Uniform rigidity sequences for weakly mixing diffeomorphisms on $\mathbb{D}^m$, $\mathbb{T}^m$ and $\mathbb{S}^1 \times [0,1]^{m-1}$

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Abstract

In continuation of [Ku] we construct weakly mixing and uniformly rigid diffeomorphisms on $\mathbb{D}^m$, $\mathbb{T}^m$ as well as $\mathbb{S}^1 \times [0,1]^{m-1}$ $(m \geq 2)$: If a sequence of natural numbers satisfies a certain growth rate, then there is a weakly mixing C^∞-diffeomorphism that is uniformly rigid with respect to that sequence. The proof is based on a quantitative version of the Anosov-Katok-method with explicitly defined conjugation maps.

Keywords: Smooth Ergodic Theory, weakly mixing, uniformly rigid, uniform rigidity sequence

1. Introduction

To begin, we recall that an invertible measure-preserving transformation T of a non-atomic probability space $(X, \mathcal{B}, \mu)$ is called rigid if there exists an increasing sequence $(n_m)_{m\in\mathbb{N}}$ of natural numbers (a so-called rigidity sequence) such that the powers T^{n_m} converge to the identity in the strong operator topology as $m \to \infty$, i.e. $\|f \circ T^{n_m} - f\|_2 \to 0$ as $m \to \infty$ for all $f \in L^2(X, \mu)$. So rigidity along a sequence $(n_m)_{m\in\mathbb{N}}$ implies $\mu(T^{n_m} A \cap A) \to \mu(A)$ as $m \to \infty$ for all $A \in \mathcal{B}$. In [BJLR] the authors examine conditions on a sequence $(n_m)_{m\in\mathbb{N}}$ which ensure that it is a rigidity sequence for some weakly mixing systems. In this paper, we study the notion of uniform rigidity introduced in [GM] as the topological analogue of rigidity in ergodic theory:

Definition 1.1. Let $(X, \mathcal{B}, \mu)$ be a Lebesgue probability space, where X is a compact metric space with metric d. A measure-preserving homeomorphism $T : X \to X$ is called uniformly rigid if there exists an increasing sequence $(k_n)_{n\in\mathbb{N}}$ in $\mathbb{N}$ such that $d_u\left(T^{k_n}, id\right) \to 0$ as $n \to \infty$, where $d_u(S,T) = d_0(S,T) + d_0\left(S^{-1}, T^{-1}\right)$ with $d_0(S,T) \coloneqq \sup_{x\in X} d(S(x), T(x))$ is the uniform metric on the group of measure-preserving homeomorphisms on X.

In [JKLSS], Proposition 4.1., it is shown that if an ergodic map is uniformly rigid, then any uniform rigidity sequence has zero density. Afterwards, the following question is posed:

Question 1.2. Which zero density sequences occur as uniform rigidity sequences for an ergodic transformation?

Ergodicity is implied by the weak mixing property. Recall that a measure-preserving transformation T is called weakly mixing if for all $A, B \in \mathcal{B}$ we have $\frac{1}{N}\sum_{n=1}^{N} |\mu(T^n A \cap B) - \mu(A) \cdot \mu(B)| \to 0$ as $N \to \infty$. An equivalent characterization is deduced by M. Sklover ([Skl]): There is an increasing sequence $(m_n)_{n\in\mathbb{N}}$ of natural numbers such that $\lim_{n\to\infty} |\mu(B \cap T^{-m_n}(A)) - \mu(A) \cdot \mu(B)| = 0$ for every pair of measurable sets $A, B \subseteq X$.
K. Yancey considered Question 1.2 in the setting of homeomorphisms on $\mathbb{T}^2$ (see [Ya]). Given a sufficient growth rate of the sequence she proved the existence of a weakly mixing homeomorphism of $\mathbb{T}^2$ that is uniformly rigid with respect to this sequence: Let $\psi(x) = x^{x^3}$. If $(k_n)_{n\in\mathbb{N}}$ is an increasing sequence of natural numbers satisfying $\frac{k_{n+1}}{k_n} \geq \psi(k_n)$, there exists a weakly mixing homeomorphism of $\mathbb{T}^2$ that is uniformly rigid with respect to $(k_n)_{n\in\mathbb{N}}$. In her Phd thesis Yancey asked about genericity of weakly mixing and uniformly rigid homeomorphisms on an arbitrary compact manifold of dimension at least 2 ([Yab], Question 5.1.2).
In [Ku] we started to examine this problem in the smooth category. As a starting point we used the construction of weakly mixing diffeomorphisms with a prescribed Liouvillean rotation number on 2-dimensional compact connected manifolds admitting a non-trivial circle action undertaken in [FS]. Hereby, we were able to construct smooth weakly mixing diffeomorphisms on $\mathbb{D}^2$, $\mathbb{T}^2$ and $\mathbb{A} = \mathbb{S}^1 \times [0,1]$ that are uniformly rigid with respect to a given sequence under a condition on the growth rate of this sequence. This condition was less restrictive than Yancey's. Actually, the constructed diffeomorphisms were C^∞-rigid.

Definition 1.3. Let M be a smooth compact connected manifold and $k \in \mathbb{N} \cup \{\infty\}$. A C^k-diffeomorphism $f : M \to M$ is called C^k-rigid, if there exists a sequence $(k_n)_{n\in\mathbb{N}}$ in $\mathbb{N}$ such that f^{k_n} converges to the identity map in the C^k-topology.

Amongst others, C^k-rigidity of pseudo-rotations on the disc $\mathbb{D}^2$ is studied in [AFLXZ].
On the other hand, for every Liouvillean number $\alpha \in \mathbb{S}^1$ we were able to prove the genericity of weakly mixing smooth diffeomorphisms in $\mathcal{A}_\alpha(M) := \overline{\{h \circ S_\alpha \circ h^{-1} \,:\, h \in \text{Diff}^\infty(M,\nu)\}}^{C^\infty}$ on any smooth compact connected manifold M of dimension $m \geq 2$ admitting a non-trivial smooth circle action $\mathcal{S} = \{S_t\}_{t\in\mathbb{S}^1}$ preserving a smooth volume ν ([GKu], Corollary 1). These constructions were based on the "conjugation by approximation"-method introduced by D. Anosov and A. Katok in their fundamental paper [AK]: Diffeomorphisms are constructed as limits of conjugates $f_n = H_n \circ S_{\alpha_{n+1}} \circ H_n^{-1}$, where $\alpha_{n+1} = \frac{p_{n+1}}{q_{n+1}} \in \mathbb{Q}$, $H_n = H_{n-1} \circ h_n$ and h_n is a measure-preserving diffeomorphism satisfying $S_{\frac{1}{q_n}} \circ h_n = h_n \circ S_{\frac{1}{q_n}}$. While the sequence of conjugation

maps H_n does not have to converge in general, one obtains that the sequence f_n is a Cauchy sequence by choosing α_{n+1} so close to α_n that

$$\begin{aligned} f_n &= H_n \circ S_{\alpha_{n+1}} \circ H_n^{-1} = H_{n-1} \circ h_n \circ S_{\alpha_n} \circ S_{\alpha_{n+1}-\alpha_n} \circ h_n^{-1} \circ H_{n-1}^{-1} \\ &= H_{n-1} \circ S_{\alpha_n} \circ h_n \circ S_{\alpha_{n+1}-\alpha_n} \circ h_n^{-1} \circ H_{n-1}^{-1} \end{aligned}$$

is close to $f_{n-1} = H_{n-1} \circ S_{\alpha_n} \circ H_{n-1}^{-1}$. Using that method Anosov and Katok were particularly able to answer the long-standing question on the existence of an ergodic diffeomorphism on the disc $\mathbb{D}^2$ affirmatively ([AK], section 3). Nowadays, this method is one of the most powerful tools for constructing smooth diffeomorphisms with ergodic properties or non-standard smooth realizations of measure-preserving maps (e. g. [Be]). See [FK04] for more details and other results of this method.
In comparison to the original construction of weakly mixing diffeomorphisms in $\mathcal{A}(M) := \overline{\{h \circ S_t \circ h^{-1} \ : \ t \in \mathbb{S}^1, h \in \text{Diff}^{\infty}(M, \nu)\}}^{C^\infty}$ in [AK], section 5, the constructions with a prescribed Liouvillean rotation number α in [GKu] required more explicit conjugation maps and finer norm estimates in order to guarantee convergence in $\mathcal{A}_\alpha(M)$. Unfortunately, these estimates are not sufficient for our purpose because the dependence on the parameter $\varepsilon_n = \frac{1}{60n^4}$ occurring in the conjugation map in [GKu] built with the aid of "Moser's trick" is not examined. This dependence is important in order to deduce a sufficient growth rate of the uniform rigidity sequence. Therefore, we need even more explicit conjugation maps and precise norm estimates. Such a construction is provided in this paper. Hereby, we can prove the subsequent theorem:

Theorem 1. *Let $m \geq 2$, M be $\mathbb{D}^m$, $\mathbb{S}^1 \times [0,1]^{m-1}$ or $\mathbb{T}^m$ and $\varphi(n)$ be the expression*

$$\left(\frac{(m+n)!}{(m-1)!}\right)^{m\cdot(n+2)^{n+3}} \cdot \left(\frac{(2n)!}{n!} \cdot \pi^{(n+1)^2} \cdot \left((n+1)! \cdot \exp\left(400n^2\right)\right)^{10\cdot(n+1)^5}\right)^{m\cdot(n+1)^{n+2}} \cdot n^{2\cdot(m-1)\cdot(n+1)^{n+2}}.$$

If $(\tilde{q}_n)_{n\in\mathbb{N}}$ is a sequence of natural numbers satisfying

$$\tilde{q}_{n+1} \geq \varphi(n) \cdot \tilde{q}_n^{2\cdot m^2\cdot(n+1)^{n+3}},$$

there exists a weakly mixing C^∞-diffeomorphism of M that is uniformly rigid (actually C^∞-rigid) with respect to $(\tilde{q}_n)_{n\in\mathbb{N}}$.

In [Ku], Theorem 1, we have obtained a similar condition on the growth rate of the uniform rigidity sequence in case of $m = 2$. In section 9 we deduce a rougher but more handy statement:

Corollary 1. *Let $m \geq 2$, M be $\mathbb{D}^m$, $\mathbb{S}^1 \times [0,1]^{m-1}$ or $\mathbb{T}^m$. If $(\tilde{q}_n)_{n\in\mathbb{N}}$ is a sequence of natural numbers satisfying*

$$\tilde{q}_1 \geq m^2 \cdot 2^8 \cdot \exp(400) \quad \textit{as well as} \quad \tilde{q}_{n+1} \geq \tilde{q}_n^{\tilde{q}_n},$$

there exists a weakly mixing C^∞-diffeomorphism of M that is uniformly rigid (actually C^∞-rigid) with respect to $(\tilde{q}_n)_{n\in\mathbb{N}}$.

We note that our requirement on the growth rate is less restrictive than the mentioned condition in [Ya], Theorem 1.5. In fact, the proof in [Ya] shows that a condition of the form $\frac{k_{n+1}}{k_n} \geq k_n^{4k_n^2+20}$ is sufficient for her construction of a weakly mixing homeomorphism, which is uniformly rigid along $(k_n)_{n\in\mathbb{N}}$. Our requirement on the growth rate is still weaker.
If we consider only C^k-diffeomorphisms for any $k \in \mathbb{N}$, we can weaken our requirements on the uniform rigidity sequence in section 8.

Corollary 2. *Let $k \in \mathbb{N}$, $m \geq 2$ and M be $\mathbb{D}^m$, $\mathbb{S}^1 \times [0,1]^{m-1}$ or $\mathbb{T}^m$ and $\varphi_k(n)$ be the expression*

$$\left(\frac{(m+k)!}{(m-1)!}\right)^{m\cdot(k+2)^4} \cdot \left(\frac{(2k)!}{k!} \cdot \pi^{(k+1)^2} \cdot \left((k+1)! \cdot \exp\left(400n^2\right)\right)^{10\cdot(k+1)^5}\right)^{m\cdot(k+1)^3} \cdot n^{2\cdot(m-1)\cdot(k+1)^4}.$$

If $(\tilde{q}_n)_{n\in\mathbb{N}}$ is a sequence of natural numbers satisfying

$$\tilde{q}_{n+1} \geq \varphi_k(n) \cdot \tilde{q}_n^{2\cdot m^2\cdot(n+1)\cdot(k+1)^4},$$

there exists a weakly mixing C^k-diffeomorphism of M that is uniformly rigid (actually C^k-rigid) with respect to $(\tilde{q}_n)_{n\in\mathbb{N}}$.

2. Preliminaries

2.1. Definitions and notations

In this chapter we want to introduce advantageous definitions and notations as in [GKu]. Initially, we discuss topologies on the space of smooth diffeomorphisms on the manifold $M = \mathbb{S}^1 \times [0,1]^{m-1}$. Note that for diffeomorphisms $f = (f_1, ..., f_m) : \mathbb{S}^1 \times [0,1]^{m-1} \to \mathbb{S}^1 \times [0,1]^{m-1}$ the coordinate function f_1 understood as a map $\mathbb{R} \times [0,1]^{m-1} \to \mathbb{R}$ has to satisfy the condition $f_1(\theta + n, r_1, ..., r_{m-1}) = f_1(\theta, r_1, ..., r_{m-1}) + l$ for $n \in \mathbb{Z}$, where either $l = n$ or $l = -n$. Moreover, for $i \in \{2, ..., m\}$ the coordinate function f_i has to be $\mathbb{Z}$-periodic in the first component, i.e. $f_i(\theta + n, r_1, ..., r_{m-1}) = f_i(\theta, r_1, ..., r_{m-1})$ for every $n \in \mathbb{Z}$.
In order to define explicit metrics on $\text{Diff}^k\left(\mathbb{S}^1 \times [0,1]^{m-1}\right)$ and in the following the subsequent notations will be useful:

Definition 2.1.
1. For a sufficiently differentiable function $f : \mathbb{R}^m \to \mathbb{R}$ and a multiindex $\vec{a} = (a_1, ..., a_m) \in \mathbb{N}_0^m$

$$D_{\vec{a}}f := \frac{\partial^{|\vec{a}|}}{\partial x_1^{a_1} ... \partial x_m^{a_m}} f,$$

where $|\vec{a}| = \sum_{i=1}^m a_i$ is the order of $\vec{a}$.
2. For a continuous function $F : (0,1)^m \to \mathbb{R}$

$$\|F\|_0 := \sup_{z\in(0,1)^m} |F(z)|.$$

Diffeomorphisms on $\mathbb{S}^1 \times [0,1]^{m-1}$ can be regarded as maps from $[0,1]^m$ to $\mathbb{R}^m$. In this spirit the expressions $\|f_i\|_0$ as well as $\|D_{\vec{a}} f_i\|_0$ for any multiindex $\vec{a}$ with $|\vec{a}| \leq k$ have to be understood for $f = (f_1, ..., f_m) \in \text{Diff}^k\left(\mathbb{S}^1 \times [0,1]^{m-1}\right)$. Since such a diffeomorphism is a continuous map on the compact manifold and every partial derivative can be extended continuously to the boundary, all these expressions are finite. Thus, the subsequent definition makes sense:

Definition 2.2.

