Theorem (Dimension formula)

For an even positiver integer k we have

$$\dim_{\mathbb{C}} M_k = \begin{cases} \lfloor \frac{k}{12} \rfloor + 1 & , \quad k \not\equiv 2 \mod 12 \\ \lfloor \frac{k}{12} \rfloor & , \quad k \equiv 2 \mod 12 \end{cases} .$$

Proof: This will now follow by induction on k from the results in previous Proposition. For $k < 12$ the above dimension formula is already proven. Combing the results of previous Proposition we have

$$M_{k+12} = \mathbb{C}E_{k+12} \oplus \Delta \cdot M_k$$

and since $\lfloor \frac{k}{12} \rfloor + 1 = \lfloor \frac{k+12}{12} \rfloor$ the statement follows inductively.

k	0	2	4	6	8	10	12	14	16	18	20	22	24	26	28	30	32	34	36
$\dim_{\mathbb{C}} M_k$	1	0	1	1	1	1	2	1	2	2	2	2	3	2	3	3	3	3	4

Figure : Dimension of M_k for even $0 \leqslant k \leqslant 36$.

OEIS

Hurwitz Identity

$E_4^2, E_8 \in M_8$. Since $\dim_{\mathbb{C}} M_8 = 1$ there must exists a $c \in \mathbb{C}$ with $E_4^2 = cE_8$. But since both have 1 as the constant term in their Fourier expansion we deduce $c = 1$. Hence

Both E_4^3 and E_6^2 are modular forms of weight 12 having 1 as the constant term in their Fourier expansion and therefore $E_4^3 - E_6^2 \in S_{12}$. By the last Proposition v) this has to be a multiple of Δ and comparing the first few Fourier coefficients gives

$$\Delta(\tau) = \frac{E_4(\tau)^3 - E_6(\tau)^2}{1728} . = 0 + q + \dots$$

In general every modular form can be written (uniquely) as a polynomial in E_4 and E_6:

Proposition

For $k \geqslant 0$, the set $\{E_4^a E_6^b \mid a, b \geqslant 0, 4a + 6b = k\}$ is a basis of the space M_k.

Proof: We first check that the mentioned set has the correct size. Let N_k be the number of solutions to $4a + 6b = k$ in nonnegative integers a and b. For $k \leqslant 12$ one can check directly that $N_k = \dim_{\mathbb{C}} M_k$ (given in the above theorem) and for $k \geqslant 12$ one can check that $N_k = N_{k-12} + 1$. Therefore we have $N_k = \dim_{\mathbb{C}} M_k$ for all k. It remains to show that the set is linearly independent. Suppose we have a relation of the form

$$\sum_{\substack{4a+6b=k \\ a,b \geqslant 0}} \lambda_{a,b} E_4(\tau)^a E_6(\tau)^b = 0$$

for all $\tau \in \mathbb{H}$. If there is a pure E_4 term, say $\lambda_{a,0} E_4(\tau)^a$, then setting $\tau = i$ shows $\lambda_{a,0} E_4(i)^a = 0$ since $E_6(i) = 0$. Since $E_4(i) \neq 0$ (again by the valence formula) we deduce $\lambda_{a,0} = 0$. Therefore all nonzero terms in the sum have $b \geqslant 1$. As E_6 is not identically 0, we can divide by it and get

$$\sum_{\substack{4a+6b=k \\ a,b \geqslant 0}} \lambda_{a,b} E_4(\tau)^a E_6(\tau)^{b-1} = 0 ,$$

which is a linear relation in weight $k - 6$. By induction we see that the remaining coefficients are 0. $\square$

Algorithm

Starting with a modular form $f = \sum_{n=0}^{\infty} a_n q^n \in M_k$ and choosing a and b with $4a + 6b = k$, we have

$$f - a_0 E_4^a E_6^b \in S_k .$$

We have shown $S_k = \Delta \cdot M_{k-12}$, i.e. we find a $g \in M_{k-12}$ with $f = a_0 E_4^a E_6^b + \Delta \cdot g$. With the explicit expression

$$\Delta = \frac{E_4^3 - E_6^2}{1728}$$

of Δ, this gives a recursive algorithm (and in fact another way of proving the above Proposition) to write f as a polynomial in E_4 and E_6.

Proposition

The modular forms E_4 and E_6 are algebraically independent over $\mathbb{C}$.

Proof. Let $P \in \mathbb{C}[X, Y]$ be with $P(E_4(\tau), E_6(\tau)) = 0$ for all $\tau \in \mathbb{H}$. Since the weight gives a grading we can reduce this to the case where $P(E_4, E_6)$ is a sum of modular forms of the same weight k. But we know that $E_4^a E_6^b$ with $4a + 6b = k$ are linearly independent and therefore we conclude $P = 0$. □

Summarizing all the results we get the following description of the space of modular forms.

~~Corollary~~ Theorem

Let M denote the *space of all modular forms* (of level 1), then we have

$$M = \bigoplus_{k=0}^{\infty} M_k = \mathbb{C}[E_4, E_6] \cong \mathbb{C}[X, Y]\,,$$

i.e. M is a graded $\mathbb{C}$-algebra, which is isomorphic to the polynomial ring in two variables.

Derivatives and quasi modular forms

Let $f = \sum_{n=0}^{\infty} a_n q^n$ be a modular form, then we write for its holomorphic derivative w.r.t. τ

$$f' := \frac{1}{2\pi i}\frac{d}{d\tau} f = q\frac{d}{dq} f = \sum_{n=1}^{\infty} n a_n q^n\,.$$

Here the factor $2\pi i$ has been included in order to preserve the rationality properties of the Fourier coefficients.

