Theorem (Rankin-Selberg unfolding)

Let $f(\tau) = \sum_{n=1}^{\infty} a_n q^n \in S_k$ and $g(\tau) = \sum_{m=0}^{\infty} b_m q^m \in M_l$. Then, if $k - l \geqslant 4$ we have for the Petersson scalarproduct

$$\langle f(\tau), g(\tau) G_{k-l}(\tau) \rangle = \frac{(k-2)!}{(4\pi)^{k-1}} \sum_{n=1}^{\infty} \frac{a_n \overline{b_n}}{n^{k-1}}.$$

Proof: By definition we have

$$\langle f(\tau), g(\tau) G_{k-l}(\tau) \rangle = \int\limits_{\mathcal{F}} f(\tau) \overline{g(\tau) G_k(\tau)} y^k \frac{\mathrm{d}\, x \,\mathrm{d}\, y}{y^2}.$$

Because of above notation we get for the integrand

$$\begin{aligned} &\frac{(2\pi i)^k}{(k-1)!\zeta(k)} f(\tau) \overline{g(\tau) G_{k-l}(\tau)} y^k \\ &= \sum_{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \langle \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right) \rangle \backslash \mathrm{SL}_2(\mathbb{Z})} (c\tau + d)^k f(\tau) \overline{(c\tau + d)}^l \overline{g(\tau)} \frac{y^k}{\left| (c\tau + d) \right|^{2k}} \\ &= \sum_{\gamma \in \langle \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right) \rangle \backslash \mathrm{SL}_2(\mathbb{Z})} f(\gamma\tau) \overline{g(\gamma\tau)} \, \mathrm{Im}(\gamma\tau)^k. \end{aligned}$$

We apply this identity to the integral

$$\begin{aligned} &\frac{(2\pi i)^k}{(k-1)!\zeta(k)} \int\limits_{\mathcal{F}} f(\tau) \overline{g(\tau) G_{k-l}(\tau)} y^k \frac{\mathrm{d}\, x \,\mathrm{d}\, y}{y^2} \\ &= \int\limits_{\mathcal{F}} \sum_{\gamma \in \langle \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right) \rangle \backslash \mathrm{SL}_2(\mathbb{Z})} \mathrm{Im}(\gamma\tau)^k f(\gamma\tau) \overline{g(\gamma\tau)} \frac{\mathrm{d}\, x \,\mathrm{d}\, y}{y^2}. \end{aligned}$$

Unfolding the domain of integration, i.e. interchanging the integral and the sum, leads to

$$\begin{aligned} \int\limits_0^{\infty} \int\limits_0^{1} f(x+y) \overline{g(x+iy)} \,\mathrm{d}\, x y^k \frac{\mathrm{d}\, y}{y^2} &= \int\limits_0^{\infty} \sum_{n=1}^{\infty} a_n \overline{b_n} e^{-4\pi n y} y^{k-2} \,\mathrm{d}\, y \\ &= \frac{(k-2)!}{(4\pi)^{k-1}} \sum_{n=1}^{\infty} \frac{a_n \overline{b_n}}{n^{k-1}} \end{aligned}$$

□

Using the "holomorphic projection principle" the formula also holds for $f(\tau) = \sum_{n=1}^{\infty} a_n q^n \in S_k$ and $g(\tau) = \sum_{m=0}^{\infty} b_m q^m \in M_{k-2}$, namely we have

$$\langle f(\tau), g(\tau) G_2^*(\tau) \rangle = \frac{(k-2)!}{(4\pi)^{k-1}} \sum_{n=1}^{\infty} \frac{a_n \overline{b_n}}{n^{k-1}}.$$

Proposition

Let $f(\tau) = \sum_{n=1}^{\infty} a_n q^n$ be a Hecke Eigenform of weight k. Then we have

$$\sum_{n=1}^{\infty} \frac{a_n \sigma_{k-1}(n)}{n^s} = \frac{L(f,s) L(f, s-k+1)}{\zeta(2s+2)}$$

Proof: Use the Euler product & $a_n \in \mathbb{R}$ ... □

For example we get

$$(\Delta, G_4 G_8) = * \frac{L(\Delta, 11) L(\Delta 9)}{\zeta(6)}$$

this will be used for the Theorem of Eichler-Shimura-Manin.

