On the height conjecture for algebraic points on curves defined over number fields

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Abstract

We study the basic height conjecture for points on curves defined over number fields and show: On any algebraic curve defined over a number field the set of algebraic points contains an unrestricted subset of infinite cardinality such that for all of its points their canonical height is bounded in terms of a small power of their root discriminant. In addition, if we assume GRH, then the upper bound is, as it is conjectured, linear in the logarithm of the root discriminant.

1 Introduction

Let X be a smooth projective curve defined over a number field. Then we have the Arakelov height function with respect to the metrized canonical bundle

$$\mathrm{ht}_{\overline{\omega}} : X(\overline{\mathbb{Q}}) \longrightarrow \mathbb{R},$$

whose definition will be given in the main text below, and the logarithmic root discriminant

$$\mathrm{disc} : X(\overline{\mathbb{Q}}) \longrightarrow \mathbb{R}.$$

For the later map we associate to a point $P \in X(\overline{\mathbb{Q}})$ the number field $k(P)$ and we set $\mathrm{disc}(P) = \log(\Delta_{k(P)})$. Here $\Delta_K = |D_{K/Q}|^{1/[K:\mathbb{Q}]}$ denotes the root discriminant of a number field K. The above two maps are conjecturally related as follows.

1.1. Conjecture. *Let X be a smooth projective curve defined over a number field. Let $\varepsilon > 0$, then there exists a constant $C(X, \varepsilon)$ such that for P varying over all algebraic points of X we have*

$$\mathrm{ht}_{\overline{\omega}}(P) \leq (1 + \varepsilon)\,\mathrm{disc}(P) + C(X, \varepsilon).$$

This conjectural height inequality is special case of Vojta's conjectures [La] and also refered to as effective Mordell theorem [MB]. We remark that this conjecture is equivalent to a uniform *abc*-conjecture for all number fields [Fr]. For a long list describing the relations of the abc conjecture to other conjectures in arithmetic geometry and analytic number theory we refer to [Go] and [Ni].

A subset $\mathcal{V} \subseteq X(\overline{\mathbb{Q}})$ is called unrestricted if for all $d, r > 0$ the cardinality of the set $\mathcal{V}_{d,r} = \{P \in \mathcal{V} \,|\, [k(P) : \mathbb{Q}] \geq d,\, \mathrm{disc}(P) \geq r\}$ is infinite. The purpose of this note is to show the following theorem.

1.2. Theorem. *Let X be a smooth projective curve of genus $g \geq 2$ defined over a number field. Let $\varepsilon, \delta > 0$, then there exists an unrestricted subset $\mathcal{V} \subseteq X(\overline{\mathbb{Q}})$ and a constant $C(X, \varepsilon, \delta, \mathcal{V})$ such that for all $P \in \mathcal{V}$ we have*

$$\mathrm{ht}_{\overline{\omega}}(P) \leq \varepsilon \exp(\delta\, \mathrm{disc}(P)) + C(X, \varepsilon, \delta, \mathcal{V}). \tag{1.2.1}$$

If in addition the Dirichlet series $L(\chi_D, s)$ for the characters $\left(\frac{D}{\cdot}\right)$, where D is a negative prime number, have no zeros in a ball of radius $1/4$ around 0, then we have for all $P \in \mathcal{V}$

$$\mathrm{ht}_{\overline{\omega}}(P) \leq \varepsilon\, \mathrm{disc}(P) + C(X, \varepsilon, \mathcal{V}). \tag{1.2.2}$$

Finally we like to mention that our results only hold for an infinite subset of $X(\overline{\mathbb{Q}})$ and the method of proof seems not to be general enough to cover all algebraic points simultaneously.

2 Heights

The height of an algebraic point P on a smooth projective curve defined over a number field K can be defined by means of Arakelov theory as follows.