Proposition

The derivative of a modular form $f \in M_k$ satisfies for all $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$:

$$f'\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^{k+2} f'(\tau) + \frac{k}{2\pi i} c (c\tau + d)^{k+1} f(\tau)\,.$$

Idea of proof: a straightforward calculation □

Definition

For a modular form $f \in M_k$, we define the *Serre derivative* by

$$\partial_k f := f' - \frac{k}{12} E_2 f\,.$$

Proposition

For a modular form $f \in M_k$ we have $\partial_k f \in M_{k+2}$.

Proof: We set $g(\tau) = f'(\tau) - \frac{k}{12} E_2(\tau) f(\tau)$, we obtain for $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$

$$\begin{aligned} g\left(\frac{a\tau + b}{c\tau + d}\right) &= f'\left(\frac{a\tau + b}{c\tau + d}\right) - \frac{k}{12} E_2\left(\frac{a\tau + b}{c\tau + d}\right) f\left(\frac{a\tau + b}{c\tau + d}\right) \\ &= (c\tau + d)^{k+2} f'(\tau) + \frac{k}{2\pi i} c (c\tau + d)^{k+1} f(\tau) \\ &\quad - \frac{k}{12}\left((c\tau + d)^2 E_2(\tau) - \frac{6}{\pi} i c (c\tau + d)\right)(c\tau + d)^k f(\tau) \\ &= (c\tau + d)^{k+2}\left(f'(\tau) - \frac{k}{12} E_2(\tau) f(\tau)\right) = (c\tau + d)^{k+2} g(\tau)\,. \end{aligned}$$

Since g is also holomorphic in $\mathbb{H}$ and at ∞ we obtain $g \in M_{k+2}$. □

Definition

The ring of *quasimodular forms* is defined by $\widetilde{M} = \mathbb{C}[E_2, E_4, E_6]$.

Proposition P, Q, R in Ramanujan's word

The ring of quasimodular forms is closed under differentiation and we have

$$E_2' = \frac{E_2^2 - E_4}{12}\,, \quad E_4' = \frac{E_2 E_4 - E_6}{3}\,, \quad E_6' = \frac{E_2 E_6 - E_4^2}{2}\,.$$

Proof: We have

$$\begin{aligned} \partial_4 E_4 &= E_4' - \frac{1}{3} E_2 E_4 \in M_6, \\ \partial_6 E_6 &= E_6' - \frac{1}{2} E_2 E_6 \in M_8. \end{aligned}$$

$2\pi i \dfrac{d}{d\tau} = q\dfrac{d}{dq}$

Since both spaces are one-dimensional with basis E_6 and E_4^2 respectively we get the second and third equation after comparing the first Fourier coefficients. Using again the modularity formula of E_2 and a straightforward calculation (as in the proof of modularity for ∂_k) we find

$$E_2' - \frac{1}{12} E_2^2 \in M_4.$$

Therefore this is also a multiple of E_4, which turns out to be $-\frac{1}{12}$ by comparing the Fourier coefficients. □

Modular forms and L-series

Outlock

The L-series attached to modular forms have many nice aspects ("motivic","automorphic",...). We will study here just some of their basic properties.

We first study the growth of the Fourier coefficients a_n of $f(\tau) = \sum a_n q^n \in M_k$ w.r.t. k.

Proposition

Let $f(\tau) = G_k(\tau)$, then there are $A, B \in \mathbb{R}$ with $A \geqslant 1$ and $B \leqslant \zeta(k-1)$such that

$$An^{k-1} \leqslant a_n \leqslant Bn^{k-1}.$$

Proof: We have

$$G_k = \frac{B_{2k}}{k-1} + \sum \sigma_{k-1}(n)\, q^n,$$

where $\sigma_{k-1}(n) = \sum_{d|n} d^{k-1}$. Clearly $n^{k-1} \leqslant \sigma_{k-1}(n)$. In addition we have,

$$\frac{a_n}{n^{k-1}} = \frac{\sigma_{k-1}(n)}{n^{k-1}} = \sum_{d|n} \frac{d^{k-1}}{n^{k-1}} = \sum_{d|n} \frac{1}{d^{k-1}} \leqslant \sum_{d=1}^{\infty} \frac{1}{d^{k-1}} = \zeta(k-1). \qquad \square$$

Theorem (Hecke bound)

If $f = \sum_{n\geqslant 1} a_n q^n \in S_k$, then[a] $a_n \in \mathcal{O}(n^{\frac{k}{2}})$.

[a] Here the "Big-O-Notation" means that $\left|\frac{a_n}{n^{\frac{k}{2}}}\right|$ is bounded for $n \to \infty$.

Proof: We will show $|a_n| \leqslant \kappa n^{\frac{k}{2}}$ with a suitable constant $\kappa = \kappa(f)$. The k-th Fourier coefficient of f is given by

$$a_n = \int_0^1 f(\tau) e^{-2\pi i n\tau}\, \mathrm{d}\, x \qquad (\tau = x + iy).$$

Hence $|a_n| \leqslant e^{2\pi ny} \sup_{0\leqslant x\leqslant 1} |f(x+iy)|$. We set $\|f\|(\tau) = \big|f(\tau)\big| y^{\frac{k}{2}}$, then

$$\|f\|^2(\gamma\tau) = \|f\|^2(\tau)$$

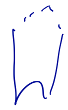
← pointwise
Petersson norm

for all $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ and therefore

$$\sup_{\tau\in\mathbb{H}} \|f\|(\tau) = \sup_{\tau\in\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}} \|f\|(\tau).$$

Since $f \in S_k$ we have in addition

$$\lim_{\tau\to\infty} \|f\|^2(\tau) = \lim_{\tau\to\infty}\left(\Big|\sum_{n\geqslant 1} a_n q^n\Big|^2 y^k\right) = \lim_{\tau\to\infty}\left(|q|^2\Big|\sum_{n\geqslant 1} a_n q^{n-1}\Big|^2 wy^k\right)$$
$$= \lim_{\tau\to\infty}(e^{-4\pi y} y^k) a_n = 0.$$