Modular forms and their periods

Outlock

Periods are countable set of complex numbers

$$\mathbb{Q} \subseteq \overline{\mathbb{Q}} \subseteq \{ \text{ periods } \} \subsetneq \mathbb{C},$$

which still obey many arithmetic properties. We will show that the finitely many periods given by the special values of the L-series $L(f,s)$ attached to a modular f contains as much information as the infinite set $\{a_n\}_{n \in \mathbb{N}}$ given by its Fourier coefficients.

Definition (n-th period)

Let $f \in S_k$, then its n-th period is defined by

$$r_n(f) = \int\limits_0^{i\infty} f(\tau) \tau^n \,\mathrm{d}\, \tau.$$

Proposition

Let $f \in S_k$, then we have
$$r_n(f) = i^{n+1} R(f, n+1),$$
where $R(f,s) = (2\pi)^{-s}\Gamma(s)L(f,s) = \pm R(f, k-s)$.

Proof: Similar as we have seen before
$$\int_0^{i\infty} f(\tau)\tau^n \,\mathrm{d}\,\tau = \int_0^{\infty} f(it)(it)^n \,\mathrm{d}\,t = \ldots = i^{n+1} R(f,s)\big|_{s=n+1}.$$

□

Definition (slash operator)

Let $f : \mathbb{C} \to \mathbb{C} \in \mathrm{Maps}(\mathbb{C}, \mathbb{C})$. Then we define the slash operator by
$$\big|_k : \mathrm{Maps}(\mathbb{C},\mathbb{C}) \times \mathrm{SL}_2(\mathbb{Z}) \to \mathrm{Maps}(\mathbb{C},\mathbb{C})$$
$$\left(f(\tau), \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\right) \mapsto f\left(\frac{a\tau+b}{c\tau+d}\right)(c\tau+d)^{-k} = f\big|_k \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)(\tau).$$

If $f \in M_k$, then $f\big|_k \gamma = f$.

Ex: $x^l\big|_{2-k} T = 1^{k-2-l}(x+1)^l$

$x^l\big|_{2-k} T^{-1} = 1^{k-2-l}(x-1)^l$

$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

$T^{-1} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$

Definition

Es sei $V_k = \{f(x) \in \mathbb{C}[X] \,|\, \deg f(x) \leqslant k-2\}$.

Lemma

The group $\mathrm{SL}_2(\mathbb{Z})$ operates on V_k via $\big|_{2-k}$.

Proof: Consider $f(x) \in \mathbb{C}[x]$ by $f(x) \mapsto (x \mapsto f(x))$ as an element of $\mathrm{Maps}(\mathbb{C}, \mathbb{C})$. Clearly is $\big|_{2-k}$ a linear map, thus it remains to show for all $0 \leqslant l \leqslant k-2$:
$$x^l\big|_{2-k}\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) = (cx+d)^{k-2}\left(\frac{ax+b}{cx+d}\right)^l = (cx+d)^{k-2-l}(ax+b)^l \in V_k.$$

□

Definition (Period polynomials

Let $f \in S_k$, then its period polynomial is defined by
$$r_f(x) = \int_0^{i\infty} f(\tau)(\tau - x)^{k-2} \,\mathrm{d}\,\tau$$

Remark

$$r_f(x) = \sum_{n=0}^{\infty} (-1)^n \binom{k-2}{n} r_n(f) x^{k-2-n} \in \mathbb{C}[x].$$

Proposition

Let $f \in S_k$ and $\gamma \in \mathrm{SL}_2(\mathbb{Z})$, then we have
$$\left(r_f\big|_{2-k}\gamma\right)(x) = \int_{\gamma^{-1}(0)}^{\gamma^{-1}(i\infty)} f(\tau)(x-\tau)^{k-2} \,\mathrm{d}\,\tau.$$