1. For $f, g \in \text{Diff}^k\left(\mathbb{S}^1 \times [0,1]^{m-1}\right)$ with coordinate functions f_i resp. g_i we define

$$\tilde{d}_0(f,g) = \max_{i=1,..,m} \left\{ \inf_{p \in \mathbb{Z}} \|(f-g)_i + p\|_0 \right\}$$

as well as

$$\tilde{d}_k(f,g) = \max\left\{ \tilde{d}_0(f,g), \|D_{\vec{a}}(f-g)_i\|_0 \ : \ i = 1,...,m \ , \ 1 \leq |\vec{a}| \leq k \right\}.$$

2. Using the definitions from 1. we define for $f, g \in \text{Diff}^k\left(\mathbb{S}^1 \times [0,1]^{m-1}\right)$:

$$d_k(f,g) = \max\left\{ \tilde{d}_k(f,g) \ , \ \tilde{d}_k\left(f^{-1}, g^{-1}\right) \right\}.$$

Obviously d_k describes a metric on $\text{Diff}^k\left(\mathbb{S}^1 \times [0,1]^{m-1}\right)$ measuring the distance between the diffeomorphisms as well as their inverses. As in the case of a general compact manifold the following definition connects to it:

Definition 2.3.

1. A sequence of $\text{Diff}^\infty\left(\mathbb{S}^1 \times [0,1]^{m-1}\right)$-diffeomorphisms is called convergent in $\text{Diff}^\infty\left(\mathbb{S}^1 \times [0,1]^{m-1}\right)$ if it converges in $\text{Diff}^k\left(\mathbb{S}^1 \times [0,1]^{m-1}\right)$ for every $k \in \mathbb{N}$.
2. On $\text{Diff}^\infty\left(\mathbb{S}^1 \times [0,1]^{m-1}\right)$ we declare the following metric

$$d_\infty(f,g) = \sum_{k=1}^{\infty} \frac{d_k(f,g)}{2^k \cdot (1 + d_k(f,g))}.$$

It is a general fact that $\text{Diff}^\infty\left(\mathbb{S}^1 \times [0,1]^{m-1}\right)$ is a complete metric space with respect to this metric d_∞.
Again considering diffeomorphisms on $\mathbb{S}^1 \times [0,1]^{m-1}$ as maps from $[0,1]^m$ to $\mathbb{R}^m$ we add the adjacent notation:

Definition 2.4. Let $f \in \text{Diff}^k\left(\mathbb{S}^1 \times [0,1]^{m-1}\right)$ with coordinate functions f_i be given. Then

$$\|Df\|_0 := \max_{i,j \in \{1,...,m\}} \|D_j f_i\|_0,$$

$$\|f\|_k := \max\left\{ \inf_{p \in \mathbb{Z}} \|f_i - p\|_0, \|D_{\vec{a}} f_i\|_0 \ : \ i = 1,...,m, \ \vec{a} \text{ multiindex with } 1 \leq |\vec{a}| \leq k \right\}$$

and

$$|||f|||_k := \max \left\{ \|f\|_k \, , \left\| f^{-1} \right\|_k \right\}.$$

Remark 2.5. By the above-mentioned observations for every multiindex $\vec{a}$ with $|\vec{a}| \geq 1$ and every $i \in \{1, ..., m\}$ the derivative $D_{\vec{a}} h_i$ is $\mathbb{Z}$-periodic in the first variable. Since in case of a diffeomorphism $g = (g_1, ..., g_m)$ on $\mathbb{S}^1 \times [0,1]^{m-1}$ regarded as a map $[0,1]^m \to \mathbb{R}^m$ the coordinate functions g_j for $j \in \{2, ..., m\}$ satisfy $g_j([0,1]^m) \subseteq [0,1]$, it holds:

$$\sup_{z \in (0,1)^m} |(D_{\vec{a}} h_i)(g(z))| \leq |||h|||_{|\vec{a}|}.$$

Analogously we can define the same expressions in the case of the torus $\mathbb{T}^m$. In the case of $\mathbb{D}^m$ the $\text{Diff}^k(\mathbb{D}^m)$-topologies are defined in a natural way with the aid of the supremum norms. Subsequently, M is $\mathbb{S}^1 \times [0,1]^{m-1}$, $\mathbb{D}^m$ or $\mathbb{T}^m$. Concerning the composition of functions the next results are useful:

Lemma 2.6. *Let $s \in \mathbb{N}$ and g, h be C^s-functions on M. Then we have*

$$\|g \circ h\|_s \leq \frac{(m+s-1)!}{(m-1)!} \cdot \|g\|_s \cdot \|h\|_s^s.$$

Proof. By induction on $k \in \mathbb{N}$ we will prove the following observation:
Claim: *For any multiindex $\vec{a} \in \mathbb{N}_0^m$ with $|\vec{a}| = k$ and $i \in \{1, ..., m\}$ the partial derivative $D_{\vec{a}} [g \circ h]_i$ consists of at most $\frac{(m+k-1)!}{(m-1)!}$ summands, where each summand is the product of one derivative of g of order at most k and at most k derivatives of h of order at most k.*

- *Start:* $k = 1$
  For $i_1, i \in \{1, ..., m\}$ we compute:

  $$D_{x_{i_1}} [g \circ h]_i (x_1, ..., x_m) = \sum_{j_1=1}^{m} \left(D_{x_{j_1}} [g]_i \right) (h(x_1, ..., x_m)) \cdot D_{x_{i_1}} [h]_{j_1} (x_1, ..., x_m)$$

  Hence, this derivative consists of $m = \frac{(m+1-1)!}{(m-1)!}$ summands and each summand has the announced form.

- *Induction assumption:* The claim holds for $k \in \mathbb{N}$.

- *Induction step:* $k \to k+1$
  Let $i \in \{1, .., m\}$ and $\vec{b} \in \mathbb{N}_0^m$ be any multiindex of order $\left|\vec{b}\right| = k+1$. There are $j \in \{1, ..., m\}$ and a multiindex $\vec{a}$ of order $|\vec{a}| = k$ such that $D_{\vec{b}} = D_{x_j} D_{\vec{a}}$. By the induction assumption the partial derivative $D_{\vec{a}} [g \circ h]_i$ consists of at most $\frac{(m+k-1)!}{(m-1)!}$ summands, at which the summand with the most factors is of the subsequent form:

  $$D_{\vec{c}_1} [g]_i (h(x_1, ..., x_m)) \cdot D_{\vec{c}_2} [h]_{i_2} (x_1, ..., x_m) \cdot ... \cdot D_{\vec{c}_{k+1}} [h]_{i_{k+1}} (x_1, ..., x_m),$$

where each $\vec{c}_i$ is of order at most k. Using the product rule we compute how the derivative D_{x_j} acts on such a summand:

$$\left(\sum_{j_1=1}^{m} D_{x_{j_1}} D_{\vec{c}_1} [g]_i \circ h \cdot D_{x_j} [h]_{j_1} D_{\vec{c}_2} [h]_{i_2} \cdot ... \cdot D_{\vec{c}_{k+1}} [h]_{i_{k+1}}\right) +$$
$$D_{\vec{c}_1} [g]_i \circ h \cdot D_{x_j} D_{\vec{c}_2} [h]_{i_2} \cdot ... \cdot D_{\vec{c}_{k+1}} [h]_{i_{k+1}} + ... + D_{\vec{c}_1} [g]_i \circ h \cdot D_{\vec{c}_2} [h]_{i_2} \cdot ... \cdot D_{x_j} D_{\vec{c}_{k+1}} [h]_{i_{k+1}}$$

Thus, each summand is the product of one derivative of g of order at most $k+1$ and at most $k+1$ derivatives of h of order at most $k+1$. Moreover, we observe that $m+k$ summands arise out of one. So the number of summands can be estimated by $(m+k) \cdot \frac{(m+k-1)!}{(m-1)!} = \frac{(m+k)!}{(m-1)!}$ and the claim is verified.

Using this claim we obtain for $i \in \{1, ..., m\}$ and any multiindex $\vec{a} \in \mathbb{N}_0^m$ of order $|\vec{a}| = k$:

$$\left\| D_{\vec{a}} [g \circ h]_i \right\|_0 \leq \frac{(m+k-1)!}{(m-1)!} \cdot \|g\|_k \cdot \|h\|_k^k$$

□

Lemma 2.7. *Let $s \in \mathbb{N}$ and $f_1, ..., f_l$ be C^s-functions on M. Then we have*

$$\|f_l \circ ... \circ f_1\|_s \leq \left(\frac{(m+s-1)!}{(m-1)!}\right)^{l-1} \cdot \|f_l\|_s \cdot \|f_{l-1}\|_s^s \cdot ... \cdot \|f_1\|_s^s$$

Proof. By several applications of Lemma 2.6 we conclude:

$$\begin{aligned}
\|f_l \circ ... \circ f_1\|_s &\leq \frac{(m+s-1)!}{(m-1)!} \cdot \|f_l \circ ... \circ f_2\|_s \cdot \|f_1\|_s^s \\
&\leq \frac{(m+s-1)!}{(m-1)!} \cdot \frac{(m+s-1)!}{(m-1)!} \cdot \|f_l \circ ... \circ f_3\|_s \cdot \|f_2\|_s^s \cdot \|f_1\|_s^s \\
&\leq \left(\frac{(m+s-1)!}{(m-1)!}\right)^{l-1} \cdot \|f_l\|_s \cdot \|f_{l-1}\|_s^s \cdot ... \cdot \|f_1\|_s^s
\end{aligned}$$

□

Lemma 2.8. *Let $s \in \mathbb{N}$ and $f_1, ..., f_l$ be C^s-diffeomorphisms on M. Then we have*

$$|||f_l \circ ... \circ f_1|||_s \leq \left(\frac{(m+s-1)!}{(m-1)!}\right)^{l-1} \cdot |||f_l|||_s^s \cdot ... \cdot |||f_1|||_s^s$$

Proof. Applying Lemma 2.7 on $f_l \circ ... \circ f_1$ as well as $f_1^{-1} \circ ... \circ f_l^{-1}$ yields the statement. □

2.2. Outline of the proof

Let $\mathbb{S}^1 \times [0,1]^{m-1}$ be equipped with Lebesgue measure μ and smooth circle action $\mathcal{R} = \{R_t\}_{t \in \mathbb{S}^1}$ comprising of the maps $R_t(\theta, r_1, ..., r_{m-1}) = (\theta + t, r_1, ..., r_{m-1})$. The aimed diffeomorphisms are constructed as limits of conjugates $f_n = H_n \circ R_{\alpha_{n+1}} \circ H_n^{-1}$, where $\alpha_{n+1} = \frac{p_{n+1}}{q_{n+1}} \in \mathbb{Q}$, $H_n = H_{n-1} \circ h_n$ and h_n is a measure-preserving diffeomorphism satisfying $R_{\frac{1}{q_n}} \circ h_n = h_n \circ R_{\frac{1}{q_n}}$. In each step the conjugation map h_n is composed of two measure-preserving diffeomorphisms: $h_n = g_n \circ \phi_n$. The step-by-step defined map ϕ_n is constructed in section 3 with the aid of several maps. In fact, $\phi_n = \bar{\phi}^{(m)}_{\lambda_m, \delta_n} \circ ... \circ \bar{\phi}^{(1)}_{\lambda_1, \delta_n}$ is a composition of maps $\bar{\phi}^{(j)}_{\lambda,\delta} = C_\lambda^{-1} \circ \tilde{\phi}^{(j)}_\delta \circ C_\lambda$, where $C_\lambda(\theta, r_1, ..., r_{m-1}) = (\lambda \cdot \theta, r_1, ..., r_{m-1})$ causes a stretch by λ in the first coordinate and $\tilde{\phi}^{(j)}_\delta$ is a "quasi-rotation", i.e. a measure-preserving diffeomorphism that coincides with the rotation by $\frac{\pi}{2}$ in the $x_1 - x_j$-plane in the interior and with the identitity in a neighbourhood of the boundary of $[0,1]^m$. Descriptively, $\bar{\phi}^{(j)}_{\lambda,\delta}$ maps a cuboid of x_1-length l_1 and x_j-length l_j onto one with x_1-length $\lambda^{-1} l_j$ and x_j-length λl_1. Additionally, we introduce a sequence of partial partitions η_n converging to the decomposition into points in subsection 3.6. These constructions are exhibited in such a way that $\Phi_n := \phi_n \circ R^{m_n}_{\alpha_{n+1}} \circ \phi_n^{-1}$ with a specific sequence $(m_n)_{n \in \mathbb{N}}$ of natural numbers (see section 4) satisfies the requirements of a criterion for weak mixing based on the notion of a $(\gamma, \delta, \epsilon)$-distribution. This criterion is stated in section 5 and is similar to the one deduced in [GKu]. In order to apply it, the map g_n shall introduce shear in the θ-direction. Therefore, we choose

$$g_n(\theta, r_1, ..., r_{m-1}) = (\theta + n \cdot q_n \cdot r_1, r_1, ..., r_{m-1}).$$

Moreover, Φ_n has to map each element of the partial partition η_n on a set of almost full length in the $r_1, ..., r_{m-1}$-coordinates in an almost uniform way. In order to produce such a mapping behaviour, there will be n different sections in a fundamental domain $\left[0, \frac{1}{q_n}\right] \times [0,1]^{m-1}$ with carefully chosen parameters λ_j of the map ϕ_n and shapes of partition elements in η_n. This can be described as an "adaptive version" of the approximation by conjugation-method and is the novelty in the constructions of [GKu].

