Fachbereich Mathematik
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Topics for talks

Seminar on characteristic classes

The seminar will mainly follow the book *Characteristic classes* by Milnor and Stasheff [4], with occasional supplementary material. Here is the preliminary list of topics.

(0) (optional)
review of basic definitions for and constructions with vector bundles
[4, §2 and §3]

(1) Stiefel-Whitney classes
axioms, first consequences, examples: canonical line bundle over $\mathbb{R}P^n$, total Stiefel-Whitney class of $\mathbb{R}P^n$, applications of the computations; Stiefel-Whitney numbers, characterization of nullcobordant closed manifolds
[4, §4]

(2) Grassmannians and universal bundles
definitions, construction of classifying map of a real vector bundle, well-definedness up to homotopy, general concept of characteristic class for a real vector bundle
[4, §5]

(3) Cell structure of Grassmannians and mod 2 cohomology ring of G_n
definition of Schubert cells, basic properties, CW decomposition of $G_{n,m}$, examples; structure of $H^*(G_n;\mathbb{Z}_2)$, uniqueness of classes satisfying the axioms for Stiefel-Whitney classes
[4, §6 and §7]

(4) Steenrod squares and existence of Stiefel Whitney classes
discussion of Steenrod squares and their basic properties, proof of existence of classes satisfying the axioms for Stiefel-Whitney classes
[4, §8] and [2, §4.L]

(5) Euler class for oriented vector bundles and the Thom isomorphism
definition of the Euler class of an oriented vector bundle, basic properties, statement and proof of the Thom isomorphism theorem
[4, §9 and §10] and [1, §11 and §12]

(6) Computations in a smooth manifold
embedding obstructions from Stiefel-Whitney classes, Euler class of the tangent bundle and Euler characteristic, Wu's formula for the Stiefel-Whitney classes of a smooth manifold, Problem 11.D
[4, §11]

(7) Characteristic classes as obstructions
Stiefel-Whitney classes as (mod 2) obstructions to the existence tuples of linearly independent cross sections over skeleta, the Gysin sequence and the mod 2 cohomology ring of the oriented Grassmannians, Euler class as an obstruction to the existence of a nonvanishing section
[4, §12]

(8) Complex vector bundles and Chern classes
basic definitions, construction of Chern classes, relation to cohomology of complex Grassmannians, basic properties, examples
[4, §13 and 14]

(9) Pontryagin classes, Chern and Pontryagin numbers
definition and basic properties of Pontrjagin classes, integral cohomology ring of the oriented Grassmannian, definition of Chern and Pontrjagin numbers, computations in examples and applications
[4, §15 and 16]

(10) Multiplicative sequences and the signature theorem
preliminaries on multiplicative sequences, definition and basic properties of the signature of a $4n$-dimensional manifold, the signature theorem and its proof
[4, §19]

(11) Basic Chern-Weil theory
construction of Chern classes of a complex vector bundle using a complex connection
[4, App. C] and [3, §17 and §18]

(12) An application: the cohomology ring of a smooth hypersurface in $\mathbb{C}P^n$
discuss how characteristic classes are used to compute the cohomology ring of a smooth hypersurface in $\mathbb{C}P^n$, following [5] (this includes tracking down proofs or references for all non-obvious claims made there)

Bibliography

[1] Raoul Bott and Loring W. Tu, *Differential Forms in Algebraic Topology*, Springer Graduate Texts in Mathematics 82, corrected printing 1995

[2] Allen Hatcher, *Algebriac Topology*, Cambridge University Press, 2002 and later, `https://pi.math.cornell.edu/~hatcher/AT/ATpage.html`

[3] Ib Madsen and Jørgen Tornehave, *From Calculus to Cohomology: De Rham cohomology and characteristic classes*, Cambridge University Press, 2001

[4] John W. Milnor and James D. Stasheff, *Characteristic classes*, Annals of Mathematics studies, Princeton University Press, 1974

[5] Qiaochu Yuan, Blogpost on Wordpress, `https://qchu.wordpress.com/2014/06/16/hypersurfaces-4-manifolds-and-characteristic-classes/`, last accessed April 7, 2024