Let $\pi : \mathcal{X} \to \mathrm{Spec}\, \mathcal{O}_K$ be a regular model for X over the ring of integers $\mathcal{O}_K$ of K, i.e. $\mathcal{X}$ is a projective, regular scheme flat over $\mathrm{Spec}\, \mathcal{O}_K$. In this note a hermitian line bundle $\overline{\mathcal{L}} = (\mathcal{L}, \|\cdot\|)$ on $\mathcal{X}$ is a line bundle on $\mathcal{X}$ together with a continuous hermitian metric on the induced complex line bundle $\mathcal{L}_\infty$ over the complex manifold $\mathcal{X}_\infty = \prod_{\sigma: K \to \mathbb{C}} \mathcal{X}_\sigma(\mathbb{C})$. A particular hermitian line bundle is the canonical bundle equipped with the Arakelov metric. We denoted this distinguished hermitian line bundle by $\overline{\omega}$, see e.g. [La].

In the sequel we also allow that the metric associated with $\overline{\mathcal{L}}$ has logarithmic singularities at a finite set $\mathcal{S}$ of algebraic points on $\mathcal{X}(\overline{\mathbb{Q}})$ of the following type: near a singular point P any section l of $\mathcal{L}$ has an expansion in a local coordinate t

$$\|l\|(t) = |t|^{\mathrm{ord}_P(l)} \phi(t) (-\log |t|)^\alpha,$$

where $\phi(t)$ is a continuous non-vanishing function and $\alpha \in \mathbb{R}$. If $\alpha > 0$ for all singular points P, then the metric is called a positive logarithmically singular metric.

Let P be an algebraic point on X and $\overline{\mathcal{L}}$ be a hermitian line bundle. After possibly replacing K by a finite extension, we may assume that the algebraic point P, $\mathcal{S}$ and X are all defined over K. Since the arithmetic surface $\mathcal{X}$ is proper, we have $\mathcal{X}(K) = \mathcal{X}(\mathcal{O}_K)$. Therefore the Zariski closure $\mathcal{P}$ of P in $\mathcal{X}$ determines a section $s_P : \operatorname{Spec} \mathcal{O}_K \to \mathcal{X}$. With the above notation we define the height of a point $P \in X(K) \setminus \mathcal{S}$ with respect to $\overline{\mathcal{L}}$ by

$$\mathrm{ht}_{\overline{\mathcal{L}}}(P) = \frac{1}{[K:\mathbb{Q}]}\Big(\log \#(s_P^*\mathcal{L}/(s_P^* l)) - \sum_{\sigma:K\to\mathbb{C}} \log \|l\|(P^\sigma)\Big),$$

here l is a regular section of $\mathcal{L}$ which is non zero at P. Observe the height does not depend on the choice of l nor of K. If we denote by p a local equation for $\mathcal{P}$, then we have an equality

$$\log \#(s_P^*\mathcal{L}/(s_P^* l)) = \sum_{x\in\mathcal{X}} \log \#(\mathcal{O}_{\mathcal{X},x}/(p,l))$$

The above quantity is also denoted by $(\mathcal{P}, \mathrm{div}(l))_{\mathrm{fin}}$ and there are only finitely many $x \in \mathcal{X}$ that give non zero contribution to $(\mathcal{P}, \mathrm{div}(l))_{\mathrm{fin}}$.

We will need the following basic facts on heights.

2.1. Proposition. *Let $\overline{\mathcal{L}}$, $\overline{\mathcal{M}}$ be hermitian line bundles on $\mathcal{X}$. Assume* $\deg(\mathcal{L}) = \deg(\mathcal{M}) > 0$. *If the metric on $\mathcal{L}$ is continuous and the metric on $\mathcal{M}$ is positive logarithmically singular metric, then for all $\varepsilon > 0$ we can find a constant $C(\varepsilon, \mathcal{X}, \overline{\mathcal{L}}, \overline{\mathcal{M}})$ such that*

$$\mathrm{ht}_{\overline{\mathcal{L}}}(P) \le (1+\varepsilon)\,\mathrm{ht}_{\overline{\mathcal{M}}}(P) + C(\varepsilon, \mathcal{X}, \overline{\mathcal{L}}, \overline{\mathcal{M}})$$

Proof. It is well known (see e.g. [Si], Proposition 3.6) that in the case where both metrics are continuous we find a constant $C(\varepsilon, \mathcal{X}, \overline{\mathcal{L}}, \overline{\mathcal{M}})$ such that for all $\varepsilon > 0$

$$\mathrm{ht}_{\overline{\mathcal{L}}}(P) \le (1+\varepsilon)\,\mathrm{ht}_{\overline{\mathcal{M}}}(P) + C(\varepsilon, \mathcal{X}, \overline{\mathcal{L}}, \overline{\mathcal{M}}). \tag{2.1.1}$$