Hence $\|f\|(\tau)$ is bounded on $\mathbb{H}$ and thus we have a maximum

$$c_f := \max_{\tau\in\mathbb{H}} \|f\|(\tau),$$

which in turn yields

$$|f(x+iy)| = \|f\|(\tau) y^{-\frac{k}{2}} \leqslant c_f y^{-\frac{k}{2}}.$$

Now we always have $|a_n| \leqslant e^{2\pi ny} y^{-\frac{k}{2}} c_f$ and in particular this holds for $y = \frac{1}{n}$. We conclude

$$|a_n| \leqslant e^{2\pi n\frac{1}{n}} \left(\frac{1}{n}\right)^{-\frac{k}{2}} c_f = \kappa n^{\frac{k}{2}}.$$

$\square$

Remark

- If $f \in M_k$, then $a_n = \mathcal{O}(n^k)$. Only for $f \in S_k$ we have the bound $\mathcal{O}(n^{\frac{k}{2}})$.
- The bounds for $f \in S_k$ are far from beeing optimal, indeed it holds

$$a_n = \mathcal{O}(n^{\frac{k-1}{2}})\sigma_0(n)\,.$$

This very deep result follows from Deligne's proof of the Weil conjectures. He was able to connect the Fourier coefficients to the number of points of certain varieties mod p.

Definition

The L-series of a modular form $f(\tau) = \sum_{n\geqslant 1} a_n q^n \in S_k$ is defined to be the Dirichlet-series given by

$$L(f,s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}, \qquad s \in \mathbb{C}.$$

Theorem (Hecke)

i) $L(f,s)$ converges for $\mathrm{Re}(s) > \frac{k}{2} + 1$ to a holomorphic function.

ii) $L(f,s)$ has a holomorphic continuation to the whole of $\mathbb{C}$ and satisfies a the functional equation. Set $R(f,s) = (2\pi)^{-s}\Gamma(s)L(f,s)$, then for all $s \in \mathbb{C}$

$$R(f,k-s) = (-1)^{\frac{k}{2}} R(f,s).$$

Proof: i) By the Hecke bounds $|a_n| \leqslant \kappa n^{\frac{k}{2}}$ we have

$$|a_n n^{-s}| = |a_n|\, n^{-\mathrm{Re}(s)} \leqslant \kappa n^{\frac{k}{2} - \mathrm{Re}(s)}.$$

As $\zeta(s) = \sum_{n\geqslant 1} \frac{1}{n^s}$ converges absolutely for $\mathrm{Re}(s) > 1$ we deduce $L(f,s)$ converges absolutely for $\mathrm{Re}(s) > \frac{k}{2} + 1$.

ii) We first recall

$$\Gamma(s) = \int_{-\infty}^{\infty} t^{s-1} e^{-t}\,\mathrm{d}\,t$$

$$n^{-s} = (2\pi)^s \Gamma(s)^{-1} \int_0^{\infty} t^{s-1} e^{-2\pi n t}\,\mathrm{d}\,t.$$

This yields for $L(f,s)$

$$\begin{aligned} L(f,s) = \sum_{n\geqslant 1} a_n n^{-s} &= \sum_{n\geqslant 1} a_n (2\pi)^s \Gamma(s)^{-1} \int_0^{\infty} t^{s-1} e^{-2\pi n t}\,\mathrm{d}\,t \\ &= (2\pi)^s \Gamma(s)^{-1} \int_0^{\infty} t^{s-1} \sum_{n\geqslant 1} a_n e^{-2\pi n t}\,\mathrm{d}\,t \\ &= (2\pi)^s \Gamma(s)^{-1} \int_0^{\infty} t^{s-1} f(it)\,\mathrm{d}\,t. \end{aligned}$$

Observe $L(f,s)$ is nothing else than what is also called a *Mellin-transform*.

We further get

$$\begin{aligned} (2\pi)^{-s}\Gamma(s)L(f,s) = \int_0^{\infty} t^{s-1} f(it)\,\mathrm{d}\,t &= \int_0^{1} t^{s-1} f(it)\,\mathrm{d}\,t + \int_1^{\infty} t^{s-1} f(it)\,\mathrm{d}\,t \\ &= \int_1^{\infty} \left(\frac{1}{t}\right)^{s-1} f\left(\left(\frac{1}{t}\right)\right) \mathrm{d}\left(\frac{1}{t}\right) + \int_1^{\infty} t^{s-1} f(it)\,\mathrm{d}\,t \\ &\overset{\heartsuit}{=} \int_1^{\infty} (-1)^{\frac{k}{2}} t^{k-s-1} f(it)\,\mathrm{d}\,t + \int_1^{\infty} t^{s-1} f(it)\,\mathrm{d}\,t \\ &= \int_1^{\infty} \left((-1)^{\frac{k}{2}} t^{k-s-1} + t^{s-1}\right) f(it)\,\mathrm{d}\,t =: \int_1^{\infty} \varepsilon(s,t) f(it)\,\mathrm{d}\,t. \end{aligned}$$

Since $\varepsilon(k-s,t) = (-1)^{\frac{k}{2}}\varepsilon(s,t)$ we deduce the functional equation

$$R(f,k-s) = (-1)^{\frac{k}{2}} R(f,s).$$

Moreover since $(2\pi)^s$ and $\Gamma(s)^{-1}$ are holomorphic and since $\int_1^{\infty} \varepsilon(s,t) f(it)\,\mathrm{d}\,t$ has a holomorphic continuation the last claim follows.

