Multiple Eisenstein series, multiple divisor functions and applications to multiple q-zeta values

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Multiple zeta values

Definition

Let $s_1 \geq 2, s_2, \ldots, s_l \geq 1$ be natural numbers, then we call sums of the type

$$\zeta(s_1, \ldots, s_l) = \sum_{n_1 > \ldots > n_l > 0} \frac{1}{n_1^{s_1} \ldots n_l^{s_l}}$$

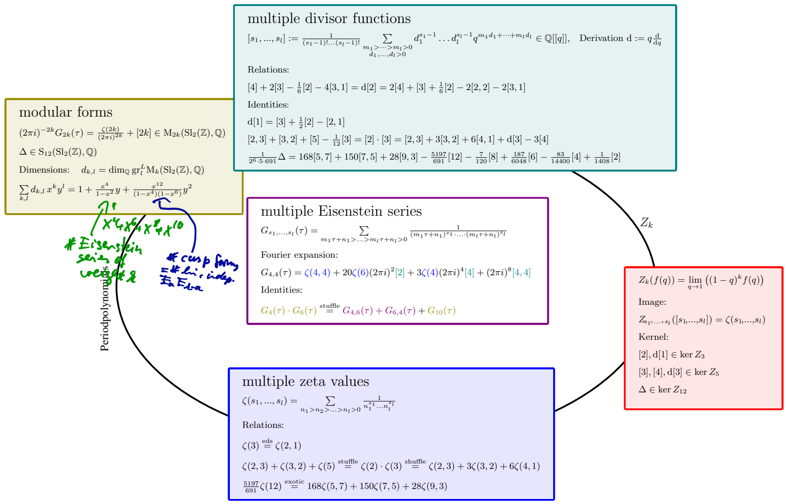
a multiple zeta value (MZV) of weight $s_1 + \ldots + s_l$ and depth l.

- The product of two MZV can be expressed as a linear combination of MZV with the same weight (stuffle relation). e.g:

$$\zeta(r) \cdot \zeta(s) = \zeta(r, s) + \zeta(s, r) + \zeta(r + s)\,.$$

- MZV can be calculated by iterated integrals. This gives another way (shuffle relation) to express the product of two MZV as a linear combination of MZV.
- These two ways to express products give a lot of $\mathbb{Q}$-relations between MZV (double shuffle relations).

$$\zeta(r)\cdot\zeta(s) = \sum_{n>0}\frac{1}{n^r}\cdot\sum_{m>0}\frac{1}{m^s} = \sum_{n>m>0}\frac{1}{n^r m^s} + \sum_{m>n>0}\frac{1}{m^s n^r} + \sum_{l=n=m>0}\frac{1}{l^{s+r}}$$



Multiple zeta-values

Example:

$$\zeta(2,3) + 3\zeta(3,2) + 6\zeta(4,1) \overset{\text{shuffle}}{=} \zeta(2)\cdot\zeta(3) \overset{\text{stuffle}}{=} \zeta(2,3) + \zeta(3,2) + \zeta(5)\,.$$

$$\Longrightarrow 2\zeta(3,2) + 6\zeta(4,1) \overset{\text{double shuffle}}{=} \zeta(5)\,.$$

But there are more relations between MZV. e.g.:

$$\zeta(2,1) = \zeta(3).$$

These follow from the "extended double shuffle relations" where one use the same combinatorics as above for "$\zeta(1)\cdot\zeta(2)$" in a formal setting. The extended double shuffle relations are conjectured to give all relations between MZV.

Let $\mathcal{A}_z$ be a finite alphabet $\{z_1, z_2, z_3, ...\}$. A word is an ordered sequence $w = z_{i_1}...z_{i_l}$ of elements taken from $\mathcal{A}_z$, with repetition allowed. We include the empty word $\emptyset$ (or 1). We use the concatenation product $w \cdot w'$ and denote by $\mathcal{A}_z^*$ the set of all words. We take $\mathcal{A}_z^*$ as a basis of the vector space $\mathbb{Q} < \mathcal{A}_z >$ of noncommutative polynomials. The concatenation of words defines by linearity a multiplication on $\mathbb{Q} < \mathcal{A}_z >$.

Definition

The stuffle product $*$ on $\mathbb{Q} < \mathcal{A}_z >$ is defined by linear extension of the recursion given by

$$1 * w = w * 1 = w$$

for all $w \in \mathcal{A}_z^*$ and for words w, w' and letters z_1, z_j by

$$z_i w * z_j w' = z_{i_1} \cdot (w * z_j w') + z_j \cdot (z_i w * w') + z_{i+j} \cdot (w * w').$$

in general a pairing

Proposition (Hoffmann)

The algebra $\left(\mathbb{Q} < \mathcal{A}_z >, *\right)$ is a commutative and associative $\mathbb{Q}$-algebra.