Proof: Let $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$. The with the change of variable $\gamma(\tau) = u$ we get
$$\begin{aligned}
r_f\big|_{2-k}\gamma(x) &= \int_0^{i\infty} f(\tau)(x-\tau)^{k-2}\big|_{2-k}\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \mathrm{d}\,\tau \\
&= \int_0^{i\infty} f(\tau)(cx+d)^{k-2}\left(\frac{ax+b}{cx+d} - \tau\right)^{k-2} \mathrm{d}\,\tau \\
&= \int_0^{i\infty} f(\tau)(c\tau+d)^{k-2}\left(x - \frac{a\tau+b}{c\tau+d}\right)^{k-2} \mathrm{d}\,\tau \\
&= \int_0^{i\infty} f(\gamma(\tau))(x-\gamma(\tau))^{k-2} \,\mathrm{d}\,\gamma(\tau) \\
&= \int_{\gamma^{-1}(0)}^{\gamma^{-1}(\infty)} f(u)(x-u) \,\mathrm{d}\,u.
\end{aligned}$$

□

Let G be a group and R be a ring. Then

$$R[G] := \Big\{ \sum_{g \in G} R[g] \Big\}$$

equipped with addition and multiplication

$$\sum_{g\in G} a_g[g] + \sum_{g\in G} b_g[g] = \sum_{g\in G} (a_g + b_g)[g]$$
$$\sum_{g\in G} a_g[g] + \sum_{g'\in G} b_{g'}[g'] = \sum_{g,g'\in G} a_g b_{g'}[gg']$$

is a ring. We call $R[G]$ the group ring of G with coefficients in R.

Remark

- $R[G]$ is a R-module.
- The action of $\mathrm{SL}_2(\mathbb{Z})$ via $|_{2-k}$ on V_k extends to a group ring action of $\mathbb{Z}[\mathrm{SL}_2(\mathbb{Z})]$ on V_k as follows:
  For $\sum r_g[g] \in \mathbb{Z}[\mathrm{SL}_2(\mathbb{Z})]$ and $p(x) \in V_k$, we set

$$p|_{2-k} \sum r_g[g](x) = \sum r_g (p|_{2-k} g)(x).$$

$\bar{S}^2 = 1 \quad , \bar{U}^3 = 1$

Let $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $U = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$. By the above Proposition, we then get for the period polynomial r_f associated with a modular form $f \in S_k$

$\int_0^\infty + \int_\infty^0 = \int_0^\infty - \int_0^\infty$

$$r_f|_{2-k} 1 + S = 0$$
$$r_f|_{2-k} 1 + U + U^2 = 0.$$

Definition (Space of period polynomials)

We define the space of period polynomials by

$$W_k := \{p(x) \in V_k \,|\, p|_{2-k} 1 + S = p|_{2-k} 1 + U + U^2 = 0\}$$

Remark

If k is clear from the context we simply write $p|\gamma$ instead of $p|_{2-k}\gamma$.

Proposition (trivial Period relations)

For $f \in S_{w+2}$ and $r_k(f) = \int_0^{i\infty} f(\tau)\tau^k \,\mathrm{d}\,\tau$ with $k = 0, \dots, w$ we have the period relations:

$$0 = r_k(f) + (-1)^k r_{w-k}(f)$$
$$0 = r_k(f) + (-1)^k \sum_{\substack{0 \leqslant i \leqslant k \\ i \equiv 0 \mod 2}} \binom{k}{i} r_{w-k+i}(f) + (-1)^k \sum_{\substack{0 \leqslant i \leqslant w-k \\ i \equiv k \mod 2}} \binom{w-k}{i} r_i(f)$$
$$0 = \sum_{\substack{1 \leqslant i \leqslant k \\ i \equiv 1 \mod 2}} \binom{k}{i} r_{-k+i}(f) + \sum_{\substack{0 \leqslant i \leqslant w-k \\ i \not\equiv k \mod 2}} \binom{w-k}{i} r_i(f)$$

Proof: The first follows by comparing the coefficients of $r_f|_{2-k} 1 + S = 0$. For the other we decompose $r_f|_{2-k} 1 + U + U^2 = 0$ w.r.t. to the ω-eigenspaces. $\square$

We aim to understand better the map

$$S_k \to V_k$$
$$f \mapsto r_f.$$

Via the involution

$$V_k \to V_k$$
$$p(x) \mapsto p(-x)$$

induced by $x \mapsto -x$ we obtain a decomposition $V_k = V_k^+ \oplus V_k^-$ where

$$V_k^+ = \{p \in V_k \,|\, p(x) = p(-x)\}$$
$$V_k^- = \{p \in V_k \,|\, p(x) = -p(-x)\}.$$

In fact we have also a decomposition

$\varepsilon = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$

$$W_k = W_k^+ \oplus W_k^-$$

Finally, for $f \in S_k$ we have the decomposition $r_f = r_f^+ + r_f^-$.

