Universität Hamburg Winter 2014/15
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Symplectic Geometry

Problem Set 4

1. *(The group of Hamiltonian diffeomorphisms)*
Let $\varphi_t : (M, \omega) \to (M, \omega)$ be the family of diffeomorphisms determined by the time-dependent Hamiltonian function $H : [0, 1] \times M \to \mathbb{R}$ via

$$\dot{\varphi}_t = X_{H_t} \circ \varphi_t.$$

a) For each $t \in (0, 1)$, write φ_t as time one map of a family of diffeomorphisms determined by a new Hamiltonian function built from H.

b) Find a time-dependent Hamiltonian function whose time one map is $(\varphi_1)^{-1}$.

c) Now suppose ψ is the time one map of a second family ψ_t determined by $F : [0, 1] \times M \to \mathbb{R}$. Find a time-dependent Hamiltonian function which generates the isotopy $\psi_t \circ \varphi_t$.

In summary, you have shown that Hamiltonian diffeomorphisms form a connected subgroup $\mathrm{Ham}(M, \omega) \subset \mathrm{Symp}(M, \omega)$ inside the group of all symplectomorphisms.

2. *(This exercise implements a suggestion by D. Salamon.)*
Consider $(\mathbb{R}^2, \omega_{\mathrm{st}} = dx \wedge dy)$.

a) Find explicit autonomous Hamiltonian functions $H_i : \mathbb{R}^2 \to \mathbb{R}$ such that the time-one-maps of the corresponding Hamiltonian flows φ_t^i are

$$\varphi_1^1 \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2x \\ \frac{1}{2}y \end{pmatrix} \qquad \text{and} \qquad \varphi_1^2 \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -x \\ -y \end{pmatrix}.$$

b) Prove that $\psi = \varphi_2^1 \circ \varphi_1^1$ cannot be generated by an autonomous Hamiltonian function (and in fact is not the time-one-map of any flow!).
Hint: Assume the contrary and first argue that 0 must be a fixpoint of the flow, then consider the linearization of the flow at this fixpoint to obtain a contradiction.

This clearly illustrates the need for time-dependent Hamiltonians in the definition of $\mathrm{Ham}(M, \omega)$.
In fact, general Hamiltonian diffeomorphisms often behave very differently from those generated by an autonomous function - for example they could have dense orbits, which clearly cannot happen in the autonomous case (why?).