For simplicity of the argument we assume that the metric $\|\cdot\|$ on $\mathcal{M}$ has only $Q \in X(\overline{\mathbb{Q}})$ as singular point. Let 1_Q be the canonical section of $\mathcal{O}(Q)$. Then we can find continuous hermitian metrics $\|\cdot\|'$ on $\mathcal{M}$ and $\|\cdot\|$ on $\mathcal{O}(Q)$ such that for all $P \in X(\mathbb{C}) \setminus \{Q\}$ and all sections m of $\mathcal{M}$

$$\|m\|(P) = \|m\|'(P) \cdot (-\log \|1_Q\|(P))^\alpha.$$

Let $\mathcal{Q}$ be the Zariski closure of Q. Then, since $\alpha > 0$, we obtain

$$\begin{aligned}
\mathrm{ht}_{\overline{\mathcal{M}}}(P) &= \mathrm{ht}_{\overline{\mathcal{M}}'}(P) - \alpha \log(-\log \|1_Q\|(P)) \\
&\ge \mathrm{ht}_{\overline{\mathcal{M}}'}(P) - \alpha \log(-\log \|1_Q\|(P) + (\mathcal{P}, \mathcal{Q})_{\mathrm{fin}}) \\
&= \mathrm{ht}_{\overline{\mathcal{M}}'}(P) - \alpha \log \mathrm{ht}_{\overline{\mathcal{O}(Q)}}(P) \\
&\ge (1-\varepsilon')\,\mathrm{ht}_{\overline{\mathcal{L}}}(P) - \alpha\varepsilon' \frac{1-\varepsilon'}{\deg(\mathcal{L})}\,\mathrm{ht}_{\overline{\mathcal{L}}}(P) - C'(\mathcal{X}, \varepsilon', \overline{\mathcal{L}}, \overline{\mathcal{M}})
\end{aligned}$$

For the last inequality we used (2.1.1) twice. If we take ε such that $1/(1+\varepsilon) = 1 - \varepsilon'(1 + \alpha(1-\varepsilon')/\deg(\mathcal{L}))$ we derive the claim. □

2.2. Proposition. *Let $f : \mathcal{Y} \to \mathcal{X}$ be a proper morphism of arithmetic surfaces, then we have*

$$\mathrm{ht}_{f^*\overline{\mathcal{L}}}(P) = \mathrm{ht}_{\overline{\mathcal{L}}}(f(P))$$

for any logarithmically singular hermitian line bundle $\overline{\mathcal{L}}$ on $\mathcal{X}$ and P not in the singular locus of the logarithmically singular metric on $\mathcal{L}$.

Proof. See e.g. [BoGS], Formula (3.2.1). □

3 Arithmetic properties of Heegner points

Due to the modular description the points on the modular curve $X(1)$ are well understood. Recall that $X(1)(\mathbb{C}) = \Gamma(1) \setminus \mathbb{H} \cup \{\infty\}$ and that $X(1)$ is isomorphic to $\mathbb{P}^1$. The regular model of $X(1)$ will be denoted by $\mathcal{X}(1)$. This arithmetic surface is canonically isomorphic to $\mathbb{P}^1_{\mathbb{Z}}$. On $\mathcal{X}(1)$ we have the line bundle of modular forms $\mathcal{M}_{12}$. The natural metric on this line bundle is the Petersson metric, here we use the normalization as given in [Kü], Definition 4.8. This metric gives rise to the positive logarithmically singular hermitian line bundle $\overline{\mathcal{M}}_{12}$ (see e.g. [Kü], Proposition 4.9 and 4.12). For any point $P \in X(1)(K) \setminus \{\infty\}$ we have a well-defined height with respect to $\overline{\mathcal{M}}_{12}$. It is called the *modular height*.