The stuffle product is an example for a quasi-shuffle product.

$\phi : A \times A \to A$

$(z_i, z_j) \to \phi(z_i, z_j)$

$z_i z_j * z_l = z_i (z_j * z_l) + z_l (z_i z_j * 1) + z_{i+l} (z_j * 1)$

Definition

We denote by $\mathcal{MZ} \in \mathbb{R}$ the $\mathbb{Q}$-subalgebra generated by multiple zeta values.

$= z_i (z_j * z_l) + z_l z_i z_j + z_{i+l} z_j$

Theorem (Hoffmann)

There is a unique homomorphism of $\mathbb{Q}$-algebras

$$\zeta^* : \left(\mathbb{Q} < \mathcal{A}_z >, *\right) \to \mathcal{MZ}$$

such that $\zeta(z_1) = 0$ and for all words $w = z_{i_1}...z_{i_l}$ with $i_1 \geq 2$

$$\zeta^*(z_{i_1}...z_{i_l}) = \zeta(s_{i_1}, ..., s_{i_l}).$$

Remark

There is also a similar homomorphism $\zeta^{ш} : \left(\mathbb{Q} < \{x, y\} >, ш\right) \to \mathcal{MZ}$ which can be used to describe the shuffle product. The comparison of ζ^* and $\zeta^{ш}$, then leads to the extended double shuffle relations.

$\zeta^*(z_i * z_j) = \zeta^*(z_i z_j + z_j z_i + z_{i+j})$

$= \zeta(s_i, s_j) + \zeta(s_j, s_i) + \zeta(s_{i+j})$

Dimension conjectures for $\mathcal{MZ}$

Consider the formal powerseries

$$\mathsf{E}_2(x) = \frac{x^2}{1-x^2} = x^2 + x^4 + x^6 + ... \qquad \text{"even zetas"},$$

$$\mathsf{O}_3(x) = \frac{x^3}{1-x^2} = x^3 + x^5 + x^7 + ... \qquad \text{"odd zetas"},$$

$$\mathsf{S}(x) = \frac{x^{12}}{(1-x^4)(1-x^6)} = x^{12} + x^{16} + x^{18} + ... \qquad \text{"period polynomials"}.$$

Broadhurst-Kreimer Conjecture

The $\mathbb{Q}$-algebra $\mathcal{MZ}$ of multiple zeta values is a free polynomial algebra, which is graded for the weight and filtered for the depth ("depth drop for even zetas"). The numbers $g_{k,l}$ of generators in weight $k \geq 3$ and depth l are determined by

$$BK(x,y) = \sum_{k,l \geq 0} \dim_{\mathbb{Q}} \left(\mathrm{gr}_{k,l}^{W,D} \mathcal{MZ}\right) x^k y^l = \left(1 + \mathsf{E}_2(x)\, y\right) \prod_{k \geq 3, l \geq 1} \frac{1}{(1 - x^k y^l)^{g_{k,l}}}$$

$H(\mathbb{Q}[\zeta(2)])$

where

$$BK(x,y) = \left(1 + \mathsf{E}_2(x) y\right) \frac{1}{1 - \mathsf{O}_3(x) y + \mathsf{S}(x) y^2 - \mathsf{S}(x) y^4}.$$

Dimension conjectures for $\mathcal{MZ}$

Zagier's Conjecture

The following identities hold:

$$Zag(x) = \sum_{k \geq 0} \dim_{\mathbb{Q}} \left(\mathrm{gr}_k^W \mathcal{MZ}\right) x^k = \frac{1}{1 - x^2 - x^3}.$$

Zagier's conjecture is implied by Broadhurst-Kreimer's conjecture. In order to neglect the depth we just have to set $y = 1$ and get

$$Zag(x) = BK(x,1) = \frac{1 + \mathsf{E}_2(x)}{1 - \mathsf{O}_3(x)} = \frac{1 + \frac{x^2}{1-x^2}}{1 - \frac{x^3}{1-x^2}} = \frac{1}{1 - x^2 - x^3}.$$

Brown's Theorem

The $\mathbb{Q}$-vector space of multiple zeta values is spanned by the "23"-MZV's, e.g. by those $\zeta(s_1, ..., s_l)$ with $s_i \in \{2, 3\}$.

By Brown's theorem the dimensions in Zagier's conjecture are the maximal possible ones.