□

Remark

- The functional equation for $L(f,s)$ is a direct consequence of the modular transformation w.r.t. S of f.
- Conversely there is a famous theorem of Weil: A L-series $L(s)$ with a functional equation $L(s) = L(k-s)\cancel{\Gamma}\gamma(s)$ equals $L(f,s)$ for some modular form f, if $L(s)$ has certain growth properties along vertical stripes.
- The functional equation for the Riemann zeta function is proven along the same path, more precisely one uses that the ϑ-function is a modular form for a subgroup of $\mathrm{SL}_2(\mathbb{Z})$.

Motivation: The Riemann zeta function has an *Euler product*

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \text{ prim}} \frac{1}{(1-p^{-s})},$$

as we have the geometric series expansion

$$\frac{1}{1-p^{-s}} = 1 + \frac{1}{p^s} + \frac{1}{p^{2s}} + \cdots$$

Question

Which of the L-series $L(f,s)$ do have an Euler product, i.e. have the shape

$$L(f,s) = \cancel{\sum_{n=1}^{\infty}} \prod_{p \in \mathcal{P}} L_p(f,s)$$

for some rational functions $L_p(f,s)$?

Answer: Hecke eigenforms will do.

Definition (Hecke operator)

For $n \geqslant 1$ we define

$$T(n)f(\tau) = n^{k-1} \sum_{\substack{a \geqslant 1,\, ad=n \\ 0 \leqslant b < d}} d^{-k} f\left(\frac{a\tau + b}{d}\right).$$

$$\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$$

Proposition

(i) If $f \in M_k$, then $T(n)f \in M_k$. In particular, if $f(\tau) = \sum_{n \geqslant 0} a_n q^n$ with $q = e^{2\pi i \tau}$, then

$$T(n)f(\tau) = \sum_{m=0}^{\infty} \gamma_m q^m$$

with

$$\gamma_m = \sum_{\substack{d \mid (n,m) \\ d \geqslant 1}} d^{k-1} a_{\frac{nm}{d^2}}.$$

(ii) We have

$$T(m)T(n)f = T(mn)f, \qquad \text{if } (n,m) = 1$$

$$T(p)T(p^n)f = T(p^{n+1})f + p^{k-1}T(p^{n-1})f, \qquad \text{if } p \text{ prime and } n \geqslant 1.$$

Idea of Proof: Clearly $T(n)f$ is holomorphic on $\mathbb{H}$. Let

$$M_n = \{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{Mat}_{2,2}(\mathbb{Z}) \mid ad - bc = n \},$$

then $\mathrm{SL}_2(\mathbb{Z})$ acts by multiplication from the left and

$$\mathrm{SL}_2(\mathbb{Z}) \backslash M_n = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \middle| ad = n,\ a,b,d \in \mathbb{Z},\ 0 \leqslant b < d \right\}.$$

Hence

$$T(n)f(\tau) = n^{k-1} \sum_{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z}) \backslash M_n} (c\tau + d)^{-k} f\left(\frac{a\tau + b}{c\tau + d}\right)$$

and the sum is unchanged if we replace $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ by $\gamma\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ with some $\gamma \in Sl_2(\mathbb{Z})$. Therefore $T(n)f$ satisfies the modular transformation property.

It remains to show that $T(n)f$ is holomorphic at ∞: We have

$$T(n)f(n) = n^{k-1} \sum_{\substack{ad=n \\ a\geqslant 1, 0\leqslant b<d}} d^{-k} \sum_{m=0}^{\infty} a_m e^{2\pi i \frac{a\tau+b}{d} m}.$$

Using

$$\sum_{0\leqslant b<d} e^{2\pi i m \frac{b}{d}} = \begin{cases} d, & d|m \\ 0, & \text{sonst} \end{cases}$$

Replacing m by $md = mn/a$ and using $n/d = a$, we find

$$T(n)f(\tau) = \sum_{m\in\mathbb{Z}} \sum_{ad=n, a\geqslant 1} a^{k-1} a_{\frac{mn}{a}} q^{am} = \sum_{m\in\mathbb{Z}} \sum_{\substack{d|(m,n) \\ d\geqslant 1}} d^{k-1} a_{\frac{nm}{d^2}} q^m.$$

Hence for $m < 0$ we get $\gamma_m = 0$, and therefore $T(n)f$ is holomorphic at ∞.

(ii) see reference, e.g. F. Martin & E. Royer, FORMES MODULAIRES ET PERIODES. $\square$

Proposition

The Hecke operators act on M_k and on S_k.

Proof: With above notation we have $\gamma_0 = \sigma_{k-1}(n)\, a_0$. If $f \in S_k$, then $a_0 = 0$ and therefore $T(m)f = \sum \gamma_m q^m$ is also a cusp form since $\gamma_0 = 0$. $\square$

Serre's course arithmetique

Shimura's book

Definition (Hecke algebra)

We denote by $\mathbb{T}$ the algebra generated by $T(n)$ for all $n \in \mathbb{N}$.

Remark

There are other ways to approach Hecke operators, namely double cosets, correspondences of lattices or via pullback and pushforwards maps between modular curves. Outstanding results rely deep connections of the Hecke algebra to arithmetic questions (e.g. Mazur's theorem on torsion subgroups of rational elliptic curves, Wiles's proof of Fermat's last theorem,...)

Rohrlich's article

Proposition (Some linear algebra)

Let $\mathbb{T}$ be a finite dimensional, commutative $\mathbb{C}$-algebra. Let $V \neq 0$ be a finite dimensional $\mathbb{C}$-vector space and $\mathbb{T} \hookrightarrow \mathrm{End}(V)$. Then there exists a simultaneous eigenvector $v \in V$ for all $T \in \mathbb{T}$.