In our case, the sequence of rational numbers will be

$$\alpha_{n+1} = \frac{p_{n+1}}{q_{n+1}} = \alpha_n - \frac{a_n}{q_n \cdot \tilde{q}_{n+1}},$$

where $a_n \in \mathbb{Z}$, $1 \leq a_n \leq q_n$ is chosen in such a way that $\tilde{q}_{n+1} \cdot p_n \equiv a_n \mod q_n$. Hereby, we have $|\alpha_{n+1} - \alpha_n| \leq \frac{1}{\tilde{q}_{n+1}}$ and $\tilde{q}_{n+1} \cdot \alpha_{n+1} = \frac{\tilde{q}_{n+1} \cdot p_n}{q_n} - \frac{a_n}{q_n} \equiv 0 \mod 1$, which implies $f_n^{\tilde{q}_{n+1}} = \mathrm{id}$. Hence, $(\tilde{q}_n)_{n \in \mathbb{N}}$ will be a uniform rigidity sequence of $f = \lim_{n \to \infty} f_n$ under some restrictions on the closeness between f_n and f (see subsection 6.3), which depend on the norms of the conjugation maps H_i and the distances $|\alpha_{i+1} - \alpha_i| \leq \frac{1}{\tilde{q}_{i+1}}$ for every $i > n$. In the course of the paper, we will face the following conditions:

$$q_{n+1} \geq n^2 \cdot q_n^{m \cdot n + 2}. \tag{A}$$

$$\tilde{q}_{n+1} \geq 2^n \cdot C_n \cdot q_n \cdot |||H_n|||_{n+1}^{n+1}. \tag{B}$$

$$\|DH_{n-1}\|_0 \leq \frac{q_n}{n^2} \tag{C}$$

Thus, we have to estimate the norms $|||H_n|||_{n+1}$ carefully. This will yield the subsequent requirement on the number $\tilde{q}_{n+1}$ (see the end of section 6.2):

$$\tilde{q}_{n+1} \geq \varphi(n) \cdot \tilde{q}_n^{2 \cdot m^2 \cdot (n+1)^{n+3}},$$

where $\varphi(n)$ is defined as above. This is a sufficient condition on the growth rate of the uniform rigidity sequence $(\tilde{q}_n)_{n \in \mathbb{N}}$ and we prove that f is weakly mixing using the before-mentioned criterion.

Since all the constructed diffeomporphisms coincide with the identity in a neighbourhood of the boundary, we can use these constructions on the torus $\mathbb{T}^m$ as well. In section 7 we transfer our constructions to the case of $\mathbb{D}^m$.

3. Explicit constructions

In the first subsections we aim for a measure-preserving diffeomorphism on $[-1,1]^m$ that coincides with the rotation by $\frac{\pi}{2}$ in the x_1-x_j-plane on $[-1+5\delta, 1-5\delta]^m$ and with the identity in a neighbourhood of the boundary. In [GKu], Lemma 3.6, we constructed such a pseudo-rotation $\varphi_{\delta,1,j}$ with the aid of "Moser's trick". Since we need precise norm estimates on the parameter δ, we have to find a new construction.

3.1. Bump functions

We use the smooth map

$$j(x) = \begin{cases} \exp\left(-\frac{1}{x^2}\right) & \text{for } x \geq 0 \\ 0 & \text{for } x < 0 \end{cases}$$

First of all, we find norm estimates for this function j:

Lemma 3.1. *For every $s \in \mathbb{N}$:*

$$\|j\|_s := \max_{t=0,1,\dots,s} \max_{x \in [0,1]} \left| j^{(t)}(x) \right| \leq 3^{2s} \cdot s^{1.5s} \cdot (s-1)!.$$

Proof. By direct calculation, see [Ku], Lemma 5.2. □

Using the map j we define the bump function

$$k_{a,b}(x) = \frac{j(b-x)}{j(x-a) + j(b-x)},$$

where $a, b \in (0,1)$. We examine this bump function $k_{a,b}$:

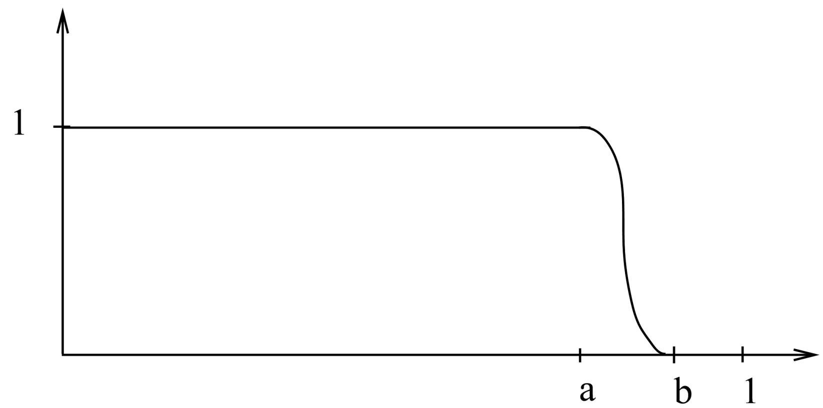

Figure 1: Qualitative shape of the bump function $k_{a,b}$

Lemma 3.2. *For every $s \in \mathbb{N}$:*

$$\|k_{a,b}\|_s \leq 2^{s-1} \cdot 3^{2s^2+2s} \cdot s^{1.5s^2+1.5s} \cdot s!^{s+2} \cdot \exp\left(\left(\frac{2}{b-a}\right)^2 \cdot (s+1)\right).$$

Proof. By direct calculation and induction arguments, see [Ku], Lemma 5.3. □

In our constructions we use $a = 1 - 3\delta$ and $b = 1 - 2\delta$. We denote the corresponding map by k_δ. In an analogous manner we define the map

$$\upsilon_{a,b,c,d}(x) = \frac{j(x-a)}{j(b-x) + j(x-a)} \cdot \frac{j(d-x)}{j(x-c) + j(d-x)}$$

The map υ_ε is introduced in case of $a = -1 + \varepsilon$, $b = -1 + 2\varepsilon$, $c = 1 - 2\varepsilon$ and $d = 1 - \varepsilon$. We find the same norm estimate.

3.2. The map $\psi_{\varepsilon,\delta,j}$

In case of $j \in \{2, ..., m\}$ we define the smooth diffeomorphism

$$\psi_{\varepsilon,\delta,j}(\theta, x_2, ..., x_{j-1}, r, x_{j+1}, ..., x_m)$$
$$= \left(\theta + \frac{\pi}{2} \cdot k_\delta(r) \cdot \upsilon_\varepsilon(x_2) \cdot ...\upsilon_\varepsilon(x_{j-1}) \cdot \upsilon_\varepsilon(x_{j+1}) \cdot ...\upsilon_\varepsilon(x_m), x_2, ..., x_{j-1}, r, x_{j+1}, ..., x_m\right)$$

We choose $\varepsilon = 2.5 \cdot \delta$ and denote the resulting map by $\psi_{\delta,j}$. As a direct consequence of the previous section we conclude:

Lemma 3.3. *For every $s \in \mathbb{N}$:*

$$|||\psi_{\delta,j}|||_s \leq \pi \cdot 2^{s-1} \cdot 3^{s^2+s} \cdot s^{1.5s^2+1.5s} \cdot s!^{s+2} \cdot \exp\left(\left(\frac{2}{\delta}\right)^2 \cdot (s+1)\right).$$

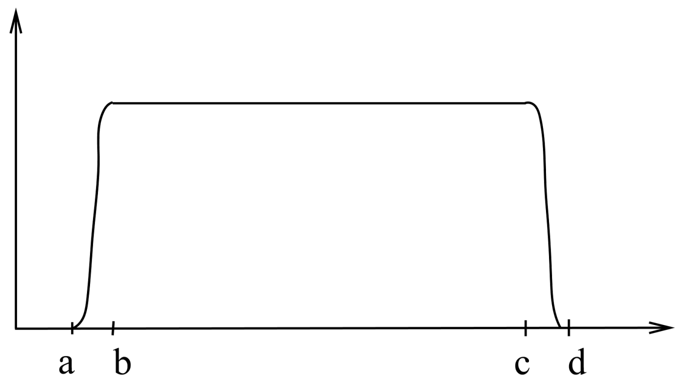

Figure 2: Qualitative shape of the bump function $v_{a,b,c,d}$

3.3. The map κ_δ

In the construction of our conjugation map φ_ε there is an angle-dependent dilation. In order to make this angle-dependence smooth we use the bump functions. We define the smooth map κ_δ:

- On $\left[0, \frac{\pi}{2}\right]$:

$$\kappa_\delta(\theta) = k_{\frac{\pi}{4}-\frac{\delta}{2},\frac{\pi}{4}+\frac{\delta}{2}}(\theta) \cdot \frac{1}{(\cos(\theta))^2} + \left(1 - k_{\frac{\pi}{4}-\frac{\delta}{2},\frac{\pi}{4}+\frac{\delta}{2}}(\theta)\right) \cdot \frac{1}{(\sin(\theta))^2}$$

- On $\left[\frac{\pi}{2}, \pi\right]$:

$$\kappa_\delta(\theta) = k_{\frac{3\pi}{4}-\frac{\delta}{2},\frac{3\pi}{4}+\frac{\delta}{2}}(\theta) \cdot \frac{1}{(\sin(\theta))^2} + \left(1 - k_{\frac{3\pi}{4}-\frac{\delta}{2},\frac{3\pi}{4}+\frac{\delta}{2}}(\theta)\right) \cdot \frac{1}{(\cos(\theta))^2}$$

- On $\left[\pi, \frac{3\cdot\pi}{2}\right]$:

$$\kappa_\delta(\theta) = k_{\frac{5\pi}{4}-\frac{\delta}{2},\frac{5\pi}{4}+\frac{\delta}{2}}(\theta) \cdot \frac{1}{(\cos(\theta))^2} + \left(1 - k_{\frac{5\pi}{4}-\frac{\delta}{2},\frac{5\pi}{4}+\frac{\delta}{2}}(\theta)\right) \cdot \frac{1}{(\sin(\theta))^2}$$

- On $\left[\frac{3\cdot\pi}{2}, 2\pi\right]$:

$$\kappa_\delta(\theta) = k_{\frac{7\pi}{4}-\frac{\delta}{2},\frac{7\pi}{4}+\frac{\delta}{2}}(\theta) \cdot \frac{1}{(\sin(\theta))^2} + \left(1 - k_{\frac{7\pi}{4}-\frac{\delta}{2},\frac{7\pi}{4}+\frac{\delta}{2}}(\theta)\right) \cdot \frac{1}{(\cos(\theta))^2}$$

Remark 3.4. We note: $\kappa_\delta\left(\theta + \frac{\pi}{2}\right) = \kappa_\delta(\theta)$.

Lemma 3.5. *For every $s \in \mathbb{N}$:*

$$\|\kappa_\delta\|_s \leq 2^{4s+2} \cdot 3^{2s^2+2s} \cdot s!^{s+3} \cdot s^{1.5s^2+1.5s} \cdot \exp\left(\frac{4}{\delta^2} \cdot (s+1)\right)$$

Proof. By direct calculation and induction arguments based on the quotient rule, see [Ku], Lemma 5.6. □

3.4. Map φ_δ

We consider the disc $\mathbb{D}^2$ equipped with symplectic polar coordinates (θ, r). For $r_1, r_2 \in (0,1)$ we define the map

$$\varphi_{r_1,r_2,\delta}(\theta, r) = \left(\theta, \kappa_\delta(\theta) \cdot r_1^2 + r - r_1\right) \text{ on } B(r_1, r_2),$$

where $B(r_1, r_2) = \{(\theta, r) : \theta \in \mathbb{R}/2\pi\mathbb{Z}, r \in [r_1, r_2]\}$. In our constructions we use $r_1 = 1 - 4\delta$ and $r_2 = 1 - \delta$. The corresponding map is called φ_δ.

3.5. Conjugation map ϕ_n

The coordinate change from symplectic polar coordinates to cartesian coordinates is given by:

$$P(\theta, r) = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \sqrt{r} \cdot \cos(\theta) \\ \sqrt{r} \cdot \sin(\theta) \end{pmatrix}$$

A direct computation yields $|\det(JP)| = \frac{1}{2}$ except at the origin.
With the aid of the maps introduced in the previous subsections we construct the smooth diffeomorphism ϕ_δ on $\mathbb{R}^2$ equipped with symplectic polar coordinates (θ, r):

$$\phi_\delta(\theta, r) = \begin{cases} \left(\theta + \frac{\pi}{2}, r\right) & \textit{inside of } \varphi_\delta\left(\mathbb{R}/2\pi\mathbb{Z} \times \{r_1\}\right) \\ \varphi_\delta \circ \psi_{\delta,2} \circ \varphi_\delta^{-1}(\theta, r) & \textit{on } \varphi_\delta\left(B(r_1, r_2)\right) \\ (\theta, r) & \textit{outside of } \varphi_\delta\left(\mathbb{R}/2\pi\mathbb{Z} \times \{r_2\}\right) \end{cases}$$

Recall that the domain $\varphi_\delta(B(r_1, r_2))$ is invariant under the rotation about arc $\frac{\pi}{2}$ due to Remark 3.4. By our choice of r_1 the map ϕ_δ is the rotation about the angle $\frac{\pi}{2}$ on $[-1+5\delta, 1-5\delta]^2$. Moreover, it coincides with the identity in a neighbourhood of the boundary of $[-1,1]^2$.
For $(\theta, \bar{r}) = \varphi_\delta(\theta, r_1)$ we have

$$\phi_\delta(\theta, \bar{r}) = \varphi_\delta \circ \psi_{\delta,2}(\theta, r_1) = \varphi_\delta\left(\theta + \frac{\pi}{2} \cdot k_\delta(r_1), r_1\right) = \left(\theta + \frac{\pi}{2}, \bar{r}\right)$$

and for $(\theta, \bar{r}) = \varphi_\delta(\theta, r_2)$ we have

$$\phi_\delta(\theta, \bar{r}) = \varphi_\delta \circ \psi_{\delta,2}(\theta, r_2) = \varphi_\delta\left(\theta + \frac{\pi}{2} \cdot k_\delta(r_2), r_2\right) = (\theta, \bar{r}).$$

Since $r_1 < a < b < r_2$ these equalities hold true on a neighbourhood of the points. Thus, ϕ_δ is a smooth diffeomorphism. Furthermore, ϕ_δ is measure-preserving because the maps φ_δ and $\psi_{\delta,2}$ are.