Theorem (Gangl&Kaneko&Zagier)

(i) The values $\zeta(\text{odd}, \text{odd})$ of weight k satisfy at least $\dim S_k$ linearly independent relations, where S_k denotes the space of cusp forms of weight k on $\mathrm{Sl}_2(\mathbb{Z})$.

(ii) For each even period polynomial an "exotic" relation as in (i) can be constructed.

Example. For $k = 12$ and $k = 16$, i.e. the first weights for which there are non-zero cusp forms, we have the identities

$$28\zeta(9,3) + 150\zeta(7,5) + 168\zeta(5,7) = \frac{5197}{691}\zeta(12)$$

$$66\zeta(13,3) + 375\zeta(11,5) + 686\zeta(9,7) + 675\zeta(7,9) + 396\zeta(5,11) = \frac{78967}{3617}\zeta(16).$$

Multiple q-zeta values

Many of the most basic concepts in mathematics have so-called q-analogues, where q is a formal variable such that the specialisation q = 1 recovers the usual concept. Attributed to Gauss are the q-integers

$$\{n\}_q = 1 + q + \ldots + q^{n-1} = \frac{1-q^n}{1-q}.$$

We will study the following q-analogues of multiple zeta values.

Definition [(modified) multiple q-zeta-value]

For $s_1, \ldots, s_l \geq 1$ and polynomials $Q_1(t) \in t\mathbb{Q}[t]$ and $Q_2(t) \ldots, Q_l(t) \in \mathbb{Q}[t]$ we define

$$\zeta_q(s_1, \ldots, s_l; Q_1, \ldots, Q_l) = \sum_{n_1 > \cdots > n_l > 0} \frac{Q_1(q^{n_1}) \ldots Q_l(q^{n_l})}{(1-q^{n_1})^{s_1} \cdots (1-q^{n_l})^{s_l}} \in \mathbb{Q}[[q]].$$

Such series can be seen as a q-analogue of multiple zeta values, since we have for $s_1 > 1$

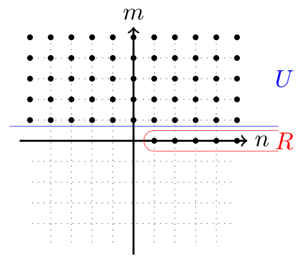
$$\lim_{q \to 1} (1-q)^{s_1 + \cdots + s_l} \zeta_q(s_1, \ldots, s_l; Q_1, \ldots, Q_l) = Q_1(1) \ldots Q_l(1) \cdot \zeta(s_1, \ldots, s_l).$$

Observe, just replacing n by $\{n\}_q$ in multiple zeta values will not work.

Multiple Eisenstein series

Let $\Lambda_\tau = \mathbb{Z}\tau + \mathbb{Z}$ be a lattice with $\tau \in \mathbb{H} := \{x + iy \in \mathbb{C} \mid y > 0\}$ and

$$P := \{m\tau + n \in \Lambda_\tau \mid m > 0 \vee (m = 0 \wedge n > 0)\} = U \cup R$$



"upper"

"right"

$\zeta(2) = \sum_{n>0} \frac{1}{n^2}$

$\sum_{\{n \in \mathbb{Z} \mid n > 0\}}$

We call P the set of positive points and we define an order $\succ$ on Λ_τ by

$$\lambda_1 \succ \lambda_2 :\Leftrightarrow \lambda_1 - \lambda_2 \in P$$

for $\lambda_1, \lambda_2 \in \Lambda_\tau$.

Multiple Eisenstein series

Definition

For $s_1 \geq 3, s_2, \ldots, s_l \geq 2$ we define the *multiple Eisenstein series* of weight $k = s_1 + \cdots + s_l$ and depth l by

$$G_{s_1,\ldots,s_l}(\tau) := \sum_{\substack{\lambda_1 \succ \cdots \succ \lambda_l \succ 0 \\ \lambda_i \in \Lambda_\tau}} \frac{1}{\lambda_1^{s_1} \ldots \lambda_l^{s_l}}.$$

- $G_k(\tau)$ with even weight k are the classical Eisenstein series.
- $G_{s_1,s_2}(\tau)$, i.e. the depth $l = 2$ cases, are due to Gangl, Kaneko and Zagier.
- general multiple Eisenstein series are considered first by Bachmann.

now ok for all $k \in \mathbb{N}$!

