Universität Hamburg
Janko Latschev
Fabian Kirchner

Winter 2014/15

Symplectic Geometry

Problem Set 6

1. We consider the two Lagrangian submanifolds of $(\mathbb{R}^{2n}, \omega_{\mathrm{st}})$ that were discussed in the lecture.

a) Given $n \geq 1$ and any fixed $0 < \epsilon < 1$, consider the image Q_1 of the Lagrangian embedding

$$\begin{aligned} \varphi : S^{n-1} \times S^1 &\to \mathbb{C}^n \cong \mathbb{R}^{2n} \\ (\xi, \lambda) &\mapsto (1 + \epsilon\lambda) \cdot \xi \end{aligned}$$

and compute the Maslov index of the loop $\gamma_0 : \mathbb{R}/\mathbb{Z} \to Q_1$, given by $\gamma_0(t) = \varphi_0(e^{2\pi i t}, (1, 0, \dots, 0))$.

b) For $n \geq 2$, consider the image Q_2 of the Lagrangian immersion

$$\begin{aligned} \varphi_1 : S^{n-1} \times S^1 &\to \mathbb{C}^n \cong \mathbb{R}^{2n} \\ (\xi, \lambda) &\mapsto \lambda \cdot \xi, \end{aligned}$$

and compute the Maslov index of the loop $\gamma_1 : \mathbb{R}/\mathbb{Z} \to Q_2$, given by $\gamma_1(t) = \varphi_1(e^{i\pi t}, (\cos(\pi t), \sin(\pi t), 0, \dots, 0))$.

2. Let (M^{2n}, ω) be symplectic and let $H : M \to \mathbb{R}$ be a function. Suppose $W := H^{-1}(0) \subset M$ is a smooth **closed** oriented hypersurface of contact type, i.e. there is a vector field Y defined near W and transverse to W such that $L_Y\omega = \omega$. As we have seen in class, this means that $\alpha := (\iota(Y)\omega)|_W$ is a contact form on W.

a) Assuming $n > 1$, prove that there is no *closed* 1-form β on W such that $\beta(X_H) > 0$ at all points of W. *Hint: You may want to use Stokes' Theorem.*

b) Use this to prove that, if $n > 1$, any other vector field Z also transverse to W and satisfying $L_Z\omega = \omega$ defines the same normal orientation of W as Y.

c) What happens for $n = 1$?

Bitte wenden!

3. Consider the following three contact forms on $\mathbb{R}^3$:

- $\lambda_1 = dz - ydx$, where (x, y, z) are cartesian coordinates,
- $\lambda_2 = dz + xdy$, where (x, y, z) are cartesian coordinates,
- $\lambda_3 = dz + r^2 d\varphi$, where (r, φ) are polar coordinates in $\mathbb{R}^2$, and z is the third coordinate.

a) Picture these contact structures and their Reeb vector fields (these will be defined on Tuesday).

b) Prove that $(\mathbb{R}^3, \text{Ker } \lambda_i)$ are pairwise contactomorphic, i.e. there are diffeomorphisms $\Phi_{ij} : \mathbb{R}^3 \to \mathbb{R}^3$ and functions $\rho_{ij} : \mathbb{R}^3 \to \mathbb{R}$ such that $\Phi_{ij}^*(\lambda_i) = \rho_{ij}\lambda_j$.

c) Prove that for any $i \in \{1, 2, 3\}$ there is a contactomorphism of $(\mathbb{R}^3, \text{Ker } \lambda_i)$ with a bounded subset $B \subset (\mathbb{R}^3, \text{Ker } \lambda_i)$.