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Symplectic Geometry

Problem Set 9

1. Suppose (ω, Y, f) is a Weinstein structure on W, where $f : W \to [0,\infty)$ is a proper Morse function on W, ω is a symplectic form and Y is a Liouville vector field for ω which is gradient-like for f, i.e. $Y(f) \geq 0$ and $Y(f) = 0$ only at the critical points of f. Prove that the index of each critical point of f is at most half the dimension of W!
Hint: Consider the flow φ_t of Y and prove that for $p \in \operatorname{Crit} f$ the stable manifold

$$W^s(p) := \{x \in W \mid \lim_{t\to\infty} \varphi_t(x) = p\}$$

must be isotropic, by proving that $\lim_{t\to\infty} (\varphi_t^*\lambda)|_{T_xW^s(p)} = 0$ for each $x \in W^s(p)$.

2. *(The Kodaira-Thurston example)* On the set $\Gamma = \mathbb{Z}^2 \times \mathbb{Z}^2$ we consider the group structure given by the operation

$$(j,k) \star (j',k') := (j + j', A_{j'}k + k'),$$

where all letters denote elements in $\mathbb{Z}^2$, and for $n = (n_1, n_2) \in \mathbb{Z}^2$ we define $A_n := \begin{pmatrix} 1 & n_2 \\ 0 & 1 \end{pmatrix}$. The resulting group Γ acts on $\mathbb{R}^4 \cong \mathbb{R}^2 \times \mathbb{R}^2$ by

$$(j,k) \cdot (z,w) = \rho_{jk}(z,w) := (z + j, A_j w + k).$$

a) Prove that the action of Γ on $\mathbb{R}^4$ preserves the standard symplectic form $\omega = dz_1 \wedge dz_2 + dw_1 \wedge dw_2$, i.e. $\rho_{jk}^*\omega = \omega$ for all $(j,k) \in \Gamma$.

b) Prove that the action is properly discontinuous. In particular, the quotient $M := \mathbb{R}^4/\Gamma$ is a compact manifold which inherits a symplectic structure. Clearly we have $\pi_1(M) \cong \Gamma$.

c) Compute the commutator subgroup $[\Gamma,\Gamma]$ of Γ, which is the smallest normal subgroup containing all commutators $[\gamma_1,\gamma_2] = \gamma_1\gamma_2\gamma_1^{-1}\gamma_2^{-1}$ with $\gamma_1,\gamma_2 \in \Gamma$. Conclude that the abelian group $\Gamma/[\Gamma,\Gamma]$ is free of rank 3.

d) Using without proof that $H^1(M;\mathbb{C}) \cong \operatorname{Hom}(\pi_1(M),\mathbb{C})$, conclude that M cannot admit a Kähler structure.

3. Show that the Gromov width c_G is the smallest symplectic capacity satisfying properties (C1)–(C3) of the lecture, in the sense that for every symplectic capacity c and every symplectic manifold (M, ω) we have

$$c_G(M, \omega) \leq c(M, \omega).$$

4. Assuming the existence of a symplectic capacity satisfying conditions (C1)–(C3) of the lecture, prove that the product of two symplectic disks $B^2(a_1) \times B^2(a_2)$ is symplectomorphic to $B^2(b_1) \times B^2(b_2)$ if and only if the sets $\{a_1, a_2\}$ and $\{b_1, b_2\}$ agree.

What does your argument prove if the polydisks have more than two factors?

5. For $H \in \mathcal{H}(\mathbb{R}^{2n}, \omega_{\mathrm{st}})$ we consider the action functional $\mathcal{A} : C^\infty(S^1, \mathbb{R}^{2n}) \to \mathbb{R}$ from the lecture, given as

$$\mathcal{A}(\gamma) := -\int_0^1 \gamma^* \left(\frac{1}{2} \sum_j (x_j dy_j - y_j dx_j) \right) - \int_0^1 H(\gamma(t)) dt.$$

Prove that $\mathcal{A}$ is unbounded both above and below, by exhibiting explicit loops γ_j with $j \in \mathbb{Z}$ satisfying

$$\frac{\mathcal{A}(\gamma_j)}{j} \geq 1.$$