Universität Hamburg
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Differential Topology

Problem Set 1

Here are some problems related to the material of the course up to this point. Some of them were already mentioned in class. Not all of them are equally difficult, or equally important. If you want to get feedback on your solution to a particular exercise, you may hand it in after any lecture, and I will try to comment within a week.

1. Prove that the subset

$$Q := \{\mathbf{z} = (z_1, \ldots, z_n) \in \mathbb{C}^n \mid \sum_{j=1}^{n} z_j^2 = 1\} \subset \mathbb{C}^n$$

is diffeomorphic to the tangent bundle of S^{n-1}, i.e. the submanifold

$$TS^{n-1} = \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^n \, : \, |x|^2 = 1, \langle x, y \rangle = 0\} \subseteq \mathbb{R}^{2n}.$$

2. Prove that every compact connected 1-dimensional manifold without boundary is diffeomorphic to S^1 and every compact connected 1-dimensional manifold with boundary is diffeomorphic to $[0, 1]$.

3. Let M be a differentiable manifold (of some class C^r, $r \geq 1$), and let $\tau : M \to M$ be a fixed point free involution, i.e. $\tau(p) \neq p$ for all $p \in M$ and $\tau \circ \tau = \mathrm{id}_M$.

a) Prove that the quotient space M/τ which is obtained by identifying every point with its image under τ is a topological manifold, and it admits a unique C^r-structure for which the projection map $\pi : M \to M/\tau$ is a local diffeomorphism.

b) Give examples of this phenomenon.

4. Give a proof of the approximation version of the Whitney embedding theorem mentioned in the lecture: For every smooth map $f : M \to \mathbb{R}^N$ of a smooth closed manifold of dimension d into $\mathbb{R}^N$ with $N \geq 2d + 1$ and every $\epsilon > 0$ there is an embedding $g : M \to \mathbb{R}^N$ such that

$$\max_{p \in M} |f(p) - g(p)| < \epsilon.$$

5. Prove directly that every product of spheres of total dimension n can be embedded into $\mathbb{R}^{n+1}$.

6. Construct an immersion of the punctured torus $S^1 \times S^1 \setminus \{pt\}$ into $\mathbb{R}^2$. Can such an immersion be injective?

Bitte wenden!

7. Prove that if M is a manifold of dimension $d < d'$, then any smooth map $f : M \to S^{d'}$ is homotopic to a constant map. In contrast, construct a map of degree 1 in case $d = d'$ and M is closed (here the degree is taken $\mod 2$ if M is not orientable).

8. Prove that if M is a connected smooth manifold, then any two distinct points $x_0 \neq x_1$ in M can be connected by a smooth embedded path, i.e. there is an embedding $\gamma : [0,1] \to M$ with $\gamma(0) = x_0$ and $\gamma(1) = x_1$.

9. Let M be a connected manifold of dimension $d \geq 2$. Prove that given two k-tupels $\{x_1, \ldots, x_k\}$ and $\{y_1, \ldots, y_k\}$ of distinct points in M, there exists a diffeomorphism $\varphi : M \to M$ satisfying $\varphi(x_j) = y_j$ for all $j = 1, \ldots k$.

10. Prove that a manifold M is orientable if and only if the restriction of the tangent bundle to every closed curve in M is an orientable vector bundle.

11. Prove that every map $S^d \to S^d$ with degree different from $(-1)^{d+1}$ has a fixed point.

12. Prove that any map $S^d \to S^d$ of odd degree maps some pair of antipodal points onto a pair of antipodal points.

13. Prove that if $f : M \to N$ and $g : N \to P$ are smooth maps between closed connected manifolds of the same dimension, then

$$\deg(g \circ f) = \deg(g) \cdot \deg(f).$$

The degrees can be interpreted as integer-valued if all three manifolds are oriented, and should be taken mod 2 otherwise.

14. The d-dimensional torus T^d is the product of d copies of S^1. Since S^1 is a Lie group, so is T^d. It is clearly orientable (as any Lie group is – why?). What is the degree of the map $T^d \to T^d$ which sends each element of T^d to its square with respect to the group structure?

15. *(Some familiarity with algebraic topology is helpful for this exercise.)*
For a closed connected oriented manifold X of dimension d, one knows that the de Rham cohomology $H^d(X)$ in degree d is 1-dimensional, and integration gives rise to an isomorphism $H^d(X) \cong \mathbb{R}$. In particular, any two d-forms with the same integral are in the same cohomology class. It follows that if M and N are two such manifolds of the same dimension and $f : M \to N$ is a smooth map, there is a unique real number $D(f)$ such that

$$\int_M f^*\mu = D(f) \cdot \int_N \mu$$

for any $\mu \in \Omega^d(N)$. Prove that this number $D(f)$ agrees with the integer valued degree of the map f as defined in the lecture.

16. Let $f_1 : M_1 \to M'$ and $f_2 : M_2 \to M'$ be two smooth maps which are *transverse* in the sense that for every pair of points $(p_1, p_2) \in M_1 \times M_2)$ with $f_1(p_1) = f_2(p_2)$ one has

$$Df_1(T_{p_1} M_1) + Df_2(T_{p_2} M_2) = T_{f_1(p_1)} M'.$$

Prove that under these conditions the *fiber product of* M_1 *and* M_2 *over* M',

$$M_1 \times_{M'} M_2 := \{(p_1, p_2) \mid f_1(p_1) = f_2(p_2)\} \subseteq M_1 \times M_2,$$

is a smooth submanifold. What is its dimension? Can you find interesting examples of this construction?