Proof: There exists a finite set of generators $T_1, \ldots, T_n$ of $\mathbb{T}$, as $\dim V < \infty$. Since the characteristic polynomial of T_1 has a zero, we have an eigen space $0 \neq V_1 \subseteq V$ for T_1. It suffices to show $T_i V_1 \subset V_1$. For all $v \in V_1$ we have $T_1 v = \lambda_1 v$ and we get

$$T_1(T_i v) = T_i(T_1 v) = T_i \lambda_1 v = \lambda_1 (T_i v).$$

Hence $T_i V_1 \subset V_1$ for all T_i.
We repeat this procedure for T_2 as operator on V_1 and obtain an eigen space $V_2 \subseteq V_1 \subseteq V$ for T_2 with $T_i V_2 \subseteq V_2$ for all T_i.
After finitely many steps we obtain an eigenvector $v_n \neq 0$ for all $T \in \mathbb{T}$. $\square$

Observe $T(n)$ are endomorphisms of the vector spaces M_k and S_k. We a searching for modular forms that are eigenfunctions for all $T(n)$:

Definition

A modular form $f \in M_k$, $f \neq 0$, which is an eigenform for all $T(n)$ is called Hecke eigenform. If in addition $a_1(f) = 1$, then f is called *normalised Hecke eigenform*. Some authors also use solely *eigenform* or *Hecke form*.

Theorem (Hecke eigenforms)

Let $f(\tau) = \sum a_n q^n$ be a modular form with $T(n)f = \lambda(n)f$ for all $n \in \mathbb{N}$ (f is called a eigenform), then

(i) $a_1 = a_1(f) \neq 0$
(ii) If f is normalised, then $a_n = \lambda(n)$.

Proof: The coefficient γ_1 of $T(n)f$ equals a_n, and by assumption $\gamma_1 = \lambda(n)a_1$, hence $a_n = \lambda(n)a_1$. $\square$
Any simultanous Hecke-Eigenspace is one dimensional, since for $f, g \in M_k$ with $T(n)f = \lambda(n)f$ and $T(n)g = \lambda(n)g$ we get $f = \kappa g$ for some $\kappa \in \mathbb{C}$.

Corollary

If $f = \sum a_n q^n$ is a normalised Hecke eigenform then

$$a_n a_m = a_{nm} \qquad \text{for } (n,m) = 1$$

$$a_p a_{p^n} = a_{p^{n+1}} + p^{k-1} a_{p^{n-1}} \qquad \text{if } p \text{ prime and } n \geqslant 1.$$

Proof: This follows from the recursive properties for the Hecke operators. □

Theorem

The L-series $L(f,s)$ of a normalised Hecke eigenform has a Euler product

$$L(f,s) = \sum_{n \geqslant 1} \frac{a_n}{n^s} = \prod_{p \text{ prim}} \frac{1}{1 - a_p p^{-s} + p^{k-1-2s}}$$

Proof: The series $\{a_n\}_{n\in\mathbb{N}}$ given by the Fourier coefficients of f determines a multiplicative function $a : \mathbb{N} \to \mathbb{C}, n \mapsto a_n$. For any such multiplicative function we get

$$L(f,s) = \prod_{p \in \mathbb{P}} \sum_{n=0}^{\infty} a_{p^n} p^{-ns}.$$

If we set $T = p^{-s}$, we have to check

$$\sum_{n=0}^{\infty} a_{p^n} T^n = \frac{1}{1 - a_p T + p^{k-1} T^2}.$$

But this is equivalent to the identity we have seen before

$$1 = (1 - a_p T + p^{k-1} T^2) \cdot \sum_{n=0}^{\infty} a_{p^n} T^n = 1 + \sum_{n=1}^{\infty} \underbrace{\left(a_{p^n} - a_p a_{p^{n-1}} + p^{k-1} a_{p^{n-2}} \right)}_{=0} T^n.$$

because it is Hecke eigenform! □

Edixhove, Couvegnes
De Jong, Bruin, ..

Theorem

The cusp form

$$\Delta(\tau) = q \prod_{n=1}^{\infty} (1 - q^n)^{24} = \sum_{n=1}^{\infty} \tau(n) q^n \in S_{12}$$

ist is a normalised Hecke eigenform.

Proof: Obvious, since $\dim S_{12} = 1$. □

Remark

By the above theorem we have that $\tau(n)$ is a multiplicative arithmetic function. For example we have

$$\tau(2)\tau(3) = -24 \cdot 252 = -6048 = \tau(6)$$

$$\tau(2)^2 = 576 = -1472 + 2048 = \tau(4) + 2^{11}.$$

Conjecture (Lehmer) $\tau(p) \neq 0 \Rightarrow \tau(n) \neq 0$

tested up to very large primes (~20 Digits)

Theorem

(i) The Eisenstein series

$$G_k = -\frac{B_k}{2k} + \sum_{n=1}^{\infty} \sigma_{k-1}(n)\, q^n$$

is a Hecke eigenform.

(ii) G_k has the L-series

$$L(G_k, s) = \sum_{n=1}^{\infty} \frac{\sigma_{k-1}(n)}{n^s} = \zeta(s)\zeta(s-k+1).$$

Proof: (i) It is straightforward to show that σ_{k-1} is a multiplicative function satisfying the required recursive property for prime powers, i.e.

$$\sigma_{k-1}(n)\,\sigma_{k-1}(m) = \sigma_{k-1}(nm) \qquad \text{for } (n,m) = 1$$

$$\sigma_{k-1}(p)\,\sigma_{k-1}(p^n) = \sigma_{k-1}(p^{n+1}) + p^{k-1}\sigma_{k-1}(p^{n-1}) \qquad \text{if } p \text{ prime and } n \geqslant 1.$$