3.1. Heegner points. Let D be a negative fundamental discriminant and $K = \mathbb{Q}(\sqrt{D})$. We briefly recall some properties of Heegner divisors. Every ideal class $[\mathfrak{a}]$ of K defines a unique point $P_{\mathfrak{a}}$ on $\Gamma(1) \setminus \mathbb{H}$ by associating with a fractional ideal $\mathfrak{a} = \mathbb{Z}a + \mathbb{Z}b$ with oriented (i.e. $\mathrm{Im}(b\bar{a}) > 0$) $\mathbb{Z}$-basis a, b the point $\rho_{\mathfrak{a}} = b/a \in \mathbb{H}$. We call $P_{\mathfrak{a}}$ the Heegner point to $\mathfrak{a}$ and sometimes write $[\rho_{\mathfrak{a}}]$ instead of $P_{\mathfrak{a}}$.
The Heegner divisor $H(D)$ on $\Gamma(1) \setminus \mathbb{H}$ consists of the sum of the $P_{\mathfrak{a}}$, where $\mathfrak{a}$ runs through all ideal classes of K, counted with multiplicity $2/w$, where w is the number of units in K. The cardinality of $H(D)$ is equal to the class number h of K, its degree is $2h(D)/w$.

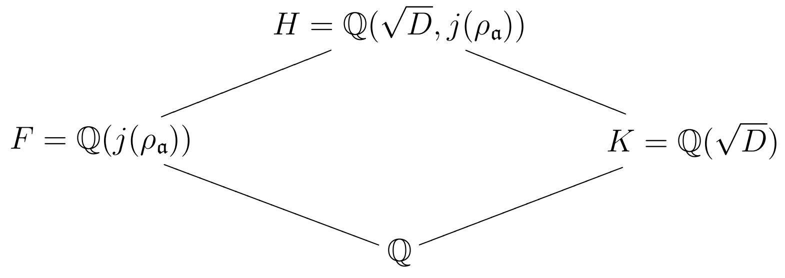
3.2. Proposition. *Let $f : X \to X(1)$ be a morphism of algebraic curves that is defined over the field over which X is defined. Let $P \in X(\overline{\mathbb{Q}})$ be a point such that $f(P)$ is contained in a Heegner divisor $H(D)$ with prime discriminant D, then we have*

$$\mathrm{disc}(P) \geq \frac{1}{2}\log|D| - \frac{55}{2}.$$

Proof. The composition formula for the discriminant implies that for all morphisms $f : X \to X(1)$ and points $P \in X(\overline{\mathbb{Q}})$ we have the inequality

$$\mathrm{disc}(P) \geq \mathrm{disc}(f(P)).$$

Thus it suffices to bound the discriminant of a Heegner point $P_{\mathfrak{a}} = f(P)$. We consider the following diagram of field extensions



By the theory of complex multiplication we have $h(D) = [H : K]$ and $D_{H|\mathbb{Q}} = D^{h(D)}$. From [Gr], Lemma 12.1.2 we deduce $\mathrm{Nm}_{F|\mathbb{Q}}(D_{H|F}) = D$. The composition formula $D_{H|\mathbb{Q}} = D_{F|\mathbb{Q}}^2 \cdot \mathrm{Nm}_{F|\mathbb{Q}}(D_{H|F})$ gives rise to the equality

$$\mathrm{disc}(P_{\mathfrak{a}}) = \frac{1}{h(D)} \log |D_{F|\mathbb{Q}}| = \left(\frac{1}{2} - \frac{1}{2h(D)}\right) \log |D|.$$

The class number of an imaginary quadratic number field with prime discriminant satisfies $h(D) > 1/55 \log |D|$ (see e.g. [Oe]). Thus we have

$$\mathrm{disc}(P_{\mathfrak{a}}) = \left(\frac{1}{2} - \frac{1}{2h(D)}\right) \log |D| \geq \frac{1}{2} \log |D| - \frac{55}{2} \tag{3.2.1}$$

□

3.3. Proposition. *Let $P_{\mathfrak{a}} \in H(D)$ be a Heegner point, then its modular height is given by*

$$\mathrm{ht}_{\overline{\mathcal{M}}_{12}}(P_{\mathfrak{a}}) = -6 \left(\frac{L'(\chi_D, 0)}{L(\chi_D, 0)} + \frac{1}{2} \log |D| \right), \tag{3.3.1}$$

here $L(\chi_D, s)$ is the Dirichlet L-function for the character $\left(\frac{D}{\cdot}\right)$.

