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Symplectic Geometry

Problem Set 7

1. Let $V = V(a_1, \ldots, a_n) \subset \mathbb{R}^{2n} \cong \mathbb{C}^n$ be the boundary of the *ellipsoid*

$$E(a_1, \ldots, a_n) := \{(z_1, \ldots, z_n) \in \mathbb{C}^n \mid \frac{|z_1|^2}{a_1} + \cdots + \frac{|z_n|^2}{a_n} \leq 1\},$$

where $0 < a_1 \leq a_2 \leq \cdots \leq a_n$ are real parameters.

a) Show that the restriction of $\lambda_0 := \frac{1}{2}\sum_j (x_j dy_j - y_j dx_j)$ to V is a contact form, and compute its Reeb vector field.

b) Assuming that the a_j are rationally independent, i.e. that all the quotients $\frac{a_j}{a_k}$ for $j \neq k$ are irrational, find all the closed Reeb orbits.

c) Compute the volume of $E(a_1, \ldots, a_n)$ with respect to the standard volume form $\frac{1}{n!}\omega_{\text{st}} = dx_1 \wedge dy_1 \wedge \ldots dx_n \wedge dy_n$.

d) Prove that for $(a_1, a_2) \neq (1, 1)$ the ellipsoid $E(a_1, a_2)$ cannot be symplectomorphic to the closed ball $\overline{B^4} = E(1, 1) \subset (\mathbb{R}^4, \omega_{\text{st}})$.
Hint: Consider the three cases $a_1 = a_2$, $\frac{a_1}{a_2} \in \mathbb{Q} \setminus \{1\}$ and $\frac{a_1}{a_2} \in \mathbb{R} \setminus \mathbb{Q}$ separately.

2. Let W be a manifold and let ξ be a coorientable contact structure on W, i.e. one which admits global contact forms. Prove that a contact vector field Y on W will be the Reeb vector field for some contact form λ defining ξ if and only if Y is everywhere transverse to ξ.

3. *(Legendrian knots)* Prove that every smooth curve $\gamma : [0, 1] \to \mathbb{R}^2$, $\gamma(t) = (x(t), y(t))$ admits a unique lift $\tilde{\gamma} : [0, 1] \to \mathbb{R}^3$ which starts at $\tilde{\gamma}(0) = (x(0), y(0), 0)$ and is tangent to the standard contact structure $\xi = \ker(dz - y dx)$. Which closed curves in the plane lift to a closed curve? Compute the lift of $\gamma(t) = (\sin 2\pi t, \sin 4\pi t)$ explicitly and sketch its image in $\mathbb{R}^3$.