Lemma 3.6. *For every $s \in \mathbb{N}$:*

$$|||\phi_\delta|||_s \leq \pi^s \cdot 2^{4s^3+3s^2+3s+3} \cdot 3^{2s^4+4s^3+4s^2+2s} \cdot s!^{s^3+4s^2+4s+4} \cdot s^{1.5s^4+3s^3+3s^2+1.5s} \cdot \exp\left(\frac{4}{\delta^2} \cdot \left(s^3+2s^2+2s+1\right)\right)$$

Proof. With the aid of the chain rule and the previous norm estimates, see [Ku], Lemma 5.7. □

We examine the coordinate change P on $B(r_1, r_2)$:

Lemma 3.7. *For every* $s \in \mathbb{N}$*:*

$$\|P\|_{s,B(r_1,r_2)} \leq \frac{(2s-2)!}{(s-1)!} \cdot \frac{1}{2^{s-0.5}}$$

Proof. By direct calculation, see [Ku], Lemma 5.8. □

For the inverse $P^{-1}|_{P(B(r_1,r_2))}$ we have the subsequent estimate:

Lemma 3.8. *For every* $s \in \mathbb{N}$*:*

$$\left\|P^{-1}\right\|_{s,P(B(r_1,r_2))} \leq 2^{3s-2} \cdot (s-1)!$$

Proof. By calculation and induction arguments based on the quotient rule, see [Ku], Lemma 5.9. □

In higher dimension we define analogously in case of $j \in \{2, ..., m\}$:

$$\phi_\delta^{(j)}(\theta, x_2, ..., x_{j-1}, r, x_{j+1}, ..., x_m)$$
$$= \begin{cases} \left(\theta + \frac{\pi}{2}, x_2, ..., x_{j-1}, r, x_{j+1}, ..., x_m\right) & \textit{inside of } \varphi_\delta\left(\mathbb{R}/2\pi\mathbb{Z} \times \mathbb{R}^{j-2} \times \{r_1\} \times \mathbb{R}^{m-j}\right) \\ \varphi_\delta \circ \psi_{\delta,j} \circ \varphi_\delta^{-1}(\theta, x_2, ..., x_{j-1}, r, x_{j+1}, ..., x_m) & \textit{on } \varphi_\delta(B(r_1, r_2)) \\ (\theta, x_2, ..., x_{j-1}, r, x_{j+1}, ..., x_m) & \textit{outside of } \varphi_\delta\left(\mathbb{R}/2\pi\mathbb{Z} \times \mathbb{R}^{j-2} \times \{r_2\} \times \mathbb{R}^{m-j}\right) \end{cases}$$

where $B(r_1, r_2) = \{(\theta, x_2, ..., x_{j-1}, r, x_{j+1}, ..., x_m) : \theta \in \mathbb{R}/2\pi\mathbb{Z}, x_i \in \mathbb{R}, r \in (r_1, r_2)\}$. Again, we observe that $\phi_\delta^{(j)}$ is a smooth measure-preserving map which coincides with the rotation in the θ-x_j-plane in $[-1+5\delta, 1-5\delta]^m$ and with the identity in a neighbourhood of the boundary of $[-1,1]^m$.

In the next step we consider the measure-preserving map $\hat{\phi}_\delta^{(j)} := P \circ \phi_\delta^{(j)} \circ P^{-1}$, where the coordinate transformation P acts in the coordinates θ and x_j: Let $s \geq 2$. Lemma 2.6 yields for $\bar{\phi} := \phi_\delta^{(j)} \circ P^{-1}$:

$$\left\|\bar{\phi}\right\|_s \leq \frac{(m+s-1)!}{(m-1)!} \cdot \left\|\phi_\delta^{(j)}\right\|_s \cdot \left\|P^{-1}\right\|_{s,P(B(r_1,r_2))}^s.$$

Again using Lemma 2.6 we obtain

$$\begin{aligned}
\left\|\hat{\phi}_{\delta}^{(j)}\right\|_{s} &\leq \frac{(m+s-1)!}{(m-1)!}\cdot\|P\|_{s,B(r_1,r_2)}\cdot\|\bar{\phi}\|_{s}^{s}\\
&\leq \left(\frac{(m+s-1)!}{(m-1)!}\right)^{s+1}\cdot\|P\|_{s,B(r_1,r_2)}\cdot\left\|\phi_{\delta}^{(j)}\right\|_{s}^{s}\cdot\left\|P^{-1}\right\|_{s,P(B(r_1,r_2))}^{s^2}\\
&\leq \left(\frac{(m+s-1)!}{(m-1)!}\right)^{s+1}\cdot\frac{(2s-2)!}{(s-1)!}\cdot\pi^{s^2}\cdot 2^{4s^4+6s^3+s^2+2s+0.5}\cdot 3^{2s^5+4s^4+4s^3+2s^2}\cdot\\
&\qquad s!^{s^4+4s^3+4s^2+4s}\cdot s^{1.5s^5+3s^4+3s^3+1.5s^2}\cdot\exp\left(\frac{4}{\delta^2}\cdot\left(s^4+2s^3+2s^2+s\right)\right)\cdot(s-1)!^{s^2}\\
&\leq \left(\frac{(m+s-1)!}{(m-1)!}\right)^{s+1}\cdot\frac{(2s-2)!}{(s-1)!}\cdot\pi^{s^2}\cdot 2^{4s^4+6s^3+s^2+2s+0.5}\cdot 9^{s^5+2s^4+2s^3+s^2}\cdot\\
&\qquad s!^{s^4+4s^3+5s^2+4s}\cdot s^{1.5s^5+3s^4+3s^3+0.5s^2}\cdot\exp\left(\frac{4}{\delta^2}\cdot\left(s^4+2s^3+2s^2+s\right)\right)
\end{aligned}$$

Let S be a dilation by factor 2 and a translation such that $\tilde{\phi}_{\delta}^{(j)} := S^{-1}\circ\hat{\phi}_{\delta}^{(j)}\circ S$ is a measure-preserving diffeomorphism on $[0,1]^m$. Then we have

$$\left\|\tilde{\phi}_{\delta}^{(j)}\right\|_{s}\leq 2^{s-1}\cdot\left\|\hat{\phi}_{\delta}^{(j)}\right\|_{s}.$$

Since $2\leq s\leq s!$ and $9\leq\exp\left(\frac{1}{\delta^2}\right)$ we continue in the following manner:

$$\begin{aligned}
&\left\|\tilde{\phi}_{\delta}^{(j)}\right\|_{s}\\
&\leq\left(\frac{(m+s-1)!}{(m-1)!}\right)^{s+1}\cdot\frac{(2s-2)!}{(s-1)!}\cdot\pi^{s^2}\cdot s!^{1.5s^5+8s^4+13s^3+6.5s^2+6s+0.5}\cdot\exp\left(\frac{1}{\delta^2}\cdot\left(s^5+6s^4+10s^3+9s^2+4s\right)\right)
\end{aligned}$$

Due to $s\geq 2$ we have $1.5s^5+8s^4+13s^3+6.5s^2+6s+0.5\leq 10s^5$ as well as $s^5+6s^4+10s^3+9s^2+4s\leq 8s^5$. Thus, we proved the following statement:

Lemma 3.9. *For every $s\in\mathbb{N}$, $s\geq 2$:*

$$|||\tilde{\phi}_{\delta}^{(j)}|||_{s}\leq\left(\frac{(m+s-1)!}{(m-1)!}\right)^{s+1}\cdot\frac{(2s-2)!}{(s-1)!}\cdot\pi^{s^2}\cdot\left(s!\cdot\exp\left(\frac{1}{\delta^2}\right)\right)^{10\cdot s^5}$$

For $\lambda\in\mathbb{N}$ we use the map $C_{\lambda}(x_1,...,x_m)=(\lambda\cdot x_1,x_2,...,x_m)$. Hereby, we define the measure-preserving diffeomorphism

$$\bar{\phi}_{\lambda,\delta}^{(j)}=C_{\lambda}^{-1}\circ\tilde{\phi}_{\delta}^{(j)}\circ C_{\lambda}.$$

For the sake of convenience we use the notation:

$$\bar{\phi}_{\lambda}^{(j)}=\bar{\phi}_{\lambda,\frac{1}{20n}}^{(j)}.$$

Then we construct the conjugation map ϕ_n on the fundamental sector $\left[0, \frac{1}{q_n}\right] \times [0,1]^{m-1}$. On $\left[\frac{k}{n \cdot q_n}, \frac{k+1}{n \cdot q_n}\right] \times [0,1]^{m-1}$ in case of $k \in \mathbb{Z}$, $0 \leq k \leq n-1$:

$$\phi_n = \bar{\phi}^{(m)}_{n \cdot q_n^{2 \cdot (m-1) \cdot (k+1)}} \circ ... \circ \bar{\phi}^{(3)}_{n \cdot q_n^{2 \cdot 2 \cdot (k+1)}} \circ \bar{\phi}^{(2)}_{n \cdot q_n^{2 \cdot (k+1)}}$$

Since ϕ_n coincides with the identity in a neighbourhood of the boundary of each individual section, ϕ_n is a smooth map. It is extended to a diffeomorhism on $\mathbb{S}^1 \times [0,1]^{m-1}$ or $\mathbb{T}^m$ by the description $\phi_n \circ R_{\frac{1}{q_n}} = R_{\frac{1}{q_n}} \circ \phi_n$.

3.6. Partial partition η_n

Remark 3.10. For convenience we will use the notation $\prod_{i=2}^{m} [a_i, b_i]$ for $[a_2, b_2] \times ... \times [a_m, b_m]$

Initially, η_n will be constructed on the fundamental sector $\left[0, \frac{1}{q_n}\right] \times [0,1]^{m-1}$. For this purpose, we divide the fundamental sector in n sections:

- In case of $k \in \mathbb{N}$ and $0 \leq k \leq n-2$ on $\left[\frac{k}{n \cdot q_n}, \frac{k+1}{n \cdot q_n}\right] \times [0,1]^{m-1}$ the partial partition η_n consists of all multidimensional intervals of the following form:

$$\Bigg[\frac{k}{n \cdot q_n} + \frac{j_1^{(1)}}{n \cdot q_n^2} + ... + \frac{j_1^{(2 \cdot m \cdot (k+1)-1)}}{n \cdot q_n^{2 \cdot m \cdot (k+1)}} + \frac{1}{2n^2 \cdot q_n^{2 \cdot m \cdot (k+1)}},$$
$$\frac{k}{n \cdot q_n} + \frac{j_1^{(1)}}{n \cdot q_n^2} + ... + \frac{j_1^{(2 \cdot m \cdot (k+1)-1)} + 1}{n \cdot q_n^{2 \cdot m \cdot (k+1)}} - \frac{1}{2n^2 \cdot q_n^{2 \cdot m \cdot (k+1)}}\Bigg]$$
$$\times \prod_{i=2}^{m} \left[\frac{j_i^{(1)}}{q_n} + \frac{j_i^{(2)}}{q_n^2} + \frac{1}{2n \cdot q_n^2}, \frac{j_i^{(1)}}{q_n} + \frac{j_i^{(2)} + 1}{q_n^2} - \frac{1}{2n \cdot q_n^2}\right],$$

  where $j_1^{(l)} \in \mathbb{Z}$ and $\left\lceil \frac{q_n}{2n} \right\rceil \leq j_1^{(l)} \leq q_n - \left\lceil \frac{q_n}{2n} \right\rceil - 1$ for $l = 1, ..., 2 \cdot m \cdot (k+1) - 1$ as well as $j_i^{(l)} \in \mathbb{Z}$ and $\left\lceil \frac{q_n}{n} \right\rceil \leq j_i^{(l)} \leq q_n - \left\lceil \frac{q_n}{2n} \right\rceil - 1$ for $i = 2, ..., m$ and $l = 1, 2$.

- On $\left[\frac{n-1}{n \cdot q_n}, \frac{1}{q_n}\right] \times [0,1]^{m-1}$ there are no elements of the partial partition η_n.

As the image under R_{l/q_n} with $l \in \mathbb{Z}$ this partial partition of $\left[0, \frac{1}{q_n}\right] \times [0,1]^{m-1}$ is extended to a partial partition of $\mathbb{S}^1 \times [0,1]^{m-1}$ or $\mathbb{T}^m$.

Remark 3.11. By construction this sequence of partial partitions converges to the decomposition into points.