(ii) We get

$$\sum_{n=1}^{\infty} \frac{\sigma_{k-1}(n)}{n^s} = \sum_{a,d \geqslant 1} \frac{a^{k-1}}{a^s d^s} = \sum_{d \geqslant 1} \frac{1}{d^s} \sum_{a \geqslant 1} \frac{1}{a^{s-k+1}} = \zeta(s)\zeta(s-k+1). \quad □$$

Examples

$$
\begin{aligned}
\sigma_3(6) &= 1+2^3+3^3+6^3 = \left(1+2^2\right)\left(1+3^3\right) = \sigma_3(2)\,\sigma_3(3)\\
\sigma_3\left(2^3\right) &= 1+2^3+4^3+8^3 = \left(1+2^3\right)\left(1+2^3+4^3\right) - 2^3\left(1+2^3\right)\\
&= \sigma_3(2)\,\sigma_3(4) - 2^3\sigma_3(2)\\
\sigma_k(p)\,\sigma_k(p^n) &= \sigma_k\left(p^{n+1}\right) + p^k\sigma_k\left(p^{n-1}\right).
\end{aligned}
$$

Observe we had not use the fact that k is odd!

We can naively search for eigenforms. For example S_{24} is a two dimensional $\mathbb{C}$-vector space spanned by

$$
\Delta^2(\tau) = \left(q\prod_{n=1}^{\infty}(1-q^n)^{24}\right)^2 = q^2 - 48q^3 + 1080q^4 + \cdots
$$

$$
\Delta(\tau)(240G_4)^3(\tau) = q + 696q^2 + 162252q^3 + 12831808q^4 + \cdots
$$

Any normalised eigenform $f \in S_{24}$ must equal

$$
f = (240G_4)^3\Delta(\tau) + \lambda\Delta(\tau)^2
$$

for some $\lambda \in \mathbb{C}$. If $T(2)f = \mu f$, then

$$
a_2^2 = a_4 + 2^{23}a_1.
$$

must hold

Since $a_2 = 696 + \lambda$ and $a_4 = 12831808 + 1080\lambda$ we get that λ is a root of

$$
\lambda^2 + \frac{3}{2}\lambda - 20736000 = 0.
$$

Therefore f is a Hecke-Eigenform if and only if

$$
f = (240G_4)^3\Delta + (156 \pm 12\sqrt{14469})\Delta^2.
$$

Corollary

There are Hecke-eigenforms in M_k resp. S_k.

Actually we aim for more

Proposition

If $(V, \langle\,,\,\rangle)$ is a hermitian $\mathbb{C}$-vector space and if $T \in \mathrm{End}(V)$ is self-adjoint, i.e.

$$
\langle Tv, w\rangle = \langle v, Tw\rangle \quad \text{for all } v, w \in V,
$$

then V has a basis given by eigenvectors for T.

Thus we need to find a hermitian scalar product on S_k s.t. the Hecke operator $T(n)$ are self-adjoint.

Theorem (Petersson)

If $f, g \in S_k$, then

$$
\langle f, g\rangle = \int_{\mathcal{F}=\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}} f(\tau)\overline{g(\tau)}y^k\frac{\mathrm{d}\,x\,\mathrm{d}\,y}{y^2}
$$

defines a positive definite hermitian scalar product on S_k. It is called the *Petersson scalar product*

Proof: We have to check

- well-definedness for $\langle\,,\,\rangle : S_k \times S_k \to \mathbb{C}$.
- $\langle v, \lambda\omega_1 + \mu\omega_2\rangle = \overline{\lambda}\langle v_1, \omega_1\rangle + \overline{\mu}\langle v_1, \omega_2\rangle$
- $\langle v, w\rangle = \overline{\langle w, v\rangle}$.
- $\langle\,,\,\rangle$ is positive definite, i.e. $\langle v, v\rangle \geqslant 0$ and $\langle v, v\rangle = 0$ if and only if $v = 0$.

Only the first claim needs some detailed justification.

At first we see that the integrand is $\mathrm{SL}_2(\mathbb{Z})$-invariant, indeed

$$f\Big(\frac{a\tau+b}{c\tau+d}\Big) = (c\tau+d)^k f(\tau)$$
$$\overline{g\Big(\frac{a\tau+b}{c\tau+d}\Big)} = \overline{(c\tau+d)}^k\,\overline{g(\tau)}$$
$$\mathrm{Im}\Big(\frac{a\tau+b}{c\tau+d}\Big) = |c\tau+d|^{-\overset{2}{k}}\,\mathrm{Im}(\tau)$$

Hence

$$f\Big(\frac{a\tau+b}{c\tau+d}\Big)g\Big(\frac{a\tau+b}{c\tau+d}\Big)\,\mathrm{Im}\Big(\frac{a\tau+b}{c\tau+d}\Big)^k = f(\tau)\overline{g}(\tau)\,\mathrm{Im}(\tau)^k.$$

Furthermore we have

$$\mathrm{d}\Big(\frac{a\tau+b}{c\tau+d}\Big) = (c\tau+d)^{-2}\,\mathrm{d}\,\tau$$
$$\overline{\mathrm{d}\Big(\frac{a\tau+b}{c\tau+d}\Big)} = (\overline{c\tau+d})^{-2}\,\mathrm{d}\,\overline{\tau},$$

and

$$\frac{\mathrm{d}\Big(\frac{a\tau+b}{c\tau+d}\Big)\,\mathrm{d}\,\overline{\Big(\frac{a\tau+b}{c\tau+d}\Big)}}{\mathrm{Im}\Big(\frac{a\tau+b}{c\tau+d}\Big)} = \frac{\mathrm{d}\,\tau\,\mathrm{d}\,\overline{\tau}}{\mathrm{Im}(\tau)^2} = 2i\frac{\mathrm{d}\,x\,\mathrm{d}\,y}{y^2}.$$

Therefore the integral is invariant under the modular transformations $\tau \mapsto \frac{a\tau+b}{c\tau+d}$.