Proof. Recall $\Delta(\tau) = q^{24} \prod_{n=1}^{\infty} (1-q)^n$, where $q = e^{2\pi i \tau}$ with $\tau \in \mathbb{H}$, is a section of $\mathcal{M}_{12}$, whose divisor equals the unique cusp ∞ of $\mathcal{X}(1)$. Its Petersson norm is determined by the formula

$$\|\Delta(\tau)\|_{Pet} = |\Delta(\tau)| (4\pi \, \mathrm{Im}(\tau))^6.$$

Therefore the modular height of a Heegner point is given by

$$\mathrm{ht}_{\overline{\mathcal{M}}_{12}}(P_{\mathfrak{a}}) = \frac{1}{[K : \mathbb{Q}]} \left((P_{\mathfrak{a}}, \infty)_{\mathrm{fin}} - \sum_{\rho_{\mathfrak{a}} \in H(D)} \log \|\Delta(\rho_{\mathfrak{a}})\|_{Pet} \right)$$

here for each embedding $\sigma : F = \mathbb{Q}(j(\rho_{\mathfrak{a}})) \to \overline{\mathbb{Q}}$ the point $\rho_{\mathfrak{a}}$ is a lift of $P_{\mathfrak{a}}^{\sigma}(\mathbb{C}) \in \Gamma(1) \setminus \mathbb{H}$ to $\mathbb{H}$. We now recall the well known Kronecker limit formula. If

$$\mathcal{E}(\tau, s) = \frac{1}{2} \sum_{\gamma \in \Gamma_\infty \setminus \Gamma_1} (\mathrm{Im}(\gamma\tau))^s$$

is the real analytic Eisenstein series for $\Gamma(1)$, then the logarithm of the Petersson norm of the Delta function is given by

$$\log\left(\|\Delta(\tau)\|_{Pet}^2\right) = -4\pi \lim_{s\to 1}\left(\mathcal{E}(\tau,s) - \frac{\Gamma(1/2)\Gamma(s-1/2)\zeta(2s-1)}{\Gamma(s)\zeta(2s)}\right) + 12\log(4\pi).$$

We also point to the identity

$$\sum_{\rho_{\mathfrak{a}}\in H(D)} \mathcal{E}(\rho_{\mathfrak{a}},s) = \frac{w}{2}\left|\frac{D}{4}\right|^{s/2}\frac{\zeta_K(s)}{\zeta(2s)},$$

where $\zeta_K(s) = \zeta(s)L(\chi_D,s)$ denotes the Dedekind zeta function of K (see [GZ] p. 210). In [BK], p. 1726, we derived from this the formulae

$$\begin{aligned}
&\sum_{\rho_{\mathfrak{a}}\in H(D)} -\log\left(|\Delta(\rho_{\mathfrak{a}})|^2(4\pi \mathrm{Im}\rho_{\mathfrak{a}})^{12}\right)\\
&= 4\pi\lim_{s\to 1}\left(\sum_{\rho_{\mathfrak{a}}\in H(D)}\mathcal{E}(\rho_{\mathfrak{a}},s) - h\frac{\Gamma(1/2)\Gamma(s-1/2)\zeta(2s-1)}{\Gamma(s)\zeta(2s)}\right) + 12h(D)\log(4\pi)\\
&= -12h(D)\left(\frac{L'(\chi_D,0)}{L(\chi_D,0)} + \frac{1}{2}\log|D|\right).
\end{aligned}$$

Since $j(\rho_{\mathfrak{a}})$ is an algebraic integer we have $(P_{\mathfrak{a}},\infty)_{\mathrm{fin}} = 0$. Thus we derived the claim. □