4. $(\gamma, \delta, \epsilon)$-distribution

We introduce the central notion of the criterion for weak mixing deduced in the next section:

Definition 4.1. Let $\Phi : M \to M$ be a diffeomorphism. We say Φ $(\gamma, \delta, \epsilon)$-distributes an element $\hat{I}$ of a partial partition if the following properties are satisfied:

- $\pi_{\vec{r}}\left(\Phi\left(\hat{I}\right)\right)$ is a $(m-1)$-dimensional interval J, i.e. $J = I_1 \times ... \times I_{m-1}$ with intervals $I_k \subseteq [0,1]$, and $1-\delta \leq \lambda(I_k) \leq 1$ for $k = 1, ..., m-1$. Here, $\pi_{\vec{r}}$ denotes the projection on the $(r_1, ..., r_{m-1})$-coordinates.
- $\Phi\left(\hat{I}\right)$ is contained in a set of the form $[c, c+\gamma] \times J$ for some $c \in \mathbb{S}^1$.
- For every $(m-1)$-dimensional interval $\tilde{J} \subseteq J$ it holds:

$$\left| \frac{\mu\left(\hat{I} \cap \Phi^{-1}\left(\mathbb{S}^1 \times \tilde{J}\right)\right)}{\mu\left(\hat{I}\right)} - \frac{\mu^{(m-1)}\left(\tilde{J}\right)}{\mu^{(m-1)}(J)} \right| \leq \epsilon \cdot \frac{\mu^{(m-1)}\left(\tilde{J}\right)}{\mu^{(m-1)}(J)},$$

  where $\mu^{(m-1)}$ is the Lebesgue measure on $[0,1]^{m-1}$.

In the next step we define the sequence of natural numbers $(m_n)_{n\in\mathbb{N}}$:

$$\begin{aligned} m_n &= \min\left\{ m \leq q_{n+1} \ : \ m \in \mathbb{N}, \ \inf_{k\in\mathbb{Z}} \left| m \cdot \frac{p_{n+1}}{q_{n+1}} - \frac{1}{n \cdot q_n} + \frac{k}{q_n} \right| \leq \frac{q_n}{q_{n+1}} \right\} \\ &= \min\left\{ m \leq q_{n+1} \ : \ m \in \mathbb{N}, \ \inf_{k\in\mathbb{Z}} \left| m \cdot \frac{q_n \cdot p_{n+1}}{q_{n+1}} - \frac{1}{n} + k \right| \leq \frac{q_n^2}{q_{n+1}} \right\} \end{aligned}$$

Lemma 4.2. *The set* $\left\{ m \leq q_{n+1} \ : \ m \in \mathbb{N}, \ \inf_{k\in\mathbb{Z}} \left| m \cdot \frac{q_n \cdot p_{n+1}}{q_{n+1}} - \frac{1}{n} + k \right| \leq \frac{q_n^2}{q_{n+1}} \right\}$ *is nonempty for every* $n \in \mathbb{N}$*, i.e.* m_n *exists.*

Proof. The number α_{n+1} was constructed by the rule $\frac{p_{n+1}}{q_{n+1}} = \frac{p_n}{q_n} - \frac{a_n}{q_n \cdot \tilde{q}_{n+1}}$, where $a_n \in \mathbb{Z}$, $1 \leq a_n \leq q_n$, i.e. $p_{n+1} = p_n \cdot \tilde{q}_{n+1} - a_n$ and $q_{n+1} = q_n \cdot \tilde{q}_{n+1}$. So $\frac{q_n \cdot p_{n+1}}{q_{n+1}} = \frac{p_{n+1}}{\tilde{q}_{n+1}}$ and the set $\left\{ j \cdot \frac{q_n \cdot p_{n+1}}{q_{n+1}} \ : \ j = 1, 2, ..., q_{n+1} \right\}$ contains $\frac{\tilde{q}_{n+1}}{\gcd(p_{n+1}, \tilde{q}_{n+1})}$ different equally distributed points on $\mathbb{S}^1$. Hence, there are at least $\frac{\tilde{q}_{n+1}}{q_n} = \frac{q_{n+1}}{q_n^2}$ different such points and so for every $x \in \mathbb{S}^1$ there is a $j \in \{1, ..., q_{n+1}\}$ such that

$$\inf_{k\in\mathbb{Z}} \left| x - j \cdot \frac{q_n \cdot p_{n+1}}{q_{n+1}} + k \right| \leq \frac{q_n^2}{q_{n+1}}.$$

In particular, this is true for $x = \frac{1}{n}$. □

Remark 4.3. We define

$$b_n = \left(m_n \cdot \frac{p_{n+1}}{q_{n+1}} - \frac{1}{n \cdot q_n} \right) \mod \frac{1}{q_n}$$

By the above construction of m_n it holds that $|b_n| \leq \frac{q_n}{q_{n+1}}$. Due to the before mentioned condition A we have $q_{n+1} \geq 8 \cdot n^2 \cdot q_n^{2n+1}$ particularly. Thus, we get:

$$|b_n| \leq \frac{1}{8 \cdot n^2 \cdot q_n^{2n}}.$$

Our constructions are done in such a way that the following property is satisfied:

Lemma 4.4. *The map* $\Phi_n := \phi_n \circ R^{m_n}_{\alpha_{n+1}} \circ \phi_n^{-1}$ *with the conjugating maps* ϕ_n *defined in section 3.5* $\left(\frac{1}{n\cdot q_n^{3m}}, \frac{1}{n}, \frac{1}{n}\right)$*-distributes the elements of the partition* η_n.

Proof. The proof is analogous to the one of [GKu], Lemma 4.5. We consider a partition element $\hat{I}_{n,k}$ on $\left[\frac{k}{n\cdot q_n}, \frac{k+1}{n\cdot q_n}\right] \times [0,1]^{m-1}$. When applying the map ϕ_n^{-1} we observe that this element is positioned in such a way that all the occuring maps $\left(\tilde{\phi}^{(j)}_{\delta}\right)^{-1}$ act as the respective rotations. Then we compute $\phi_n^{-1}\left(\hat{I}_{n,k}\right)$:

$$\left[v_1 + \frac{1}{2\cdot n^2\cdot q_n^{2\cdot(k+2)}}, v_1 + \frac{1}{n\cdot q_n^{2\cdot(k+2)}} - \frac{1}{2\cdot n^2\cdot q_n^{2\cdot(k+2)}}\right]$$
$$\times \prod_{i=2}^{m-1}\left[v_i + \frac{1}{2\cdot n\cdot q_n^{2\cdot(k+2)}}, v_i + \frac{1}{q_n^{2\cdot(k+2)}} - \frac{1}{2\cdot n\cdot q_n^{2\cdot(k+2)}}\right]$$
$$\times \left[v_m + \frac{1}{2n\cdot q_n^{2\cdot(k+1)}}, v_m + \frac{1}{q_n^{2\cdot(k+1)}} - \frac{1}{2n\cdot q_n^{2\cdot(k+1)}}\right],$$

where

$$v_1 = \frac{k}{n\cdot q_n} + \frac{j_1^{(1)}}{n\cdot q_n^2} + \dots + \frac{j_1^{(2k+1)}}{n\cdot q_n^{2\cdot(k+1)}} + \frac{j_2^{(1)}}{n\cdot q_n^{2\cdot(k+1)+1}} + \frac{j_2^{(2)}}{n\cdot q_n^{2\cdot(k+2)}}$$
$$v_i = 1 - \frac{j_1^{(2\cdot(i-1)\cdot(k+1))}}{q_n} - \dots - \frac{j_1^{(2\cdot i\cdot(k+1)-1)}}{q_n^{2\cdot(k+1)}} - \frac{j_{i+1}^{(1)}}{q_n^{2\cdot(k+1)+1}} - \frac{j_{i+1}^{(2)}+1}{q_n^{2\cdot(k+2)}}$$
$$v_m = 1 - \frac{j_1^{(2\cdot(m-1)\cdot(k+1))}}{q_n} - \dots - \frac{j_1^{(2\cdot m\cdot(k+1)-1)}+1}{q_n^{2\cdot(k+1)}}.$$

By our choice of the number m_n the subsequent application of $R^{m_n}_{\alpha_{n+1}}$ yields a translation by $\frac{1}{nq_n}$ modulo $\frac{1}{q_n}$ except for the "error term" b_n introduced in Remark 4.3. In particular, $R^{m_n}_{\alpha_{n+1}} \circ \phi^{-1}\left(\hat{I}_{n,k}\right)$ is positioned in another domain of definition of the map ϕ_n, namely $\phi_n = \bar{\phi}^{(m)}_{n\cdot q_n^{2\cdot(m-1)\cdot(k+2)}} \circ \dots \circ \bar{\phi}^{(3)}_{n\cdot q_n^{2\cdot 2\cdot(k+2)}} \circ \bar{\phi}^{(2)}_{n\cdot q_n^{2\cdot(k+2)}}$. With the aid of the bound on b_n from Remark 4.3 we can compute the image of $\hat{I}_{n,k}$ under Φ_n:

$$\left[v + \frac{1}{2n^2\cdot q_n^{2(m-1)\cdot(k+2)+2(k+1)}}, v + \frac{1}{nq_n^{2(m-1)\cdot(k+2)+2(k+1)}} - \frac{1}{2n^2\cdot q_n^{2(m-1)\cdot(k+2)+2(k+1)}}\right]$$
$$\times \left[\frac{1}{2n} + n\cdot q_n^{2\cdot(k+2)}\cdot b_n, 1 - \frac{1}{2\cdot n} + n\cdot q_n^{2\cdot(k+2)}\cdot b_n\right] \times \prod_{i=3}^{m}\left[\frac{1}{2n}, 1 - \frac{1}{2n}\right],$$

where

$$v=\frac{k+1}{n\cdot q_n}+\frac{j_1^{(1)}}{n\cdot q_n^2}+...+\frac{j_1^{(2\cdot(k+1)-1)}}{n\cdot q_n^{2\cdot(k+1)}}+\frac{j_2^{(1)}}{n\cdot q_n^{2\cdot(k+1)+1}}+\frac{j_2^{(2)}}{n\cdot q_n^{2\cdot(k+2)}}+\frac{j_1^{(2\cdot(k+1))}}{n\cdot q_n^{2\cdot(k+2)+1}}+...$$
$$+\frac{j_m^{(2)}}{n\cdot q_n^{2\cdot(m-1)\cdot(k+2)}}+\frac{j_1^{(2\cdot(m-1)\cdot(k+1))}}{n\cdot q_n^{2\cdot(m-1)\cdot(k+2)+1}}+...+\frac{j_1^{(2\cdot m\cdot(k+1)-1)}}{n\cdot q_n^{2\cdot(m-1)(k+2)+2\cdot(k+1)}}.$$

Thus, such a set $\Phi_n\left(\hat{I}_n\right)$ with $\hat{I}_n\in\eta_n$ has a θ-witdth of at most $\frac{1}{n\cdot q_n^{3m}}$. Moreover, we see that we can choose $\epsilon=0$ in the definition of a (γ,δ,ϵ)-distribution: With the notation $A_\theta\coloneqq\pi_\theta\left(\Phi_n\left(\hat{I}_n\right)\right)$ we have $\Phi_n\left(\hat{I}_n\right)=A_\theta\times J$ and so for every $(m-1)$-dimensional interval $\tilde{J}\subseteq J$:

$$\frac{\mu\left(\hat{I}_n\cap\Phi_n^{-1}\left(\mathbb{S}^1\times\tilde{J}\right)\right)}{\mu\left(\hat{I}_n\right)}=\frac{\mu\left(\Phi_n\left(\hat{I}_n\right)\cap\mathbb{S}^1\times\tilde{J}\right)}{\mu\left(\Phi_n\left(\hat{I}_n\right)\right)}=\frac{\tilde{\lambda}\left(A_\theta\right)\cdot\mu^{(m-1)}\left(\tilde{J}\right)}{\tilde{\lambda}\left(A_\theta\right)\cdot\mu^{(m-1)}\left(J\right)}=\frac{\mu^{(m-1)}\left(\tilde{J}\right)}{\mu^{(m-1)}\left(J\right)}$$

because Φ_n is measure-preserving. □

5. Criterion for weak mixing

In this section we will state a criterion for weak mixing on $M=\mathbb{S}^1\times[0,1]^{m-1}$ or $M=\mathbb{T}^m$ in the setting of the beforehand constructions. Its proof is analogous to the one in [GKu], section 6. The only difference occurs in comparison to Lemma 6.3. which in our case will be formulated in the subsequent way:

Lemma 5.1. *Consider the sequence of partial partitions $(\eta_n)_{n\in\mathbb{N}}$ constructed in section 3.6 and the diffeomorphisms $g_n(\theta,x_2,...,x_m)=(\theta+n\cdot q_n\cdot x_2,x_2,...,x_m)$. Furthermore, let $(H_n)_{n\in\mathbb{N}}$ be a sequence of measure-preserving smooth diffeomorphisms satisfying*

$$\|DH_{n-1}\|_0\leq\frac{q_n}{n^2}\tag{C}$$

for every $n\in\mathbb{N}$ and define the partial partitions $\nu_n=\left\{\Gamma_n=H_{n-1}\circ g_n\left(\hat{I}_n\right)\ :\ \hat{I}_n\in\eta_n\right\}$. Then we get $\nu_n\to\varepsilon$.

Proof. By construction $\eta_n=\left\{\hat{I}_n^i:i\in\Lambda_n\right\}$, where Λ_n is a countable set of indices. Because of $\eta_n\to\varepsilon$ it holds $\lim_{n\to\infty}\mu\left(\bigcup_{i\in\Lambda_n}\hat{I}_n^i\right)=1$. Since $H_{n-1}\circ g_n$ is measure-preserving, we conclude:

$$\lim_{n\to\infty}\mu\left(\bigcup_{i\in\Lambda_n}\Gamma_n^i\right)=\lim_{n\to\infty}\mu\left(\bigcup_{i\in\Lambda_n}H_{n-1}\circ g_n\left(\hat{I}_n^i\right)\right)=\lim_{n\to\infty}\mu\left(H_{n-1}\circ g_n\left(\bigcup_{i\in\Lambda_n}\hat{I}_n^i\right)\right)=1.$$

For any m-dimensional cube with sidelength l_n it holds: $\mathrm{diam}(W_n)=\sqrt{m}\cdot l_n$. Because every element of the partition η_n is contained in a cube of side length

$\frac{1}{q_n^2}$, it follows for every $i \in \Lambda_n$: $\operatorname{diam}\left(\hat{I}_n^i\right) \leq \sqrt{m} \cdot \frac{1}{q_n^2}$. Hence, for every $\Gamma_n^i = H_{n-1} \circ g_n\left(I_n^i\right)$ we observe:

$$\operatorname{diam}\left(\Gamma_n^i\right) \leq \|DH_{n-1}\|_0 \cdot \|Dg_n\|_0 \cdot \operatorname{diam}\left(\hat{I}_n^i\right) \leq \frac{q_n}{n^2} \cdot n \cdot q_n \cdot \frac{\sqrt{m}}{q_n^2} \leq \frac{\sqrt{m}}{n}.$$

We conclude $\lim_{n\to\infty}\operatorname{diam}\left(\Gamma_n^i\right) = 0$ and consequently $\nu_n \to \varepsilon$. □

Now we are able to formulate the aimed criterion for weak mixing.