Theorem (Eichler-Shimura)

(i) The map $r^- : S_k \to W_k^-$ is an isomorphism of vector spaces.

(ii) We have $p_k^+(x) := x^{k-2} - 1 \in W_k^+$ and the map $r^+ : S_k \to W_k^+ \big/ \mathbb{C}p_k^+$ is an isomorphism of vector spaces.

We sketch a proof of the weaker claim $M_k \oplus S_k \cong W_k$. So that we can avoid the additional action of $\epsilon = \left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, which give rise to the spitting $V = V^+ \oplus V^-$ into eigenspaces for ϵ, i.e., the space of even polynomials V^+ and the space of odd polynomials V^- .

Proposition

For all $k \geqslant 4$ we have

$$\dim W_k = \dim M_k + \dim S_k.$$

Proof: We set

$$\begin{aligned} V^S &= \{p \in V \,|\, p|_{2-k}S = p\} \\ V^U &= \{p \in V \,|\, p|_{2-k}U = p\}. \end{aligned}$$

As being an involution S is diagonalizeable, i.e.,

$$V = V^S \oplus \ker(1+S).$$

The characteristic polynomials for $U \in V_k$ is $(x-1)(x-\omega)(x-\omega^2)$, since U is a root for $X^3 - 1$. Therefore

$$V_k = V_k^U \oplus \ker(U - \omega \cdot 1) \oplus \ker(U - \omega^2 \cdot 1).$$

We further need the non-degenerate bilinear form on V_k given by

$$(X^m, X^n) = \begin{cases} (-1)^n \binom{k-2}{n}^{-1}, & \text{if } n + m = k-2 \\ 0, & \text{else.} \end{cases}$$

It is straightforward to show that

$$\left\langle p|_{2-k}\gamma, q|_{2-k}\gamma \right\rangle = \langle p, q \rangle$$

for all $\gamma \in \mathrm{SL}_2(\mathbb{Z})$.

Now we get for all subspaces $F \subseteq V_k$ the orthogonal decomposition

$$V_k = F \oplus F^\perp$$

where

$$F^\perp = \{q \in V_k \,|\, \langle q, p \rangle = 0 \quad \forall\, p \in F\}.$$

We have $V_k^S \perp \ker(1+s)$, since for $p \in V_k^S$ with $q \in \ker(1+s)$ we have

$$\langle p, q \rangle = \left\langle p|_{2-k}S, q|_{2-k}S \right\rangle = \langle p, -q \rangle = -\langle p, q \rangle = 0.$$

Analogously we get

$$\begin{aligned} V_k^U &\perp \ker(U - \omega \cdot 1) \\ V_k^U &\perp \ker(U - \omega^2 \cdot 1) \end{aligned}$$

We conclude

$$\ker(1 + U + U^2) = \ker(U - \omega \cdot 1) \oplus \ker(U - \omega^2 \cdot 1) = (V_k^U)^\perp$$

and

$$W_k = (V_k^S)^\perp \cap (V_k^U)^\perp = (V_k^S + V_k^U)^\perp.$$

Observe that $T = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right) = U^2 S$. Now any $p \in V_k^S \cap V_k^U$ must therefore be 1-periodic, i.e. $(p|_{2-k}T)(x) = p(x+1)$, and there it must be constant. By assumption we have $k > 2$ and therefore the constants are not elements of V_k^S, as otherwise

$$p(x) = x^{k-2} p(-1/x) = (p|_{2-k}S)(x)$$

is not satisfied. Moreover we have

$$W_k = (V_k^S \oplus V_k^U)^\perp.$$

All these consideration then prove the first step of the proof:

$$\dim W_k = \dim(V_k^S \oplus V_k^U)^\perp = \dim V_k - \dim(V_k^S) - \dim(V_k^U).$$

We now calculate these dimensions:

$$\dim V_k = \dim\{p \in \mathbb{C}[x] \,|\, \deg p \leqslant k-2\} = k - 1.$$

The polynomials $s_n(x) = (x-i)^n(x+i)^{k-2-n}$, $n = 0, \dots, k-2$ give rise to a basis of eigenvectors in V_k with eigenvalues $(-1)^n i^{n-2}$ w.r.t. the action of S. Hence we get

$$\begin{aligned}\dim V_k^S &= \#\{n = 0, \dots, k-2 \,|\, (-1)^n i^{k-2} = 1\} \\ &= \#\{n = 0, \dots, k-2 \,|\, 2n \equiv k-2 \mod 4\} \\ &= 1 + 2\left\lfloor \frac{k-2}{4} \right\rfloor.\end{aligned}$$

