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Symplectic Geometry

Problem Set 10

1. We consider the standard sympelctic forms on $\mathbb{R}^{2n}$, and denote by $B^4(a) \subset \mathbb{R}^4$ the open ball of Gromov width a, and by $T^2(1) = \mathbb{R}^2/\mathbb{Z}^2$ the square symplectic 2-torus of area 1.

a) Prove that, for any $a > 0$, there exists a 2-dimensional linear subspace $V_a \in \mathbb{R}^4$ such that the symplectic area of the intersection satisfies

$$0 < \operatorname{area}(B^4(a) \cap V_a) < \frac{\pi}{4}.$$

b) Given V_a as in part **a)**, find a symplectic linear map $\Psi_a \colon \mathbb{R}^4 \to \mathbb{R}^4$ which maps planes parallel to V_a to planes parallel to $\mathbb{R}^2 \times \{0\} \subset \mathbb{R}^4$, and disks parallel to V_a to disks.

c) Prove that the projection $\pi \colon \mathbb{R}^4 \to T^2(1) \times \mathbb{R}^2$ is injective on $\Psi_a(B^4(a))$.

d) Conclude that

$$c_G(T^2(1) \times \mathbb{R}^2) = \infty.$$

2. Suppose $u, v \in L^2(S^1, \mathbb{R}^{2n})$. We say that v is the *weak derivative* of u and write $v = Du$ if

$$\int_0^1 v(t)\varphi(t)dt = -\int_0^1 u(t)\varphi'(t)dt$$

for all test functions $\varphi \in C^\infty(S^1, \mathbb{R}^{2n})$.

a) Prove that the weak derivative is unique: If $w \in L^2(S^1, \mathbb{R}^{2n})$ is another weak derivative for u then $v = w$ almost everywhere.

b) Prove that the space $H^1(S^1, \mathbb{R}^{2n})$ introduced in the lecture can be characterized as the subspace of all functions in L^2 having a weak derivative in L^2, and that there is a constant $C > 0$ such that

$$\frac{1}{C}\|u\|_1 \le \|u\|_{L^2} + \|Du\|_{L^2} \le C\|u\|_1,$$

where $\|\cdot\|_1$ denotes the H^1-norm defined in class.

Remark: More generally, it is true that $u \in H^r$, $r \in \mathbb{N}$, if and only if u has weak derivatives up to order r in L^2.

Bitte wenden!

3. Prove that for any two compact symplectic manifolds (M_1, ω_1) and (M_2, ω_2) without boundary we have

$$c_{\mathrm{HZ}}(M_1 \times M_2) \geq c_{\mathrm{HZ}}(M_1) + c_{\mathrm{HZ}}(M_2).$$

Can you give counterexamples if you drop the assumption that the M_i be closed? Where does your proof fail without that assumption?

4. Suppose $\Omega \subset \mathbb{R}^m$ is a bounded open set, $f : \overline{\Omega} \to \mathbb{R}^m$ is continuous and smooth on Ω and $y \in \mathbb{R}^m \setminus f(\operatorname{Fr}\Omega)$ is a regular value of f.[1]

a) Prove that $f^{-1}(y)$ is finite.

For each $x \in f^{-1}(y)$, the differential $df_x : T_x\mathbb{R}^m \to T_y\mathbb{R}^m$ is an isomorphism, and we define $\epsilon(x)$ to be ± 1 depending on whether df_x preserves orientations or not. Now we define the Brouwer degree

$$d(f, \Omega, y) := \sum_{x \in f^{-1}(y)} \epsilon(x).$$

Prove the following two properties of the Brouwer degree:

b) If y_0 and y_1 are regular values in the same connected component of $\mathbb{R}^m \setminus f(\operatorname{Fr}\Omega)$, then

$$d(f, \Omega, y_0) = d(f, \Omega, y_1).$$

c) If $F : [0,1] \times \overline{\Omega} \to \mathbb{R}^m$ is a homotopy which is smooth on Ω and such that $y \notin F_t(\operatorname{Fr}\Omega)$ for all $t \in [0,1]$, then (assuming y is regular for F_0 and F_1)

$$d(F_0, \Omega, y_0) = d(F_1, \Omega, y).$$

Hint: For parts **b)** *and* **c)** *you will need to use basic results from differential topology, namely that*

(i) the two regular values y_0 and y_1 can be connected by a smooth path γ in the complement of $f(\operatorname{Fr}\Omega)$ to which f is transverse, and so $f^{-1}(\gamma([0,1]))$ is an oriented 1-dimensional manifold with boundary $f_1^{-1}(y) - f_0^{-1}(y)$ (as oriented manifolds).

(ii) By an arbitrarily small perturbation one can arrange that y will be a regular value for the homotopy F, so that $F^{-1}(y)$ is an oriented 1-dimensional manifold with boundary $f_1^{-1}(y) - f_0^{-1}(y)$.

If you find these assertions difficult to prove, consult any book on differential topology, e.g. Milnor's book "Topology from the differentiable viewpoint" for the case of compact manifolds without boundary. Then figure out how our assumptions on the relation of y and $\operatorname{Fr}\Omega$ are used.

[1] We denote by $\operatorname{Fr}\Omega = \overline{\Omega} \setminus \Omega$ the topological boundary of Ω.