Proposition 5.2 (Criterion for weak mixing)**.** *Let $f_n = H_n \circ R_{\alpha_{n+1}} \circ H_n^{-1}$ and the sequence $(m_n)_{n\in\mathbb{N}}$ be constructed as in the previous sections. Suppose additionally that $d_0\left(f^{m_n}, f_n^{m_n}\right) < \frac{1}{2^n}$ for every $n \in \mathbb{N}$, $\|DH_{n-1}\|_0 \leq \frac{q_n}{n^2}$ and that the limit $f = \lim_{n\to\infty} f_n$ exists.*
Then f is weakly mixing.

Proof. We just give a sketch of the proof which is analogous to the one of [GKu], Proposition 6.6.
As above, we consider the partial partitions $\nu_n = H_{n-1} \circ g_n(\eta_n)$ defined with the aid of η_n constructed in section 3.6. By Lemma 5.1 this sequence converges to the decomposition into points. In order to prove the weak mixing property of f it suffices to check that for every m-dimensional cube A and for every $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that for every $n \geq N$ and for every $\Gamma_n \in \nu_n$ we have

$$\left|\mu\left(\Gamma_n \cap f^{-m_n}(A)\right) - \mu(\Gamma_n) \cdot \mu(A)\right| \leq 3 \cdot \epsilon \cdot \mu(\Gamma_n) \cdot \mu(A). \tag{1}$$

Due to the proximity of f^{m_n} and $f_n^{m_n}$ it is enough to check (1) for f_n. Moreover, we consider m-dimensional cubes S_n of side length q_n^{-1} (instead of $q_n^{-\sigma}$ as in [GKu]) and observe for sets $C_n = H_{n-1}(S_n)$ that

$$\operatorname{diam}(C_n) \leq \|DH_{n-1}\|_0 \cdot \operatorname{diam}(S_n) \leq \frac{q_n^2}{n^2} \cdot \frac{\sqrt{m}}{q_n} \to 0 \text{ as } n \to \infty.$$

Thus, we can approximate any cube A by a countable disjoint union of sets $C_n = H_{n-1}(S_n)$ with given precision for n sufficiently large and so we can examine $\left|\mu\left(\Gamma_n \cap f_n^{-m_n}(C_n)\right) - \mu(\Gamma_n) \cdot \mu(C_n)\right|$ in order to check (1). Since $f_n^{m_n} = H_{n-1} \circ g_n \circ \Phi_n \circ g_n^{-1} \circ H_{n-1}^{-1}$ and g_n as well as H_{n-1} are measure-preserving, we get

$$\left|\mu\left(\Gamma_n \cap f_n^{-m_n}(C_n)\right) - \mu(\Gamma_n)\,\mu(C_n)\right| = \left|\mu\left(\hat{I}_n \cap \Phi_n^{-1} \circ g_n^{-1}(S_n)\right) - \mu\left(\hat{I}_n\right)\mu(S_n)\right|$$

with $\hat{I}_n \in \eta_n$. By Lemma 4.4 Φ_n $\left(\frac{1}{n \cdot q_n^{3m}}, \frac{1}{n}, \frac{1}{n}\right)$-distributes the elements of the partition η_n. Then a partition element is "almost uniformly distributed" under $g_n \circ \Phi_n$ on the whole manifold M due to the shear induced by g_n (see [GKu], Lemma 6.5, for a detailed proof of this fact). So $\left|\mu\left(\hat{I}_n \cap \Phi_n^{-1} \circ g_n^{-1}(S_n)\right) - \mu\left(\hat{I}_n\right) \cdot \mu(S_n)\right| \to 0$ as $n \to \infty$. □

Remark 5.3. In [GKu] it is demanded $\|DH_{n-1}\|_0 < \frac{\ln(q_n)}{n}$ instead of requirement C. We did this modification because the fulfilment of the original condition would lead to stricter requirements on the uniform rigidity sequence: In particular, it would require an exponential growth rate.

6. The case of $\mathbb{T}^m$ and $\mathbb{S}^1 \times [0,1]^{m-1}$

We aim for precise requirements on the growth rate of the uniform rigidity sequence to guarantee convergence of the sequence of diffeomorphisms $f_n = H_n \circ R_{\alpha_{n+1}} \circ H_n^{-1}$. For this purpose, we need norm estimates on the conjugation maps.

6.1. Properties of the conjugation maps

Lemma 6.1. *We have for every $s \in \mathbb{N}$, $s \geq 2$:*

$$|||\phi_n|||_s \leq \left(\frac{(m+s-1)!}{(m-1)!}\right)^{(m-1)\cdot(s+1)^2} \cdot \left(\frac{(2s-2)!}{(s-1)!} \cdot \pi^{s^2} \cdot \left(s! \cdot \exp\left(\frac{1}{\delta_n^2}\right)\right)^{10s^5}\right)^{(m-1)\cdot s} \cdot \left(n \cdot q_n^{m\cdot n}\right)^{(m-1)\cdot s^2}$$

Proof. Obviously, we have for $\bar{\phi}^{(j)}_{\lambda,\delta} = C_\lambda^{-1} \circ \tilde{\phi}^{(j)}_\delta \circ C_\lambda$:

$$|||\bar{\phi}^{(j)}_{\lambda,\delta}|||_s \leq \lambda^s \cdot |||\tilde{\phi}^{(j)}_\delta|||_s.$$

Lemma 2.8 yields

$$\begin{aligned} |||\phi_n|||_s &\leq \left(\frac{(m+s-1)!}{(m-1)!}\right)^{m-2} \cdot \left(\lambda_m^s \cdot |||\tilde{\phi}_\delta|||_s\right)^s \cdot \ldots \cdot \left(\lambda_2^s \cdot |||\tilde{\phi}_\delta|||_s\right)^s \\ &= \left(\frac{(m+s-1)!}{(m-1)!}\right)^{m-2} \cdot (\lambda_m \cdot \ldots \cdot \lambda_2)^{s^2} \cdot |||\tilde{\phi}_\delta|||_s^{(m-1)\cdot s}. \end{aligned}$$

By our explicit constructions in subsection 3.5 we obtain

$$\lambda_m \cdot \ldots \cdot \lambda_2 \leq n \cdot q_n^{2\cdot(m-1)\cdot n} \cdot n \cdot q_n^{2\cdot(m-2)\cdot n} \cdot \ldots \cdot n \cdot q_n^{2\cdot n} = n^{m-1} \cdot q_n^{2\cdot n \cdot \sum_{l=1}^{m-1} l} = \left(n \cdot q_n^{m\cdot n}\right)^{m-1}.$$

With the aid of Lemma 3.9 we conclude

$$\begin{aligned} &|||\phi_n|||_s \\ &\leq \left(\frac{(m+s-1)!}{(m-1)!}\right)^{m-2+(m-1)\cdot s\cdot(s+1)} \cdot \left(n \cdot q_n^{m\cdot n}\right)^{(m-1)\cdot s^2} \cdot \left(\frac{(2s-2)!}{(s-1)!} \cdot \pi^{s^2} \cdot \left(s! \cdot \exp\left(\frac{1}{\delta_n^2}\right)\right)^{10s^5}\right)^{(m-1)\cdot s}. \end{aligned}$$

□

As a direct consequence we conclude for the composition $h_n = g_n \circ \phi_n$:

Lemma 6.2. *We have for every $s \in \mathbb{N}$, $s \geq 2$:*

$$|||h_n|||_s \leq$$

$$2 \cdot \left(\frac{(m+s-1)!}{(m-1)!} \right)^{(m-1)\cdot(s+1)^2} \cdot \left(\frac{(2s-2)!}{(s-1)!} \cdot \pi^{s^2} \cdot \left(s! \cdot \exp\left(\frac{1}{\delta_n^2} \right) \right)^{10s^5} \right)^{(m-1)\cdot s} \cdot \left(n^2 \cdot q_n^{m\cdot n+1} \right)^{(m-1)\cdot s^2}.$$

Proof. At first, we estimate for the composition

$$|||h_n|||_s \leq 2 \cdot (nq_n)^s \cdot |||\phi_n|||_s = 2 \cdot n^s \cdot q_n^s \cdot |||\phi_n|||_s$$

We conclude with the aid of Lemma 6.1:

$$|||h_n|||_s \leq$$

$$2 \cdot \left(\frac{(m+s-1)!}{(m-1)!} \right)^{(m-1)\cdot(s+1)^2} \cdot \left(\frac{(2s-2)!}{(s-1)!} \cdot \pi^{s^2} \cdot \left(s! \cdot \exp\left(\frac{1}{\delta_n^2} \right) \right)^{10s^5} \right)^{(m-1)\cdot s} \cdot \left(n^2 \cdot q_n^{m\cdot n+1} \right)^{(m-1)\cdot s^2}.$$

□

Under another condition on the growth rate of the sequence $(q_n)_{n\in\mathbb{N}}$ we deduce a norm estimate on the conjugation map H_n:

Lemma 6.3. *Assume*

$$q_{n+1} \geq n^2 \cdot q_n^{m\cdot n+2}. \tag{A}$$

Then we have for every $s \in \mathbb{N}$, $s \geq 2$:

$$|||H_n|||_s \leq \varphi(s,n) \cdot \left(n^2 \cdot q_n^{m\cdot n+2} \right)^{(m-1)\cdot s^{n+1}},$$

at which $\varphi(s,n)$ is the expression

$$2^{n\cdot s^n} \cdot \left(\frac{(m+s-1)!}{(m-1)!} \right)^{m\cdot(s+1)^2\cdot n\cdot s^{n-1}} \cdot \left(\frac{(2s-2)!}{(s-1)!} \right)^{(m-1)\cdot n\cdot s^n} \cdot \pi^{(m-1)\cdot s^{2+n}\cdot n} \cdot \left(s! \cdot \exp\left(\frac{1}{\delta_n^2} \right) \right)^{(m-1)\cdot n\cdot 10\cdot s^{n+5}}.$$

Proof. We prove this result by induction on $n \in \mathbb{N}$:
Start $n = 1$: Lemma 6.2 yields the statement for $H_1 = h_1$.
Induction assumption: The claim holds true for $n \in \mathbb{N}$.
Induction step $n \to n+1$: We apply Lemma 2.8, Lemma 6.2 and the induction

assumption on the composition $H_{n+1} = H_n \circ h_{n+1}$:

$$
\begin{aligned}
& |||H_{n+1}|||_s \\
\leq & \frac{(m+s-1)!}{(m-1)!} \cdot |||H_n|||_s^s \cdot |||h_{n+1}|||_s^s \\
\leq & \frac{(m+s-1)!}{(m-1)!} \cdot 2^{n \cdot s^{n+1}} \cdot \left(\frac{(m+s-1)!}{(m-1)!}\right)^{m \cdot (s+1)^2 \cdot n \cdot s^n} \cdot \left(\frac{(2s-2)!}{(s-1)!}\right)^{(m-1) \cdot n \cdot s^{n+1}} \cdot \pi^{(m-1) \cdot s^{3+n} \cdot n} \\
& \cdot \left(s! \cdot \exp\left(\frac{1}{\delta_n^2}\right)\right)^{(m-1) \cdot n \cdot 10 \cdot s^{n+6}} \cdot q_{n+1}^{(m-1) \cdot s^{n+2}} \cdot 2^s \cdot \left(\frac{(m+s-1)!}{(m-1)!}\right)^{(m-1) \cdot (s+1)^2 \cdot s} \\
& \cdot \left(\frac{(2s-2)!}{(s-1)!} \cdot \pi^{s^2} \cdot \left(s! \cdot \exp\left(\frac{1}{\delta_{n+1}^2}\right)\right)^{10 s^5}\right)^{(m-1) \cdot s^2} \cdot \left((n+1)^2 \cdot q_{n+1}^{m \cdot (n+1)+1}\right)^{(m-1) \cdot s^3} \\
\leq & 2^{(n+1) \cdot s^{n+1}} \cdot \left(\frac{(m+s-1)!}{(m-1)!}\right)^{m \cdot (s+1)^2 \cdot (n+1) \cdot s^n} \cdot \left(\frac{(2s-2)!}{(s-1)!}\right)^{(m-1) \cdot (n+1) \cdot s^{n+1}} \cdot \pi^{(m-1) \cdot s^{3+n} \cdot (n+1)} \\
& \cdot \left(s! \cdot \exp\left(\frac{1}{\delta_{n+1}^2}\right)\right)^{(m-1) \cdot (n+1) \cdot 10 \cdot s^{n+6}} \cdot \left((n+1)^2 \cdot q_{n+1}^{m \cdot (n+1)+2}\right)^{(m-1) \cdot s^{n+2}}
\end{aligned}
$$

□

6.2. Proof of convergence of $(f_n)_{n \in \mathbb{N}}$ *in* $\mathrm{Diff}^{\infty}(M)$

For the proof of convergence of the sequence $(f_n)_{n \in \mathbb{N}}$ the next result is very useful:

Lemma 6.4. *Let* $k \in \mathbb{N}_0$ *and* h *be a* C^{k+1}*-diffeomorphism on* M*. Then we get for every* $\alpha, \beta \in \mathbb{R}$*:*

$$d_k\left(h \circ R_\alpha \circ h^{-1}, h \circ R_\beta \circ h^{-1}\right) \leq C_k \cdot |||h|||_{k+1}^{k+1} \cdot |\alpha - \beta|,$$

where $C_k = \frac{(m+k-1)!}{(m-1)!}$*.*

Indeed, this is a more precise statement than [FS], Lemma 5.6.