The multiple Eisenstein series have a Fourier expansion

$$G_{s_1,\ldots,s_l}(\tau) = \sum_{n \geq 0} a_n q^n, \qquad (q = e^{2\pi i \tau})$$

since $G_{s_1,\ldots,s_l}(\tau + 1) = G_{s_1,\ldots,s_l}(\tau)$, but in general they are not modular.

Question: What can we say about the Fourier coefficients a_n?

Multiple Eisenstein series - Fourier expansion, preliminaries

To calculate the Fourier expansion we rewrite the multiple Eisenstein series as

$$\begin{aligned} G_{s_1,\dots,s_l}(\tau) &= \sum_{\lambda_1 \succ \dots \succ \lambda_l \succ 0} \frac{1}{\lambda_1^{s_1} \dots \lambda_l^{s_l}} \\ &= \sum_{(\lambda_1,\dots,\lambda_l) \in P^l} \frac{1}{(\lambda_1 + \dots + \lambda_l)^{s_1} (\lambda_2 + \dots + \lambda_l)^{s_2} \dots (\lambda_l)^{s_l}} \end{aligned}$$

We decompose the set of tuples of positive lattice points P^l into the 2^l distinct subsets $A_1 \times \dots \times A_l \subset P^l$ with $A_i \in \{R, U\}$ and write

$$G^{A_1 \dots A_l}_{s_1,\dots,s_l}(\tau) := \sum_{(\lambda_1,\dots,\lambda_l) \in A_1 \times \dots \times A_l} \frac{1}{(\lambda_1 + \dots + \lambda_l)^{s_1} (\lambda_2 + \dots + \lambda_l)^{s_2} \dots (\lambda_l)^{s_l}}$$

this gives the decomposition

$$G_{s_1,\dots,s_l} = \sum_{A_1,\dots,A_l \in \{R,U\}} G^{A_1 \dots A_l}_{s_1,\dots,s_l} .$$

In the following we identify the $A_1 \dots A_l$ with words in the alphabet $\{R, U\}$.

Multiple Eisenstein series - Fourier expansion, depth=1

In depth $l = 1$ we have $G_k(\tau) = G_k^R(\tau) + G_k^U(\tau)$ and

$$G_k^R(\tau) = \sum_{\substack{m_1 = 0 \\ n_1 > 0}} \frac{1}{(0\tau + n_1)^k} = \zeta(k) ,$$

$$G_k^U(\tau) = \sum_{\substack{m_1 > 0 \\ n_1 \in \mathbb{Z}}} \frac{1}{(m_1\tau + n_1)^k} = \sum_{m_1 > 0} \Psi_k(m_1\tau) ,$$

where Ψ_k is the so called monotangent function defined for $k > 1$ by

$$\Psi_k(x) = \sum_{n \in \mathbb{Z}} \frac{1}{(x+n)^k} .$$

To calculate the Fourier expansion of G_k^U one uses the Lipschitz formula.

Multiple Eisenstein series - Fourier expansion, depth=1

Proposition (Lipschitz formula)

For $k > 1$ it is

$$\Psi_k(x) = \sum_{n \in \mathbb{Z}} \frac{1}{(x+n)^k} = \frac{(-2\pi i)^k}{(k-1)!} \sum_{d > 0} d^{k-1} e^{2\pi i d x} .$$

With this we get

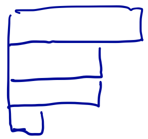
$$\begin{aligned} G_k^U(\tau) &= \sum_{m_1 > 0} \Psi_k(m_1\tau) = \sum_{m_1 > 0} \frac{(-2\pi i)^k}{(k-1)!} \sum_{d > 0} d^{k-1} e^{2\pi i m_1 d \tau} \\ &= \frac{(-2\pi i)^k}{(k-1)!} \sum_{n > 0} \sigma_{k-1}(n) q^n \\ &=: (-2\pi i)^k [k] , \end{aligned}$$

where $\sigma_{k-1}(n) = \sum_{d|n} d^{k-1}$ is the classical divisor sum.