The integral exists:

$$f(\tau)\overline{g(\tau)} = \sum_{n\geqslant 1} a_n q^n \sum_{m\geqslant 1}\overline{b_m q^m} = \Big(\sum_{n\geqslant 1} a_n e^{2\pi i n x}e^{-2\pi n y}\Big)\Big(\sum_{m\geqslant 1}\overline{b_m}e^{-2\pi i m x}e^{-2\pi m y}\Big)$$
$$= \sum_{n\geqslant 1} a_n b_n e^{-4\pi n y} + \text{„}\sum_l \text{ with } e^{2\pi i l x}\text{-terms“}$$

We now cut the Fundamental domain $\mathcal{F}$ at the line $y = T_0$.

$$\int_{\mathcal{F}} f(\tau)\overline{g(\tau)}y^k\frac{\mathrm{d}\,x\,\mathrm{d}\,y}{y^2} = \int_A f(\tau)\overline{g(\tau)}y^k\frac{\mathrm{d}\,x\,\mathrm{d}\,y}{y^2} + \int_B f(\tau)\overline{g(\tau)}y^k\frac{\mathrm{d}\,x\,\mathrm{d}\,y}{y^2}$$

For the bounded domain A the integral exists by continuity of the integrand. Because of

$$f(\tau)\overline{g(\tau)} = \cdots = \sum a_n\overline{b_n}e^{-4\pi n y} + \text{terms with } e^{2\pi i m x}$$

we get for all $m \in \mathbb{Z}\backslash\{0\}$, using $\int\limits_{-\frac12\leqslant x\leqslant\frac12} h(y)e^{2\pi i m x} = 0$,

$$\int_B f(\tau)\overline{g(\tau)}y^k\frac{\mathrm{d}\,x\,\mathrm{d}\,y}{y^2} = \lim_{T_1\to\infty}\int\limits_{\substack{-\frac12\leqslant x<\frac12\\ T_0<y<T_1}} f(\tau)\overline{g(\tau)}y^k\frac{\mathrm{d}\,x\,\mathrm{d}\,y}{y^2}$$
$$= \lim_{T_1\to\infty}\int\limits_{T_0\leqslant y\leqslant T_1}\sum_{n=1}^{\infty} a_n\overline{b_n}e^{-4\pi n y}y^{k-2}\,\mathrm{d}\,y,$$

For the remaining integral we get

$$\sum a_n\overline{b_n}e^{-4\pi n y} = e^{-4\pi y}\Big(\sum a_n\overline{b_n}e^{-4\pi(n-1)y}\Big) = e^{-4\pi y}\Big(a_1\overline{b_1} + a_2\overline{b_2} + \cdots\Big).$$

Because of the Hecke-bound for the growth of the Fourier coefficients the sum is bounded, thus

$$\lim_{T_1\to\infty}\int\limits_{T_0\leqslant y\leqslant T_1}\sum a_n\overline{b_n}e^{-4\pi n y}y^{k-2}\,\mathrm{d}\,y \leqslant \lim_{T_1\to\infty}\kappa\int\limits_{T_0\leqslant y\leqslant T_1} e^{-4\pi y}y^{k-2}\,\mathrm{d}\,ty < \infty.$$

We conclude that Petersson's scalar product is well-defined. □

Theorem (Petersson, Hecke)

The Hecke operators are self-adjoint w.r.t. the Peterson scalar product, i.e.,

$$\langle T(n)f, g\rangle = \langle f, T(n)g\rangle$$

for alle $T(n) \in \mathbb{T}(S_k)$ and $f, g \in S_k$.

Proof: This a long calculation, we refer to the literature.

Consequences from linear algebra

- S_k has an orthonormal basis given by Hecke- eigenforms.
- The Fourier coefficients of normalised Hecke eigenforms are real-algebraic numbers, since for any self-adjoint operator F with eigenvector v we get

$$\lambda\langle v, v\rangle = \langle Fv, v\rangle = \langle v, Fv\rangle = \overline{\lambda}\langle v, v\rangle,$$

hence $\lambda = \overline{\lambda}$.

Theorem

We have a bilinear and non-degenerate pairing

$$\begin{aligned} M_k \times \mathbb{T}(M_k) &\to \mathbb{C} \\ (f, T) &\mapsto a_1(Tf). \end{aligned}$$

Proof: The pairing is clearly bilinear. We have to show, if $a_1(T_n f) = 0$ for all $n \in \mathbb{N}$ then we must have $f = 0$. If $a_1(T(f)) = 0$ for all $f \in M_k$, then we have for all n and all f

$$a_1(TT_n f) = a_1(T_n(Tf)) = a_n(Tf) = 0.$$

Therefore $Tf = 0$ for all f, which implies$T = 0$. □

Corollary

The map

$$\begin{aligned} M_k &\to \mathrm{Hom}_{\mathbb{C}}(\mathbb{T}(M_k), \mathbb{C}) \\ f &\mapsto (T \mapsto a_1(T(f))) \end{aligned}$$

is an isomorphism of $\mathbb{T}(M_k)$-modules.

Of practical use[2] is the converse direction: Is $\varphi \in \mathrm{Hom}_{\mathbb{C}}(\mathbb{T}(M_k), \mathbb{C})$, then we get the modular form

$$f_\varphi = \sum_{m=0}^{\infty} \varphi(T(m))q^m.$$

For example, if

$$\mathbb{T}(M_k) \hookrightarrow \mathrm{End}_{\mathbb{C}}(M_k) \cong M_{n\times n}(\mathbb{C})$$

$$T \mapsto \begin{pmatrix} a_{11}(T) & \cdots & a_{1n}(T) \\ \vdots & \ddots & \vdots \\ a_{n1}(T) & \cdots & a_{nn}(T) \end{pmatrix}$$

then $\varphi(T) = a_{ij}(T) \in \mathrm{Hom}(\mathbb{T}, \mathbb{C})$.