3.4. Remark. Recall that $\mathcal{X}(1)\cong\mathbb{P}^1_{\mathbb{Z}}$, $\mathcal{M}_{12}\cong\mathcal{O}(1)$ and that the line bundle $\mathcal{O}(1)$ equipped with a particular metric gives rise to the naive height $\mathrm{ht}_{\mathbb{P}^1}$. This height is for a Heegner point $P_{\mathfrak{a}}\in X(1)(K)$ given by

$$\begin{aligned}
\mathrm{ht}_{\mathbb{P}^1}(P_{\mathfrak{a}}) &= \frac{1}{[K:\mathbb{Q}]}\left((P_{\mathfrak{a}},\infty)_{\mathrm{fin}} - \sum_{\rho_{\mathfrak{a}}}\log\max(1,j(\rho_{\mathfrak{a}}))\right)\\
&= 6\left(\frac{L'(\chi_D,1)}{L(\chi_D,1)} + \frac{1}{2}\log|D|\right)\left(1+O\left(\frac{\log\log|D|}{\log|D|}\right)\right)^{-1}.
\end{aligned}$$

Indeed, since $j(\rho_{\mathfrak{a}})$ is an algebraic integer we have $(P_{\mathfrak{a}},\infty)_{\mathrm{fin}} = 0$. Now the claim follows immediately from [GS] by combing their equation (7) with their Theorem 3.

3.5. Proposition. *Let $P_{\mathfrak{a}}\in H(D)$ be a Heegner Point with prime discriminant.*
(i) For all $\delta>0$ there exists a constant $S(\delta)$ such that

$$\mathrm{ht}_{\overline{\mathcal{M}}_{12}}(P_{\mathfrak{a}}) \le S(\delta)\cdot\exp\left(\delta\,\mathrm{disc}(P_{\mathfrak{a}})\right). \tag{3.5.1}$$

(ii) If the Dirichlet L-series $L(\chi_D, s)$ have no zero in the ball of radius $1/4$ around 0, then there exists constants a and b such that the modular height of a Heegner point of discriminant D satisfies

$$\mathrm{ht}_{\overline{\mathcal{M}}_{12}}(P_{\mathfrak{a}}) \leq a\,\mathrm{disc}(P_{\mathfrak{a}}) + b. \tag{3.5.2}$$

(iii) Assuming the generalized Riemann hypothesis (GRH) for the Dirichlet L-series $L(\chi_D, s)$ in question we have

$$\mathrm{ht}_{\overline{\mathcal{M}}_{12}}(P_{\mathfrak{a}}) = 6\,\mathrm{disc}(P_{\mathfrak{a}}) + o(\mathrm{disc}(P_{\mathfrak{a}})). \tag{3.5.3}$$

Proof. (i) and (ii). Let $E_{\mathcal{O}_K}$ be a elliptic curve with complex multiplication by $\mathcal{O}_K$, then the Faltings height of $E_{\mathcal{O}_K}$ equals twelve times the modular height of its modular point $P_{\mathcal{O}_K}$, see e.g. [Co] p.362 and p. 365. By means of the inequality (3.2.1) we derive that (i) is a reformulation of the corresponding formula in the remark on page 365 in [Co] and the claim (ii) is a reformulation of Theorem 6 (ii) in [Co].
(iii) Using the functional equation for $L(\chi_D, s)$ we formulate the right hand side of (3.3.1) as a special value at $s = 1$

$$-\left(\frac{L'(\chi_D,0)}{L(\chi_D,0)} + \frac{1}{2}\log|D|\right) = \left(\frac{L'(\chi_D,1)}{L(\chi_D,1)} + \frac{1}{2}\log|D| - \log(2\pi e^{\gamma})\right),$$

where γ is the Euler constant. Assuming the GRH we have

$$\frac{L'(\chi_D,1)}{L(\chi_D,1)} = O(\log\log|D|),$$

here the implied constant is uniform in D (see e.g. [GS], section 3.1) which yields

$$\mathrm{ht}_{\overline{\mathcal{M}}_{12}}(P_{\mathfrak{a}}) = 6\left(\frac{1}{2}\log|D| + O(\log\log|D|)\right) \tag{3.5.4}$$

Since $O(\log\log|D|)$ is also of order $o(\log|D|)$, we derive by means of (3.2.1) the claim. □

4 Main result

4.1. Definition. Let X be curve defined over a number field and let f be a non constant function in the function field of X. We consider f as a morphism $f : X \to \mathbb{P}^1$ and identify $\mathbb{P}^1$ with the modular curve $X(1)$. Then we define

$$\mathcal{V}(X, f) = \{P \in X(\overline{\mathbb{Q}}) \mid f(P) \text{ is a Heegner point with prime discriminant}\}.$$