The polynomials $u_n(x) = (x+\omega)^n(x+\omega)^{k-2-n}$, $n = 0, \dots, 2-k$ give rise to a basis of eigenvectors in V_k with eigenvalues ω^{k-2+n} w.r.t. the action of U. In a similar way we get

$$\dim V_k^U = \#\{n = 0, \dots, k-2 \,|\, w^{k-2+n} = 1\} = 1 + 2\left\lfloor \frac{k-2}{6} \right\rfloor.$$

Finally we get for $k \geqslant 4$ the formula

$$\begin{aligned}\dim W_k &= k - 1 - \left(1 + 2\left\lfloor \frac{k-2}{4} \right\rfloor\right) - \left(1 + 2\left\lfloor \frac{k-2}{6} \right\rfloor\right) \\ &= \begin{cases} 2\left\lfloor \frac{k}{12} \right\rfloor + 1, & \text{falls } k \not\equiv 2 \mod 12 \\ 2\left\lfloor \frac{k}{12} \right\rfloor - 1, & \text{falls } k \equiv 2 \mod 12 \end{cases}\end{aligned}$$

Comparing this with the formula for $\dim(M_k)$ resp. for $\dim(S_k)$ the claim follow. ☐

$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

$U = (\quad)$

Theorem (Haberland)

Let $k \geqslant 4$, and let $f, g \in S_k$. Then their Petersson scalar product satisfies

$$\begin{aligned}\langle f, g \rangle &= \frac{1}{6(2i)^{k-1}} \left\langle r_f|_{2-k}(T - T^{-1}), \overline{r_g} \right\rangle \\ &= \frac{1}{3(2i)^{k-1}} \sum_{\substack{0<m<n<k-2 \\ m \not\equiv n \mod 2}} (-1)^m \binom{k-2}{n}\binom{n}{m} r_{n-2-m}(f)\overline{r_m(g)}.\end{aligned}$$

Modular form

Proof: Either proved by a long calculation or by using group cohomology. Martin & Royer: Periods ☐

Corollary

The morphisms $r^+ : S_k \to W_k^+ / \mathbb{C}p_k$ and $r^- : S_k \to W_k^-$ are injektive.

This Corollary will imply the Theorem of Eichler-Shimura:
Proof: Let $f \in S_k$ with $f \subset \ker(r^-) \cup \ker(r^+)$. By the theorem of Haberland we get $\langle f, f \rangle = 0$, since by assumption all periods are 0. Since the Petersson scalar product is non-degenerate, we must have $f = 0$. We conclude the injectivity of r^- and r^+, since $p_k^+ \notin r^+(S_k)$. ☐

Theorem (Kohnen-Zagier)

For $0 \leqslant a \leqslant k/2$ let

$$E_{a,k-a} = \begin{cases} E_a E_{k-a} & \text{if } a \neq 2 \\ \pi_{hol}(E_2^* E_{k-2}) = E_2 E_{k-2} - \frac{12}{k-2} E'_{k-a} & \text{else,} \end{cases}$$

then the set $\{E_{a,k-a}\}_{0 \leqslant a \leqslant k/2}$ spans M_k.

Proof: We have $E_k = E_{0,k}$. Let $f \in S_k$ be a Hecke eigenform, then with some non-zero constants κ, κ', which only depend on k, we have by Rankin's formula for all even $0 < a \leqslant k/2$

$$\langle f, E_{a,k-a} \rangle = \kappa L(f, k-1)L(f, a) = \kappa' \underbrace{r_{k-2}(f)}_{\text{always} \neq 0} r_{a-1}(f)$$

We observe, that since $L(f, s)$ is for $s = k-1$ given by an Euler product, we have $r_{k-2}(f) \neq 0$. Thus if f is orthogonal to all such $E_{a,k-a}$, all linear equations in the odd periods of f have to vanish. Thus $r_f^- = 0$ and by the theorem of Eichler-Shimura we must have $f = 0$.
☐