Multiple Eisenstein series - Fourier expansion, U^l-case

In general the $G^{U^l}_{s_1,\dots,s_l}$ can be written as

$$\begin{aligned} G^{U^l}_{s_1,\dots,s_l}(\tau) &= \sum_{\substack{m_1 > \dots > m_l > 0 \\ n_1,\dots,n_l \in \mathbb{Z}}} \frac{1}{(m_1\tau + n_1)^{s_1} \dots (m_l\tau + n_l)^{s_l}} \\ &= \sum_{m_1 > \dots > m_l > 0} \Psi_{s_1}(m_1\tau) \dots \Psi_{s_l}(m_l\tau) \\ &= \frac{(-2\pi i)^{s_1 + \dots + s_l}}{(s_1 - 1)! \dots (s_l - 1)!} \sum_{\substack{m_1 > \dots > m_l > 0 \\ d_1,\dots,d_l > 0}} d_1^{s_1 - 1} \dots d_l^{s_l - 1} q^{m_1 d_1 + \dots + m_l d_l} \\ &=: \frac{(-2\pi i)^{s_1 + \dots + s_l}}{(s_1 - 1)! \dots (s_l - 1)!} \sum_{n > 0} \sigma_{s_1 - 1,\dots,s_l - 1}(n) q^n \\ &=: (-2\pi i)^{s_1 + \dots + s_l} [s_1, \dots, s_l] . \end{aligned}$$



We call the $\sigma_{r_1,\dots,r_l}$ **multiple divisor sums** and their generating functions

$$[s_1, \dots, s_l] \in \mathbb{Q}[[q]]$$

are called **brackets**.

Multiple Eisenstein series - Fourier expansion, R^l-case

The other special case $G^{R^l}_{s_1,\dots,s_l}$ can also be written down directly:

$$G^{R^l}_{s_1,\dots,s_l}(\tau) = \sum_{\substack{m_1=\dots=m_l=0\\ n_1>\dots>n_l>0}} \frac{1}{(0\tau+n_1)^{s_1}\dots(0\tau+n_l)^{s_l}} = \zeta(s_1,\dots,s_l)$$

What about the mixed terms in depth $l>1$?

Multiple Eisenstein series - Fourier expansion, depth=2

In depth 2 we have $G_{s_1,s_2} = G^{RR}_{s_1,s_2} + G^{UR}_{s_1,s_2} + G^{RU}_{s_1,s_2} + G^{UU}_{s_1,s_2}$ and

$$G^{UR}_{s_1,s_2} = \sum_{\substack{m_1>0,m_2=0\\ n_1\in\mathbb{Z},n_2>0}} \frac{1}{(m_1\tau+n_1)^{s_1}(0\tau+n_2)^{s_1}}$$
$$= \sum_{m_1>0}\Psi_{s_1}(m_1\tau)\sum_{n_2>0}\frac{1}{n_2^{s_2}} = (-2\pi i)^{s_1}[s_1]\zeta(s_2)\,,$$

$$G^{RU}_{s_1,s_2}(\tau) = \sum_{\substack{m_1=m_2>0\\ n_1>n_2\\ n_i\in\mathbb{Z}}} \frac{1}{(m_1\tau+n_1)^{s_1}(m_1\tau+n_2)^{s_2}} = \sum_{m>0}\Psi_{s_1,s_2}(m\tau).$$

where we call $\Psi_{s_1,s_2}(x) = \sum_{n_1>n_2}\frac{1}{(x+n_1)^{s_1}(x+n_2)^{s_2}}$ the multitangent function of depth 2.

Multiple Eisenstein series - Fourier expansion, depth=2

Using partial fraction expansion one can show that

$$\Psi_{s_1,s_2}(x) = \sum_{k_1+k_2=s_1+s_2}\left((-1)^{s_2}\binom{k_2-1}{s_2-1} + (-1)^{k_1-s_1}\binom{k_2-1}{s_1-1}\right)\zeta(k_2)\Psi_{k_1}(x).$$

and therefore

$$G^{RU}_{s_1,s_2}(\tau) = \sum_{m>0}\Psi_{s_1,s_2}(m\tau)$$
$$= \sum_{m>0}\sum_{k_1+k_2=s_1+s_2}\left((-1)^{s_2}\binom{k_2-1}{s_2-1} + (-1)^{k_1-s_1}\binom{k_1-1}{s_1-1}\right)\zeta(k_2)\Psi_{k_1}(m\tau)$$
$$= \sum_{k_1+k_2=s_1+s_2}\underbrace{\left((-1)^{s_2}\binom{k_2-1}{s_2-1} + (-1)^{k_2-s_1}\binom{k_1-1}{s_1-1}\right)}_{\text{"combinatorial coefficient"}}\zeta(k_2)(-2\pi i)^{k_1}[k_1].$$

Multiple Eisenstein series - Fourier expansion, depth=2

Therefore we obtain

Proposition (Gangl-Kaneko-Zagier)