Proof. Let $i \in \{1, ..., m\}$ and $\vec{a} \in \mathbb{N}_0^m$ be a multiindex of order $|\vec{a}| = k$. Based on the observations in the proof of Lemma 2.6 the derivative $D_{\vec{a}}\left[h \circ R_\alpha \circ h^{-1}\right]_i$ consists of at most $\frac{(m+k-1)!}{(m-1)!}$ summands, where each summand is the product of one derivative of h of order at most k and at most k derivatives of h^{-1} of order at most k.

Furthermore, with the aid of the mean value theorem we can estimate for any multiindex $\vec{a} \in \mathbb{N}_0^2$ with $|\vec{a}| \leq k$ and $i \in \{1, ..., m\}$:

$$\left|D_{\vec{a}}\,[h]_i\left(R_\alpha \circ h^{-1}(x_1, ..., x_m)\right) - D_{\vec{a}}\,[h]_i\left(R_\beta \circ h^{-1}(x_1, ..., x_m)\right)\right| \leq |||h|||_{k+1} \cdot |\alpha - \beta|.$$

Since $\left(h \circ R_\alpha \circ h^{-1}\right)^{-1} = h \circ R_{-\alpha} \circ h^{-1}$ is of the same form, we obtain in conclusion:

$$\begin{aligned} d_k\left(h \circ R_\alpha \circ h^{-1}, h \circ R_\beta \circ h^{-1}\right) &\leq \frac{(m+k-1)!}{(m-1)!} \cdot |||h|||_{k+1} \cdot |||h|||_k^k \cdot |\alpha - \beta| \\ &\leq \frac{(m+k-1)!}{(m-1)!} \cdot |||h|||_{k+1}^{k+1} \cdot |\alpha - \beta| . \end{aligned}$$

□

With the aid of the subsequent lemma we are able to prove convergence of the sequence $(f_n)_{n\in\mathbb{N}}$ under a condition on the proximity of α_{n+1} and α_n:

Lemma 6.5. *We assume*

$$|\alpha_{n+1} - \alpha_n| \leq \frac{1}{2^n \cdot C_n \cdot q_n \cdot |||H_n|||_{n+1}^{n+1}}. \tag{B'}$$

Then the diffeomorphisms $f_n = H_n \circ R_{\alpha_{n+1}} \circ H_n^{-1}$ satisfy:

- *The sequence $(f_n)_{n\in\mathbb{N}}$ converges in the Diff$^\infty(M)$-topology to a measure-preserving diffeomorphism f.*
- *We have for every $n \in \mathbb{N}$ and $m \leq q_{n+1}$:*

$$d_0\left(f^m, f_n^m\right) < \frac{1}{2^n}.$$

Proof. Analogous to [Ku], Lemma 6.5. □

Since $|\alpha_{n+1} - \alpha_n| = \frac{a_n}{q_n \cdot \tilde{q}_{n+1}} \leq \frac{1}{\tilde{q}_{n+1}}$ this requirement B' can be met if we demand

$$\tilde{q}_{n+1} \geq 2^n \cdot C_n \cdot q_n \cdot |||H_n|||_{n+1}^{n+1}. \tag{B}$$

By Lemma 6.3 this condition is fulfilled under the requirement

$$\begin{aligned} \tilde{q}_{n+1} \geq & 2^n \cdot C_n \cdot q_n \cdot 2^{n\cdot(n+1)^{n+1}} \cdot \left(\frac{(m+n)!}{(m-1)!}\right)^{m\cdot(n+2)^2\cdot n\cdot(n+1)^n} \cdot \left(\frac{(2n)!}{n!}\right)^{(m-1)\cdot n\cdot(n+1)^{n+1}} \\ & \cdot \pi^{(m-1)\cdot(n+1)^{3+n}\cdot n} \cdot \left((n+1)! \cdot \exp\left(\frac{1}{\delta_n^2}\right)\right)^{(m-1)\cdot n\cdot 10\cdot(n+1)^{n+6}} \cdot \left(n^2 \cdot q_n^{m\cdot n+2}\right)^{(m-1)\cdot(n+1)^{n+2}} . \end{aligned}$$

Hereby, condition A is satisfied, too.
Using $q_n = q_{n-1} \cdot \tilde{q}_n < \tilde{q}_n^2$ we can fulfill the requirement if we demand

$$\tilde{q}_{n+1} \geq \varphi(n) \cdot \tilde{q}_n^{2\cdot m^2\cdot(n+1)^{n+3}},$$

at which $\varphi(n)$ is the expression (recall $\delta_n = \frac{1}{20n}$)

$$\left(\frac{(m+n)!}{(m-1)!}\right)^{m\cdot(n+2)^{n+3}} \cdot \left(\frac{(2n)!}{n!} \cdot \pi^{(n+1)^2} \cdot \left((n+1)! \cdot \exp\left(400n^2\right)\right)^{10\cdot(n+1)^5}\right)^{m\cdot(n+1)^{n+2}} \cdot n^{2\cdot(m-1)\cdot(n+1)^{n+2}}.$$

This condition is satisfied by the assumptions of Theorem 1. Hence, we can apply Lemma 6.5 and obtain convergence of the sequence $(f_n)_{n\in\mathbb{N}}$ in the $\mathrm{Diff}^{\infty}(M)$-topology to a measure-preserving diffeomorphism f. In the following subsections we will prove that f is the aimed diffeomorphism as asserted in Theorem 1, namely uniformly rigid with respect to $(\tilde{q}_n)_{n\in\mathbb{N}}$ and weakly mixing.

6.3. Proof of uniform rigidity along the sequence $(\tilde{q}_n)_{n\in\mathbb{N}}$

By definition $\tilde{q}_{n+1} \leq q_{n+1}$. Hence, the second statement of Lemma 6.5 implies $d_0\left(f_n^{\tilde{q}_{n+1}}, f^{\tilde{q}_{n+1}}\right) < \frac{1}{2^n}$. Since the number α_{n+1} was chosen in such a way that $f_n^{\tilde{q}_{n+1}} = \mathrm{id}$, we have $d_0\left(\mathrm{id}, f^{\tilde{q}_{n+1}}\right) < \frac{1}{2^n}$ which converges to zero as $n \to \infty$. Thus, $(\tilde{q}_n)_{n\in\mathbb{N}}$ is a uniform rigidity sequence of f.

6.4. Proof of weak mixing

In our criterion for weak mixing in Proposition 5.2 we need $\|DH_{n-1}\|_0 \leq \frac{q_n}{n^2}$. This condition is satisfied if we require condition B. Moreover, the required proximity $d_0\left(f^{m_n}, f_n^{m_n}\right) < \frac{1}{2^n}$ is fulfilled by Lemma 6.5 for the sequence $(m_n)_{n\in\mathbb{N}}$ introduced in section 4. Hence, we can apply the criterion for weak mixing deduced in section 5 and conclude that f is weakly mixing.

7. The case of $M = \mathbb{D}^m$

First of all, we introduce the coordinate change $J : \mathbb{S}^1 \times [0,1]^{m-1} \to \mathbb{D}^m$, $J(\theta, r_1, r_2, ..., r_{m-1}) = \vec{x}$, to m-dimensional polar coordinates:

$$\begin{aligned} x_1 &= r_1 \cdot \cos(\pi r_2) \\ x_i &= r_1 \cdot \prod_{j=2}^{i} \sin(\pi r_j) \cdot \cos(\pi r_{i+1}) \quad \text{for } i = 2, ..., m-2 \\ x_{m-1} &= r_1 \cdot \prod_{j=2}^{m-1} \sin(\pi r_j) \cdot \cos(2\pi\theta) \\ x_m &= r_1 \cdot \prod_{j=2}^{m-1} \sin(\pi r_j) \cdot \sin(2\pi\theta). \end{aligned}$$

Then we can define a sequence of smooth diffeomorphisms $\tilde{f}_n = J \circ f_n \circ J^{-1}$ on $\mathbb{D}^m \setminus \{(0,...,0)\}$, where f_n is constructed as in the previous section. Since these diffeomorphisms satisfy $f_n = R_{\alpha_{n+1}}$ on $\mathbb{S}^1 \times \left[0, \frac{1}{40n}\right]^{m-1}$, we observe for any $k \in \mathbb{N}$

$$d_k\left(\tilde{f}_n, \tilde{f}_{n-1}\right) \leq \frac{(m+k-1)!}{(m-1)!} \cdot |||J \circ H_n|||^{k+1}_{k+1, \mathbb{S}^1 \times \left[\frac{1}{40n}, 1\right]^{m-1}} \cdot |\alpha_{n+1} - \alpha_n| \,.$$

Under the condition $|\alpha_{n+1} - \alpha_n| < \frac{1}{2^n \cdot \frac{(m+n-1)!}{(m-1)!} \cdot q_n \cdot |||J \circ H_n|||^{n+1}_{n+1, \mathbb{S}^1 \times \left[\frac{1}{40n}, 1\right]^{m-1}}}$ we can prove convergence of the sequence $\left(\tilde{f}_n\right)_{n\in\mathbb{N}}$ in $\mathrm{Diff}^{\infty}(\mathbb{D}^m)$ as before and

the limit diffeomorphism $\tilde{f}$ can be extended to the origin smoothly. This diffeomorphism is weakly mixing with respect to the measure $J_*\mu$, where μ is the Lebesgue measure on $\mathbb{S}^1 \times [0,1]^{m-1}$ and $J_*\mu(A) = \mu\left(J^{-1}(A)\right)$ for any Lebesgue measurable set $A \subset \mathbb{D}^m$. By [AK], Theorem 1.2, there is a C^∞-diffeomorphism $G : \mathbb{D}^m \to \mathbb{D}^m$ such that $(G \circ J)_* \mu = G_* (J_*\mu) = \lambda$, where λ is the Lebesgue measure on $\mathbb{D}^m$. Hence, the diffeomorphism $G \circ \tilde{f} \circ G^{-1}$ is weakly mixing with respect to λ.
In order to find estimates on $|||J \circ H_n|||_{n+1, \mathbb{S}^1 \times \left[\frac{1}{40n}, 1\right]^{m-1}}$ we use the same techniques and estimates as in the previous sections. In particular, we have $\|J\|_{s, \mathbb{S}^1 \times [0,1]^{m-1}} = 1$ for every $s \in \mathbb{N}$. For the inverse transformation we deduce the subsequent norm estimate:

Lemma 7.1. *For any* $s \in \mathbb{N}$

$$\left\|J^{-1}\right\|_{s, J\left(\mathbb{S}^1 \times \left[\frac{1}{40n}, 1\right]^{m-1}\right)} \leq s! \cdot (40n)^{4sm}.$$

Proof. We have

$$J^{-1}(x_1, ..., x_m) = \begin{pmatrix} \frac{1}{2\pi} \arccos\left(\frac{x_{m-1}}{\sqrt{x_m^2 + x_{m-1}^2}}\right) \\ \sqrt{x_1^2 + ... + x_m^2} \\ \frac{1}{\pi} \arccos\left(\frac{x_1}{\sqrt{x_1^2 + ... + x_m^2}}\right) \\ \frac{1}{\pi} \arccos\left(\frac{x_2}{\sqrt{x_2^2 + ... + x_m^2}}\right) \\ \vdots \\ \frac{1}{\pi} \arccos\left(\frac{x_{m-2}}{\sqrt{x_{m-2}^2 + x_{m-1}^2 + x_m^2}}\right) \end{pmatrix} \text{ in case of } x_m \geq 0$$

and

$$J^{-1}(x_1, ..., x_m) = \begin{pmatrix} 1 - \frac{1}{2\pi} \arccos\left(\frac{x_{m-1}}{\sqrt{x_m^2 + x_{m-1}^2}}\right) \\ \sqrt{x_1^2 + ... + x_m^2} \\ \frac{1}{\pi} \arccos\left(\frac{x_1}{\sqrt{x_1^2 + ... + x_m^2}}\right) \\ \frac{1}{\pi} \arccos\left(\frac{x_2}{\sqrt{x_2^2 + ... + x_m^2}}\right) \\ \vdots \\ \frac{1}{\pi} \arccos\left(\frac{x_{m-2}}{\sqrt{x_{m-2}^2 + x_{m-1}^2 + x_m^2}}\right) \end{pmatrix} \text{ in case of } x_m < 0.$$

We examine the derivatives of $\arccos\left(\frac{x_1}{\sqrt{x_1^2 + ... + x_m^2}}\right)$. The first partial derivative with respect to x_i in case of $i = 2, ..., m$ is $\frac{x_1 \cdot x_i}{\left(x_1^2 + ... + x_m^2\right) \cdot \sqrt{x_2^2 + ... + x_m^2}}$. The further derivatives are found with the aid of the quotient rule. For this purpose, we consider

$$\varphi_s(x_1,...,x_m) = \frac{P_s(x_1,...,x_m)}{(x_1^2+...+x_m^2)^{n_s} \cdot \sqrt{x_2^2+...+x_m^2}^{b_s}},$$

where P_s is a polynomial of degree d_s with z_s summands. With the aid of the quotient rule we see that the partial derivative of $\frac{P_s(x_1,...,x_m)}{\sqrt{x_2^2+...+x_m^2}^{b_s}}$ with respect to x_i is of the form

$$\frac{\tilde{P}_s(x_1,...,x_m)}{\sqrt{x_2^2+...+x_m^2}^{b_s+2}}, \text{ where } \tilde{P}_{s+1}(x_1,...,x_m) = \frac{\partial P_s}{\partial x_i}(x_1,...,x_m)\cdot(x_2^2+...+x_m^2) - P_s(x_1,...,x_m)\cdot b_s \cdot x_i$$

is a polynomial of degree d_s+1 with at most $d_s \cdot z_s \cdot (m-1) + b_s \cdot z_s$ summands. Then the quotient rule yields

$$\frac{\partial \varphi_s}{\partial x_i}(x_1,...,x_m) = \frac{\tilde{P}_{s+1}(x_1,...,x_m)\cdot(x_1^2+...+x_m^2) - P_s(x_1,...,x_m)\cdot 2n_s \cdot x_i \cdot (x_2^2+...+x_m^2)}{(x_1^2+...+x_m^2)^{n_s+1} \cdot \sqrt{x_2^2+...+x_m^2}^{b_s+2}}.$$