[2] e.g. application in SAGE as descripted in the notes by W. Stein.

Rankin-Selberg method

For the Eisenstein series we have a decomposition

$$G_k = \frac{(k-1)!}{(2\pi i)^k}\frac{1}{2}\sum_{m,n}{}' \frac{1}{(m\tau+n)^k} = \frac{(k-1)!}{(2\pi i)^k}\frac{1}{2}\sum_{r=1}^{\infty}\frac{1}{r^k}\sum_{\substack{m,n\in\mathbb{Z}\\(m,n)=1}}\frac{1}{(m\tau+n)^k}$$

$$= \frac{(k-1)!}{(2\pi i)^k}\frac{1}{2}\zeta(k)\sum_{\substack{m,n\in\mathbb{Z}\\(m,n)=1}}\frac{1}{(m\tau+n)^k}$$

This works because of Euklid's lemma: For every $(c, d) \in \mathbb{Z}^2$ with $(c, d) = 1$ there exist a $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$. Given such a solution, then

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) = \begin{pmatrix} a+c & b+d \\ c & d \end{pmatrix}$$

gives the equation$(a+c)d + (b+d)c = 1$. If we plug $m = c$ and $n = d$ into G_k, then we derive

$$G_k = \frac{(k-1)!}{(2\pi i)^k}\frac{1}{2}\zeta(k)\sum_{\langle\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)\rangle\backslash\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\in\mathrm{SL}_2(\mathbb{Z})}\frac{1}{(c\tau+d)^k}.$$

parabolic subgroup

Theorem (Rankin-Selberg unfolding)

Let $f(\tau) = \sum_{n=1}^{\infty} a_n q^n \in S_k$ and $g(\tau) = \sum_{m=0}^{\infty} b_m q^m \in M_l$. Then, if $k - l \geqslant 2$ we have for the Petersson scalarproduct

$l \leq k$

$$\langle f(\tau), g(\tau) G_{k-l}(\tau) \rangle = \frac{(k-2)!}{(4\pi)^{k-1}} \sum_{n=1}^{\infty} \frac{a_n \overline{b_n}}{n^{k-1}}.$$

Proof: By definition we have $k = l \Rightarrow G_0 = 1$

$$\langle f(\tau), g(\tau) G_{k-l}(\tau) \rangle = \int_{\mathcal{F}} 1 f(\tau) \overline{1 g(\tau) G_{k-l}(\tau)} y^k \frac{\mathrm{d}\, x \,\mathrm{d}\, y}{y^2}.$$

Because of above notation we get for the integrand

$$\frac{(2\pi i)^k}{(k-1)!\zeta(k)} f(\tau) \overline{g(\tau) G_{k-l}(\tau)} y^k$$

$$= \sum_{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \langle \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right) \rangle \backslash \mathrm{SL}_2(\mathbb{Z})} (c\tau + d)^k f(\tau) \overline{(c\tau + d)^{l} g(\tau)} \underbrace{\frac{y^k}{\left|(c\tau + d)\right|^{2k}}}$$

$$= \sum_{\gamma \in \langle \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right) \rangle \backslash \mathrm{SL}_2(\mathbb{Z})} f(\gamma\tau) \overline{g(\gamma\tau)}\, \mathrm{Im}(\gamma\tau)^k \cdot (c\tau+d)^k \, (\overline{c\tau+d})^l \, (\overline{c\tau+d})^{k-l}$$

"Introducing a "$1 = \frac{c\tau+d}{c\tau+d}$""

We apply this identity to the integral

$$\frac{(2\pi i)^k}{(k-1)!\zeta(k)} \int_{\mathcal{F}} f(\tau) \overline{g(\tau) G_{k-l}(\tau)} y^k \frac{\mathrm{d}\, x \,\mathrm{d}\, y}{y^2}$$

$$= \int_{\mathcal{F}} \sum_{\gamma \in \langle \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right) \rangle \backslash \mathrm{SL}_2(\mathbb{Z})} \mathrm{Im}(\gamma\tau)^k f(\gamma\tau) \overline{g(\gamma\tau)} \frac{\mathrm{d}\, x \,\mathrm{d}\, y}{y^2}.$$

Unfolding the domain of integration, i.e. interchanging the integral and the sum, leads to

$$\int_0^{\infty} \int_0^{1} f(x+y) \overline{g(x+iy)} \,\mathrm{d}\, x y^k \frac{\mathrm{d}\, y}{y^2} = \int_0^{\infty} \sum_{n=1}^{\infty} a_n \overline{b_n} e^{-4\pi n y} y^{k-2} \,\mathrm{d}\, y$$

$$= \frac{(k-2)!}{(4\pi)^{k-1}} \sum_{n=1}^{\infty} \frac{a_n \overline{b_n}}{n^{k-1}}$$

□

$\langle f, G_{k-a} G_a \rangle$

Proposition

Let $f(\tau) = \sum_{n=1}^{\infty} a_n q^n$ be a Hecke Eigenform of weight k. Then we have

$$\sum_{n=1}^{\infty} \frac{a_n \sigma_{k-1}(n)}{n^s} = \frac{L(f,s) L(f, s-k+1)}{\zeta(2s+2)}$$

Proof: Use the Euler product & $a_n \in \mathbb{R}$ ... □

For example we get

$$(\Delta, G_4 G_8) = * \frac{L(\Delta, 11) L(\Delta 9)}{\zeta(6)}$$

this will be used for the Theorem of Eichler-Shimura-Manin.