Theorem (Fukuhara)

Let $d_k = \dim S_k$, then a basis for M_k is given by

$$\begin{cases} \{E_k\} \cup \{E_{4i}E_{k-4i} \mid i = 1, \dots, d_k\} & \text{if } k \equiv 0 \pmod 4 \\ \{E_k \cup \{E_{4i+2}E_{k-4i-2} \mid i = 1, \dots, d_k\} & \text{if } k \equiv 2 \pmod 4. \end{cases}$$

For example for M_{36}, we have the basis

$$\{E_{36}, E_4E_{32}, E_8E_{28}, E_{12}E_{24}\}.$$

Tasara et. al give another basis
using a clever partial fraction expansion

Theorem (Eichler, Shimura, Manin)

Let $f = \sum a_n(f) q^n \in S_k$ be a Hecke eigenform. Then there exist to non-zero complex numbers $\omega_f^+ \in i\mathbb{R}$ and $\omega_f^- \in \mathbb{R}$, such that

$$\frac{r_f^-}{\omega_f^-}, \frac{r_f^+}{\omega_f^+} \in \mathbb{Q}(a_2(f), a_3(f), \dots)[X] = \mathbb{Q}(f)[X].$$

Moreover, one may choose ω_f^+ and ω_f^- such that

$$\omega_f^+ \omega_f^- = i\langle f, f\rangle.$$

Proof: This combines the trivial Period relations plus relations obtain by Rankin's method for calulating $< f, E_a E_{k-a} >$. ☐

We make explicit the above results for

$$f = \Delta(\tau) = q \prod_{n=1}^{\infty} (1-q^n)^{24} = q - 24q^2 + 252q^3 + \dots \in S_{12}.$$

The trivial period relations imply

$$\begin{aligned} L(\Delta, 2) &= L(\Delta, 10) \\ 48L(\Delta, 4) &= 48L(\Delta, 9) = 25L(\Delta, 10) \\ 12L(\Delta, 6) &= 5L(\Delta, 10) \\ L(\Delta, 1) &= L(\Delta, 11) \\ L(\Delta, 3) &= L(\Delta, 9) \\ 14L(\Delta, 5) &= 14L(\Delta, 7) = 9L(\Delta, 9). \end{aligned}$$

This suffices to show

$$r_\Delta^-(X) = \frac{5}{2} L(\Delta, 11)\big(4(X^9 + X) - 25(X^7 + X^3) + 42X^5\big).$$

Using Rankin-Selberg unfolding gives

$$\beta\langle \Delta, \Delta\rangle = \cancel{\langle \Delta, \alpha G_{12} + \beta\Delta\rangle} = \langle \Delta, G_8 G_4\rangle = \frac{1}{2^{11}} L(\Delta, 1) L(\Delta, 4)$$

for $\beta = \frac{15}{2764}$.

With another modular form $g = F_2(G_6, G_2) \in M_{12}$, where $F_2(*, *)$ denotes the second Rankin-Cohen bracket, one calculates

$$\beta\langle \Delta, \Delta\rangle = \langle \Delta, g\rangle = -\frac{45}{2048} L(\Delta, 3) L(\Delta, 4)$$

for $\beta = -\frac{5}{48}$.
These two additional identities give rise to

$$1620 L(\Delta, 9) = 691 L(\Delta, 11).$$

We conclude

$$r_\Delta^+(x) = \frac{691}{36} i L(\Delta, 11)\Big(\frac{36}{691}(X^{10} - 1) - X^2(X^2 - 1)^3\Big).$$

For the second claim of the theorem we note

$$\begin{aligned} r_\Delta^-(X) &= q^- \omega_\Delta^- \big(4(X^9 + X) + \dots\big) \\ r_\Delta^+(X) &= q^+ \omega_\Delta^+ \Big(\frac{36}{691}(X^{10} + 1) + \dots\Big) \end{aligned}$$

mit

$$\begin{aligned} q^- \omega_\Delta^- &= \frac{5}{2} L(\Delta, 10) \\ q^+ \omega_\Delta^+ &= \frac{691}{36} i L(\Delta, 11) \end{aligned}$$

and

$$\omega_\Delta^- \omega_\Delta^+ = i\langle \Delta, \Delta\rangle.$$

This implies

$$q^- q^+ = \frac{3455}{72} \frac{L(\Delta, 10) L(\Delta, 11)}{\langle \Delta, \Delta\rangle} = 1024.$$

