ for the bundles $T^{1,0}(\mathbb{C}^n)$ and $T^{0,1}(\mathbb{C}^n)$, respectively.

(b) The standard symplectic form $\omega_{\text{st}} = \sum_k dx_k \wedge dy_k$ is given in these complex coordinates as

$$\omega_{\text{st}} = \frac{i}{2} \sum_k dz_k \wedge d\overline{z}_k.$$

(c) Writing a smooth complex valued function $f : \mathbb{C}^n \to \mathbb{C}$ as $f = f(z, \overline{z})$, one has

$$\partial f = \sum_k (\partial_k f) dz_k \qquad \text{and} \qquad \overline{\partial} f = \sum_k (\overline{\partial}_k f) d\overline{z}_k.$$

Such a function is holomorphic if and only if $\overline{\partial} f = 0$.