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Symplectic Geometry

Problem Set 3

1. Let $\Psi : \mathbb{R}/\mathbb{Z} \to Sp(2n, \mathbb{R})$ be a loop of symplectic linear maps. Let $\Gamma_\Psi : \mathbb{R}/\mathbb{Z} \to \mathcal{L}(2n)$ be the loop of Lagrangian subspaces in $(\mathbb{R}^{2n} \oplus \mathbb{R}^{2n}, (-\omega_{\text{st}}) \oplus \omega_{\text{st}})$ given by

$$\Gamma_\Psi(t) = \{(v, \Psi(t)v) \mid v \in \mathbb{R}^{2n}\}.$$

a) Prove that $\Gamma_\Psi(t)$ is indeed a Lagrangian subspace for all t!

b) Prove that the Maslov indices of these two loops are related by

$$\mu(\Gamma_\Psi) = 2\mu(\Psi).$$

2. A diffeomorphism $\varphi : Q \to Q'$ between manifolds lifts to a diffeomorphism $\Phi : T^*Q \to T^*Q'$ given by the formula

$$\Phi(x, \alpha) := \left(\varphi(x), (d\varphi(x)^{-1})^*(\alpha)\right).$$

a) Prove that $\Phi^*(\lambda_{\text{can}}) = \lambda_{\text{can}}$, and so Φ is a symplectomorphism from T^*Q to T^*Q'!

b) Let $Y : Q \to TQ$ be a complete vector field, and denote by ψ_t its flow. Let $X : T^*Q \to T(T^*Q)$ be the vector field generating the corresponding flow Ψ_t on T^*Q. Prove that X is the Hamiltonian vector field associated to the function $H : T^*Q \to \mathbb{R}$ defined as

$$H(x, \alpha) := \alpha(Y(x)).$$

3. Show that if $\gamma : M \to M$ is any symplectomorphism of (M, ω) and $H : M \to \mathbb{R}$ is smooth, then the Hamiltonian vector fields of the functions H and $H \circ \gamma^{-1}$ are related by

$$X_{H\circ\gamma^{-1}}(\gamma(x)) = \gamma_*(X_H(x))$$

where $\gamma_* : TM \to TM$ is the differential of γ.

4. Consider a Hamiltonian function $H : B^2(0, 1) \to \mathbb{R}$ of the form $H = y \cdot \rho(r)$, where $\rho : B^2(0, 1) \to [0, 1]$ is a nonincreasing smooth function of the radius $r = \sqrt{x^2 + y^2}$ which equals 1 for $0 \le r \le \frac{1}{2}$ and equals 0 for $\frac{3}{4} \le r \le 1$. Describe the image of the ball $B^2(0, \frac{1}{100})$ under the time-t-map φ_t of the Hamiltonian flow of H for $t = 10$, $t = 10^2$ and $t = 10^5$ qualitatively!