The Fourier expansion of the double Eisenstein series is given by

$$G_{s_1,s_2}(\tau) = G^{RR}_{s_1,s_2} + G^{UR}_{s_1,s_2} + G^{RU}_{s_1,s_2} + G^{UU}_{s_1,s_2}$$
$$= \zeta(s_1,s_2) + (-2\pi i)^{s_1}[s_1]\zeta(s_2)$$
$$+ \sum_{k_1+k_2=s_1+s_2} C^{k_2}_{s_1,s_2}\zeta(k_2)(-2\pi i)^{k_1}[k_1] + (-2\pi i)^{s_1+s_2}[s_1,s_2]\,.$$

where

$$C^{k_2}_{s_1,s_2} := (-1)^{s_2}\binom{k_2-1}{s_2-1} + (-1)^{k_2-s_1}\binom{k_2-1}{s_1-1}.$$

Multiple Eisenstein series - Fourier expansion, word reduction

In the case G^{UR} we saw that we could write it as G^{U} multiplied with a zeta value.

In general having a word w of depth l ending in the letter R, i.e. there is a word w' ending in U with $w = w'R^r$ and $1 \leq r \leq l$ we can write

$$G^{w}_{s_1,\dots,s_l}(\tau) = G^{w'}_{s_1,\dots,s_{l-r}}(\tau) \cdot \zeta(s_{l-r+1},\dots,s_l)\,.$$

Example: $G^{RUURR}_{3,4,5,6,7} = G^{RUU}_{3,4,5} \cdot \zeta(6,7)$

Hence one can concentrate on the words ending in U when calculating the Fourier expansion of a multiple Eisenstein series.

Multiple Eisenstein series - Fourier expansion, multitangent fct's

Let $w = R^{r_1}UR^{r_2}U\dots R^{r_j}U$, then using multitangent functions one can write

$$G^{w}_{s_1,\dots,s_l}(\tau) = \sum_{m_1>\dots>m_j>0} \Psi_{s_1,\dots,s_{r_1+1}}(m_1\tau)\cdot\Psi_{s_{r_1+2},\dots}(m_2\tau)\dots\Psi_{s_{l-r_j},\dots,s_l}(m_j\tau)\,.$$

Definition

For $s_1,\dots,s_l \geq 2$ the multitangent function of depth l is defined by

$$\Psi_{s_1,\dots,s_l}(x) = \sum_{\substack{n_1>\dots>n_l\\ n_i\in\mathbb{Z}}} \frac{1}{(x+n_1)^{s_1}\dots(x+n_l)^{s_l}}\,.$$

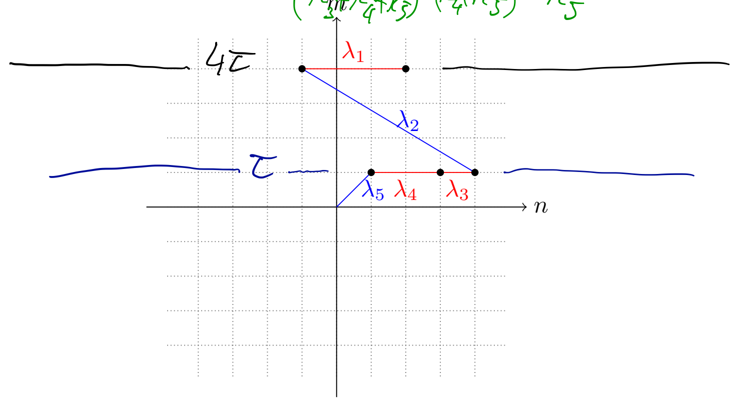
In the case $l = 1$ we also refer to these as monotangent function.

Let us consider an example...

Example: Let $w = RURRU$, then a typical summand of $G^{RURRU}_{s_1,\dots,s_5}$ is

$$\frac{1}{(2+3\tau-5+1+2+\tau+1)^{s_1}(3\tau-5+1+2+\tau+1)^{s_2}(1+2+\tau+1)^{s_3}(2+\tau+1)^{s_4}(\tau+1)^{s_5}}\,.$$

$(\lambda_3+\lambda_4+\lambda_5)$ $(\lambda_4+\lambda_5)$ λ_5



and therefore

$$G^{RURRU}_{s_1,\dots,s_5} = \sum_{m_1>m_2>0} \Psi_{s_1,s_2}(m_1\tau)\Psi_{s_3,s_4,s_5}(m_2\tau).$$

Multiple Eisenstein series - Fourier expansion, multitangent fct's

To calculate the Fourier expansion of such terms we need the following theorem which reduces the multitangent functions into monotangent functions.