We set for an arbitrary $q_- \in \mathbb{Q}^*$

$$\begin{aligned} \omega_\Delta^- &= \frac{5}{2}\frac{1}{q_-} L(\Delta, 10) \\ \omega_\Delta^+ &= \frac{691}{36 \cdot 1024} q_- L(\Delta, 11). \end{aligned}$$

In particular for $q_- = 9$ we get for example

$$\omega_\Delta^- \approx 0{,}00102991957\ldots$$
$$\omega_\Delta^+ \approx 0{,}001005264371\ldots \cdot i.$$

Remark

Write $\Delta(\tau) = \sum_{n\geqslant 1} \tau(n) q^n$, then

$$L(\Delta, s) = \sum_{n=1}^{\infty} \frac{\tau(n)}{n^s}$$

converges for $\mathrm{Re}(s) > \frac{12}{2} + 1 = 7$ quite quickly. Using this fact one can calculate $\omega_\Delta^- \approx L(\Delta, 10)$ and $\omega_\Delta^+ \approx L(\Delta, 11)$ very fast and very precise. Thus this also holds for $\langle \Delta, \Delta \rangle = \omega_\Delta^+ \omega_\Delta^-$. Conversely, numerical calculations of the integral

$$\langle \Delta, \Delta \rangle = \int_{\mathcal{F}} \Delta(\tau)\overline{\Delta(\tau)} y^{12} \frac{\mathrm{d}\,x\,\mathrm{d}\,y}{y^2}.$$

are far away of producing such precise approximations in a reasonable time.

Definition

Let

$$M_n = \{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in M_2(\mathbb{Z}) \,|\, ad - bc = n \}$$

and

$$\mathcal{M}_n = \bigoplus_{\gamma \in M_n} \mathbb{Z}\gamma.$$

then an operation of $\mathcal{M}_n$ on V_k is defined by the linear extension of

$$p(x)\big|_{2-k} \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) = (cx+d)^{k-2} p\left(\frac{ax+b}{cx+d}\right).$$

Theorem (Manin, Zagier)

For $n \in \mathbb{N}$ we define $\tilde{T}_n \in \mathcal{M}_n$ by

$$\tilde{T}_n = \sum \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right),$$

where in the sum only those $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in M_n$contribute, for that $a > |c|$, $d > |b|$ and $bc < 0$, or if $b = 0$, then $-a < 2c \leqslant a$, or if $c = 0$, then $-d < 2b \leqslant d$.
The we have

$$\tilde{T}_n W_k^+ \subset W_k^+ \quad \text{und} \quad \tilde{T}_n W_k^- \subset W_k^-$$

for all $k, n \in \mathbb{N}$. Furthermore we have

$$r_{T_n f}(X) = r_f\big|_{2-k} \tilde{T}_n.$$

i.e. the action of $T_n$on S_k resp. on M_k corresponds to the action of $\tilde{T}_n$ on W_k.

Example

We have

$$\tilde{T}_2 = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} + \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix} + \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix}$$

Corollary

If f is a Hecke eigenform, then r_f is an eigenvector for the action of $\tilde{T}_n$ on W_k.

It is instructive to study the action of $\tilde{T}_2, \tilde{T}_4, \tilde{T}_6$ on r_Δ^+ and r_Δ^-.

Corollary

Let

$$f = \sum_{n=1}^{\infty} a_n q^n = q + \sum_{n=2}^{\infty} a_n q^n$$

be a Hecke eigenform, then we have

$$a_l = \sum_{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \tilde{T}_l} \sum_{\substack{0 \leqslant n \leqslant l-1 \\ n \text{ gerade}}} \binom{k-2}{n} \frac{r_n(f)}{r_0(f)} (b^{k-2-n} d^n - b^n d^{k-2-n}).$$

even

Proof: We have

$$a_l r_f(X) = r_{T_l f}(X) = r_f\big|_{2-k} \tilde{T}_l(X),$$

in particular this holds for $X = 0$, which is the claimed formula. $\square$
Exercise:Determine the Ramanujan tau function for $\tau(2), \tau(3), \tau(4)$ and $\tau(6)$ by this method.