4.2. Proposition. *The subset $\mathcal{V}(X,f) \subseteq X(\overline{\mathbb{Q}})$ is unrestricted.*

Proof. The set of Heegner points with prime discriminant on $X(1)$ is, as we have seen already in the proof of Proposition 3.2, unrestricted. The composition formula for the discriminant implies that for all morphisms $f : X \to X(1)$ and points $P \in X(\overline{\mathbb{Q}})$ we have the inequality

$$\operatorname{disc}(f(P)) \le \operatorname{disc}(P).$$

Therefore the set $\mathcal{V}(X,f)$ is also unrestricted. $\square$

4.3. Theorem. *Let X be a curve of genus $g \ge 2$ defined over a number field. Let f be a non constant function in the function field of X and let $\varepsilon, \delta > 0$.*
(i) There exists constants $S(\delta)$ and $C(X,\varepsilon,\mathcal{V}(X,f))$ such all $P \in \mathcal{V}(X,f)$ satisfy

$$\operatorname{ht}_{\overline{\omega}}(P) \le (1+\varepsilon)\frac{S(\delta)(2g-2)}{\deg(f)} \exp\left(\delta \operatorname{disc}(P)\right) + C(X,\varepsilon,\mathcal{V}(X,f)). \tag{4.3.1}$$

(ii) Assume that $\operatorname{ht}_{\overline{\mathcal{M}}_{12}}(P_{\mathfrak{a}}) \le a \operatorname{disc}(P_{\mathfrak{a}}) + b$ for all Heegner points $P_{\mathfrak{a}}$ with prime discriminant D, then for all $P \in \mathcal{V}(X,f)$ we have

$$\operatorname{ht}_{\overline{\omega}}(P) \le (1+\varepsilon)\frac{a(2g-2)}{\deg(f)} \operatorname{disc}(P) + C(X,\varepsilon,\mathcal{V}(X,f)). \tag{4.3.2}$$

Proof. Let $f : \mathcal{X} \to \mathcal{X}(1)$ be an extension of the morphism $f : X \to X(1)$ given by f. The degrees of the line bundles $\omega^{\otimes \deg(f)}$ and $(f^*\mathcal{M}_{12})^{\otimes(2g-2)}$ are equal and positive. We endow $\mathcal{M}_{12}$ with with the Petersson metric and by pull-back we obtain the positive logarithmically singular line bundle $f^*\overline{\mathcal{M}}_{12}$ on $\mathcal{X}$. Then by Proposition 2.1 and Proposition 2.2 we get for all $P \in X(\overline{\mathbb{Q}}) \setminus \{f^{-1}(\infty)\}$

$$\operatorname{ht}_{\overline{\omega}}(P) \le (1+\varepsilon')\frac{2g-2}{\deg(f)} \operatorname{ht}_{\overline{\mathcal{M}}_{12}}(f(P)) + C'(X,\varepsilon',\mathcal{V}(X,f));$$

here we wrote $C'(X,\varepsilon',\mathcal{V}(X,f))$ instead of $C'(\varepsilon',\mathcal{X},\overline{\omega}, f^*\overline{\mathcal{M}}_{12})$. If $P \in \mathcal{V}(X,f) \subseteq X(\overline{\mathbb{Q}})$ then $f(P)$ is a Heegner point with prime discriminant. Thus (4.3.1) follows immediately from (3.5.1). Finally (4.3.2) is an easy consequence of the assumed bound for the modular height of $f(P)$. $\square$

4.4. Remark. (i) In Theorem 4.3 we can choose f with arbitrary large degree. If we let $\deg(f) \ge (1+\varepsilon)\cdot S(\delta)\cdot(2g-2)/\varepsilon$ we derive formula (1.2.1) of Theorem 1.2. If we let $\deg(f) \ge (1+\varepsilon)\cdot a\cdot(2g-2)/\varepsilon$ we obtain formula (1.2.2).
(ii) We note that because of [Fr] the exponential height inequality (1.2.1) should somehow be related to the exponential *abc*-inequality [SY], [Su]. We remark also that (1.2.2) could be seen as a converse to a theorem of Granville and Stark [GS] saying that the *abc*-conjecture implies that there are no Siegel zeros.

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