Theorem (Bouillot 2011, Bachmann 2012)

Let $\mathcal{MZ}_k$ be the $\mathbb{Q}$-vector space spanned by all MZVs of weight k. Then for $s_1,\dots,s_l \geq 2$ and $k = s_1+\dots+s_l$ the multitangent function can be written as

$$\Psi_{s_1,\dots,s_l}(x) = \sum_{h=2}^{k} c_{k-h}(s_1,\dots,s_l)\Psi_h(x)$$

with $c_{k-h}(s_1,\dots,s_l) \in \mathcal{MZ}_{k-h}$.

Proof idea: Use partial fraction decomposition.

Multiple Eisenstein series - Fourier expansion, general case

To summarize one can compute the Fourier expansion of the multiple Eisenstein series $G_{s_1,\dots,s_l}$ in the following way

- Split up the summation into 2^l distinct parts $G^w_{s_1,\dots,s_l}$ where w are a words in $\{R,U\}$.
- For w being a word ending in R one can write $G^w_{s_1,\dots,s_l}$ as $G^{w'}_{s_1,\dots} \cdot \zeta(\dots,s_l)$ with a word w' ending in U.
- For w being a word ending in U one can write $G^w_{s_1,\dots,s_l}$ as

$$G^w_{s_1,\dots,s_l}(\tau) = \sum_{m_1>\dots>m_j>0} \Psi_{s_1,\dots}(m_1\tau)\dots\Psi_{\dots,s_l}(m_l\tau)\,.$$

- Using the reduction theorem for multitangent functions this can be written as a MZV-linear combination of sums of the form

$$\sum_{m_1>\dots>m_j>0} \Psi_{k_1}(m_1\tau)\dots\Psi_{k_j}(m_j\tau) = (2\pi i)^{k_1+\dots+k_l}[k_1,\dots,k_l]$$

for which the Fourier expansions are known.

Multiple Eisenstein series - Summary and Examples

Theorem (Bachmann, master thesis 2012)

- Multiple Eisenstein series are holomorphic functions on the upper half plane, which are defined as a sum over ordered lattice points.
- They have a Fourier expansion where the constant term is given by the corresponding multiple zeta value and the remaining terms are rational linear combinations of products of multiple zeta values and multiple divisor functions.

A few examples:

$$G_{4,4}(\tau) = \zeta(4,4) + 20\zeta(6)(2\pi i)^2[2] + 3\zeta(4)(2\pi i)^4[4] + (2\pi i)^8[4,4]\,,$$

$$\begin{aligned}G_{3,2,2}(\tau) =& \zeta(3,2,2) + \left(\frac{54}{5}\zeta(2,3) + \frac{51}{5}\zeta(3,2)\right)(2\pi i)^2[2]\\ &+ \frac{16}{3}\zeta(2,2)(2\pi i)^3[3] + 3\zeta(3)(2\pi i)^4[2,2] + 4\zeta(2)(2\pi i)^5[3,2]\\ &+ (2\pi i)^7[3,2,2]\,.\end{aligned}$$

Multiple Eisenstein series - some open questions

The multiple Eisenstein series fulfill the **stuffle product**, for example it is

$$G_4(\tau)\cdot G_6(\tau) = G_{4,6}(\tau) + G_{6,4}(\tau) + G_{10}(\tau)\,.$$

This follows using the same combinatorial argument as in the MZV case, but the **shuffle product** can't be fulfilled because for example it is

$$\zeta(4)\zeta(6) = \zeta(4,6) + 4\zeta(4,6) + 11\zeta(6,4) + 26\zeta(7,3) + 56\zeta(8,2) + 112\zeta(9,1)$$

and this equation does not make sense in terms of multiple Eisenstein series. In fact, because of convergence problems we haven't defined $G_{9,1}$ yet.

We have two options to define and study non convergent multiple Eisenstein series:

- use analytical regularization (Bouillot and Bachmann)
- use formal Fourier expansions (Bachmann-Tasaka, 2017)

The second approach was our motivation to study the brackets in its own.

generating series for multiple divisor sums

Recall the **multiple divisor sum** is for any integers $s_1,\dots,s_l \geq 0$ defined by

$$\sigma_{s_1,\dots,s_l}(n) := \sum_{\substack{u_1v_1+\dots+u_lv_l=n\\ u_1>\dots>u_l>0}} v_1^{s_1}\dots v_l^{s_l}\,.$$

and its generating series for multiple divisor sums are denoted by the brackets

$$[s_1,\dots,s_l] := \frac{1}{(s_1-1)!\dots(s_l-1)!}\sum_{n>0}\sigma_{s_1-1,\dots,s_l-1}(n)q^n \in \mathbb{Q}[[q]]\,.$$

bounded denominators

Example

$$[2] = \sum_{n>0}\sigma_1(n)q^n = q + 3q^2 + 4q^3 + 7q^4 + 6q^5 + 12q^6 + 8q^7 + \dots$$

$$[4,2] = \frac{1}{6}\sum_{n>0}\sigma_{3,1}(n)q^n = \frac{1}{6}\left(q^3 + 3q^4 + 15q^5 + 27q^6 + 78q^7 + \dots\right)$$

Vector space spanned by brackets $[s_1, \dots, s_l]$ and $[\emptyset] = 1$

Theorem (Bachmann-K.)