Hence, P_{s+1} is a polynomial of degree d_s+3 with at most $(d_s z_s \cdot (m-1) + b_s z_s)\cdot m + z_s \cdot 2n_s \cdot (m-1)$ summands. Since $d_1 = 2$, $b_1 = 1$ and $n_1 = 1$ we get $d_s = 3s-1$, $b_s = 2s-1$ and $n_s = s$. Hereby, we have $z_{s+1} \leq z_s \cdot s \cdot m \cdot (3m+1)$. By $z_1 = 1$ this implies $z_s \leq (s-1)! \cdot m^{s-1} \cdot (3m+1)^{s-1}$.
Analogously, we consider the partial derivative of an expression of the form

$$\frac{P_s(x_1,...,x_m)}{\sqrt{x_2^2+...+x_m^2}^{b_s} \cdot (x_1^2+...+x_m^2)^{n_s}}$$

with respect to x_1 (note that in case of the first partial derivative with respect to x_1 we have $b_1 = -1$):

$$\frac{\frac{\partial P_s}{\partial x_1}(x_1,...,x_m)\cdot\left(x_1^2+...+x_m^2\right) - P_s(x_1,...,x_m)\cdot 2x_1 \cdot n_s}{\sqrt{x_2^2+...+x_m^2}^{b_s} \cdot (x_1^2+...+x_m^2)^{n_s+1}}.$$

Hence, P_{s+1} is a polynomial of degree d_s+1 with at most $z_s \cdot (d_s m + 2n_s)$ summands. We get $d_s \leq s+2$, $n_s = s$ and $z_s \leq 2^{s-1} \cdot s! \cdot m^{s-1}$.

Altogether, we conclude an estimate for the derivative of order s of the following form

$$\frac{s! \cdot m^{s-1} \cdot (3m+1)^{s-1}}{(x_1^2+...+x_m^2)^{s} \cdot \sqrt{x_2^2+...+x_m^2}^{2s-1}}.$$

Additionally, we observe on $J\left(\mathbb{S}^1 \times \left[\frac{1}{40n}, 1\right]^{m-1}\right)$

$$x_{m-k}^2 + ... + x_m^2 = r_1^2 \cdot \prod_{j=2}^{m-k} \sin^2(\pi r_j) \geq \left(\frac{1}{40n}\right)^{2(m-k)}.$$

Since

$$s!\cdot k^{s-1}\cdot(3k+1)^{s-1}\cdot(40n)^{2s(m-k)}\cdot(40n)^{(2s-1)\cdot(m-k+1)}\leq s!\cdot(40n)^{4sm}$$

we obtain

$$\left\|J^{-1}\right\|_{s,J\left(\mathbb{S}^1\times\left[\frac{1}{40n},1\right]^{m-1}\right)}\leq s!\cdot(40n)^{4sm}.$$

□

With the aid of Lemma 2.7 and Lemma 6.3 we have

$$\begin{aligned}
&2^n\cdot\frac{(m+n-1)!}{(m-1)!}\cdot q_n\cdot\|J\circ H_n\|_{n+1,\mathbb{S}^1\times\left[\frac{1}{40n},1\right]^{m-1}}\\
\leq&2^n\cdot\frac{(m+n-1)!}{(m-1)!}\cdot q_n\cdot\frac{(m+n)!}{(m-1)!}\cdot\|J\|_{n+1,\mathbb{S}^1\times\left[\frac{1}{40n},1\right]^{m-1}}\cdot\|H_n\|_{n+1}^{n+1}\\
\leq&2^n\cdot\frac{(m+n-1)!}{(m-1)!}\cdot q_n\cdot\frac{(m+n)!}{(m-1)!}\cdot 2^{n\cdot(n+1)^{n+1}}\cdot\left(\frac{(m+n)!}{(m-1)!}\right)^{m\cdot(n+2)^2\cdot n\cdot(n+1)^n}\cdot\left(\frac{(2n)!}{n!}\right)^{(m-1)\cdot n\cdot(n+1)^{n+1}}\\
&\cdot\pi^{(m-1)\cdot(n+1)^{3+n}\cdot n}\cdot\left((n+1)!\cdot\exp\left(400n^2\right)\right)^{(m-1)\cdot n\cdot 10\cdot(n+1)^{n+6}}\cdot\left(n^2\cdot q_n^{m\cdot n+2}\right)^{(m-1)\cdot(n+1)^{n+2}}
\end{aligned}$$

as well as

$$\begin{aligned}
&2^n\cdot\frac{(m+n-1)!}{(m-1)!}\cdot q_n\cdot\left\|H_n^{-1}\circ J^{-1}\right\|_{n+1,J\left(\mathbb{S}^1\times\left[\frac{1}{40n},1\right]^{m-1}\right)}\\
\leq&2^n\cdot\frac{(m+n-1)!}{(m-1)!}\cdot q_n\cdot\frac{(m+n)!}{(m-1)!}\cdot|||H_n|||_{n+1}\cdot\left\|J^{-1}\right\|^{n+1}_{n+1,J\left(\mathbb{S}^1\times\left[\frac{1}{40n},1\right]^{m-1}\right)}\\
\leq&2^n\cdot\frac{(m+n-1)!}{(m-1)!}\cdot q_n\cdot\frac{(m+n)!}{(m-1)!}\cdot 2^{n\cdot(n+1)^{n}}\cdot\left(\frac{(m+n)!}{(m-1)!}\right)^{m\cdot(n+2)^2\cdot n\cdot(n+1)^{n-1}}\cdot\left(\frac{(2n)!}{n!}\right)^{(m-1)\cdot n\cdot(n+1)^{n}}\\
&\cdot\pi^{(m-1)\cdot(n+1)^{2+n}\cdot n}\cdot\left((n+1)!\cdot\exp\left(400n^2\right)\right)^{(m-1)\cdot n\cdot 10\cdot(n+1)^{n+5}}\cdot\left(n^2\cdot q_n^{m\cdot n+2}\right)^{(m-1)\cdot(n+1)^{n+1}}\\
&\cdot(n+1)!^{n+1}\cdot(40n)^{4(n+1)^2m}.
\end{aligned}$$

By the same arguments as above we find the sufficient condition on the growth rate

$$\tilde{q}_{n+1}\geq\varphi(n)\cdot\tilde{q}_n^{2\cdot m^2\cdot(n+1)^{n+3}}.$$

Since this condition is fulfilled due to our assumptions of Theorem 1, we obtain convergence of the sequence $\left(\tilde{f}_n\right)_{n\in\mathbb{N}}$ in $\text{Diff}^\infty\left(\mathbb{D}^m\right)$ to a limit diffeomorphism $\tilde{f}$. As argued above, $G\circ\tilde{f}\circ G^{-1}$ is weakly mixing with respect to the Lebesgue measure on $\mathbb{D}^m$ and uniformly rigid with respect to $\left(\tilde{q}_n\right)_{n\in\mathbb{N}}$. Hence, Theorem 1 is also proven in the case of the disc $\mathbb{D}^m$.

8. Proof of Corollary 2

In order to prove Corollary 2 we only need the proximity

$$d_k\left(f_n,f_{n-1}\right)\leq C_k\cdot|||H_n|||_{k+1}^{k+1}\cdot|\alpha_{n+1}-\alpha_n|<\frac{1}{2^n},$$

which is satisfied if we demand

$$\tilde{q}_{n+1} \geq 2^n \cdot \frac{(m+k)!}{(m-1)!} \cdot q_n \cdot |||H_n|||_{k+1}^{k+1}. \tag{2}$$

We find a new norm estimate $|||H_n|||_{k+1}^{k+1}$: Since $\tilde{q}_n \leq q_n$ we estimate with the aid of Lemma 2.8, equation 2 and Lemma 6.2

$$|||H_n|||_{k+1} = |||H_{n-1} \circ h_n|||_{k+1} \leq \frac{(m+k)!}{(m-1)!} \cdot |||H_{n-1}|||_{k+1}^{k+1} \cdot |||h_n|||_{k+1}^{k+1} \leq q_n \cdot |||h_n|||_{k+1}^{k+1}$$

$$\leq q_n \cdot 2^{k+1} \cdot \left(\frac{(m+k)!}{(m-1)!}\right)^{(m-1)\cdot(k+2)^2\cdot(k+1)} \cdot \left(\frac{(2k)!}{k!} \cdot \pi^{(k+1)^2} \cdot \left((k+1)! \cdot \exp\left(\frac{1}{\delta_n^2}\right)\right)^{10\cdot(k+1)^5}\right)^{(m-1)\cdot(k+1)^2}$$
$$\cdot n^{2\cdot(m-1)\cdot(k+1)^3} \cdot q_n^{(m\cdot n+1)\cdot(m-1)\cdot(k+1)^3}$$

By equation 2 we conclude the requirement

$$\tilde{q}_{n+1} \geq \left(\frac{(m+k)!}{(m-1)!}\right)^{m\cdot(k+2)^4} \cdot \left(\frac{(2k)!}{k!} \cdot \pi^{(k+1)^2} \left((k+1)! \cdot \exp\left(\frac{1}{\delta_n^2}\right)\right)^{10\cdot(k+1)^5}\right)^{m\cdot(k+1)^3}$$
$$\cdot n^{2\cdot(m-1)\cdot(k+1)^4} \cdot q_n^{m^2\cdot(n+1)\cdot(k+1)^4}$$

Due to $q_n < \tilde{q}_n^2$ the condition from Corollary 2 is sufficient. □

9. Proof of Corollary 1

We recall the assumptions $\tilde{q}_1 \geq m^2 \cdot 2^8 \cdot \exp(400)$ and $\tilde{q}_{n+1} \geq \tilde{q}_n^{\tilde{q}_n}$ on the sequence $(\tilde{q}_n)_{n\in\mathbb{N}}$.

Claim: Under these assumptions the numbers $\tilde{q}_n$ satisfy $\tilde{q}_n \geq m^2 \cdot (n+1)^{n+7} \cdot \exp\left(400n^2\right)$.

Proof with the aid of complete induction:

- *Start $n = 1$:* $\tilde{q}_1 \geq m^2 \cdot 2^8 \cdot \exp(400) = m^2 \cdot (1+1)^{1+7} \cdot \exp(400)$
- *Assumption:* The claim is true for $n \in \mathbb{N}$.
- *Induction step $n \to n+1$:* We calculate

$$\begin{aligned}\tilde{q}_{n+1} &\geq \tilde{q}_n^{\tilde{q}_n} \geq \left(m^2 \cdot (n+1)^{n+7} \cdot \exp\left(400n^2\right)\right)^{m^2\cdot(n+1)^{n+7}} \\ &\geq m^2 \cdot (n+1)^{(n+7)\cdot m^2\cdot(n+1)^{n+7}} \cdot \exp\left(400n^2 \cdot m^2 \cdot (n+1)^{n+7}\right) \\ &\geq m^2 \cdot (n+2)^{n+8} \cdot \exp\left(400 \cdot (n+1)^2\right)\end{aligned}$$

  using the relation $(n+1)^{m^2} \geq n+2$ in the last step.

Hereby, we have due to $\exp\left(400n^2\right) \geq 14$:

$$\begin{aligned}
\tilde{q}_{n+1} \geq & \tilde{q}_n^{\tilde{q}_n} \geq \tilde{q}_n^{14\cdot m^2\cdot(n+1)^{n+7}} = \tilde{q}_n^{2\cdot m^2\cdot(n+1)^{n+7}} \cdot \tilde{q}_n^{12\cdot m^2\cdot(n+1)^{n+7}} \\
\geq & \tilde{q}_n^{2\cdot m^2\cdot(n+1)^{n+7}} \cdot \left(m^2 \cdot (n+1)^{n+7} \cdot \exp\left(400n^2\right)\right)^{12\cdot m^2\cdot(n+1)^{n+7}} \\
\geq & \tilde{q}_n^{2\cdot m^2\cdot(n+1)^{n+7}} \cdot (n+1)^{(n+7)\cdot 10\cdot m^2\cdot(n+1)^{n+7}} \cdot \exp\left(400n^2\right)^{10\cdot m^2\cdot(n+1)^{n+7}} \cdot (mn+m)^{2\cdot m^2\cdot(n+1)^{n+7}} \\
& \cdot (m\cdot(n+1))^{2\cdot m^2\cdot(n+1)^{n+7}} \cdot (n+1)^{2\cdot m^2\cdot(n+1)^{n+7}} \cdot \exp\left(400n^2\right)^{2\cdot m^2\cdot(n+1)^{n+7}} \\
\geq & \tilde{q}_n^{2\cdot m^2\cdot(n+1)^{n+7}} \cdot ((n+1)!)^{10\cdot m^2\cdot(n+1)^{n+7}} \cdot \exp\left(400n^2\right)^{10\cdot m^2\cdot(n+1)^{n+7}} \cdot \left(\frac{(m+n)!}{(m-1)!}\right)^{2\cdot m^2\cdot(n+1)^{n+6}} \\
& \cdot \left(\frac{(2n)!}{n!}\right)^{2\cdot m^2\cdot(n+1)^{n+6}} \cdot n^{2\cdot m^2\cdot(n+1)^{n+6}} \cdot \pi^{m^2\cdot(n+1)^{n+4}} \\
\geq & \varphi_1(n) \cdot \tilde{q}_n^{2\cdot m^2\cdot(n+1)^{n+3}}.
\end{aligned}$$

Hence, the requirement of the Theorem is met. □

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