(i) The $\mathbb{Q}$-vector space $\mathcal{MD}$ has the structure of a $\mathbb{Q}$-Algebra $(\mathcal{MD}, \cdot)$, where the multiplication is the natural multiplication of formal power series, which is filtered w.r.t. the weight and the depth.

(ii) The ring of quasi-modular forms is a subalgebra of $\mathcal{MD}$

(iii) The multiplication is a (homomorphic image of a) quasi-shuffle algebra in the sense of Hoffman.

The first products of multiple divisor functions are given by

$$
\begin{aligned}
[1]\cdot[1] &= 2[1,1] + [2] - [1] \quad \text{← smaller weight} \\
[1]\cdot[2] &= [1,2] + [2,1] + [3] - \frac{1}{2}[2]\,, \\
[1]\cdot[2,1] &= [1,2,1] + 2[2,1,1] + [2,2] + [3,1] - \frac{3}{2}[2,1]\,.
\end{aligned}
$$

← depth 1

Multiple divisor functions - multiplicative structure

Idea of Proof: At first we rewrite the multiple divisor functions. For this we define a *normalized polylogarithm* by

$$\widetilde{\mathrm{Li}}_s(z) := \frac{\mathrm{Li}_{1-s}(z)}{\Gamma(s)}\,,$$

where for $s, z \in \mathbb{C}$, $|z| < 1$ the polylogarithm $\mathrm{Li}_s(z)$ of weight s is given by

$$\mathrm{Li}_s(z) = \sum_{n>0} \frac{z^n}{n^s}\,.$$

Proposition

For $q \in \mathbb{C}$ with $|q| < 1$ and for all $s_1, \dots, s_l \in \mathbb{N}$ we can write the multiple divisor functions as

$$[s_1, \dots, s_l] = \sum_{n_1 > \dots > n_l > 0} \widetilde{\mathrm{Li}}_{s_1}(q^{n_1}) \dots \widetilde{\mathrm{Li}}_{s_l}(q^{n_l})\,.$$

We remark for later use that, by the definiton of eulerian polynomials $P_s(q) \in \mathbb{Q}[q]$,

$$\widetilde{\mathrm{Li}}_s(q) = \frac{1}{(s-1)!}\frac{qP_{s-1}(q)}{(1-q)^s}$$

is in fact a rational function in q if $s \in \mathbb{N}$.

Multiple divisor functions - multiplicative structure

The product of $[s_1]$ and $[s_2]$ can thus be written as

$$
\begin{aligned}
[s_1]\cdot[s_2] &= \sum_{n_1>0} \widetilde{\mathrm{Li}}_{s_1}(q^{n_1}) \cdot \sum_{n_2>0} \widetilde{\mathrm{Li}}_{s_2}(q^{n_1}) \\
&= \sum_{n_1>n_2>0} \dots + \sum_{n_2>n_1>0} \dots + \sum_{n_1=n_2>0} \widetilde{\mathrm{Li}}_{s_1}(q^{n_1})\,\widetilde{\mathrm{Li}}_{s_2}(q^{n_1}) \\
&= [s_1, s_2] + [s_2, s_1] + \sum_{n>0} \widetilde{\mathrm{Li}}_{s_1}(q^{n})\,\widetilde{\mathrm{Li}}_{s_2}(q^{n})\,.
\end{aligned}
$$

In order to prove that this product is an element of $\mathcal{MD}$ we will show that the product $\widetilde{\mathrm{Li}}_{s_1}(q^n)\,\widetilde{\mathrm{Li}}_{s_2}(q^n)$ is a rational linear combination of $\widetilde{\mathrm{Li}}_j(q^n)$ with $1 \le j \le s_1 + s_2$.

Multiple divisor functions - multiplicative structure

Lemma

For $a, b \in \mathbb{N}$ we have

$$\widetilde{\mathrm{Li}}_a(z)\cdot\widetilde{\mathrm{Li}}_b(z) = \widetilde{\mathrm{Li}}_{a+b}(z) + \sum_{j=1}^{a} \lambda_{a,b}^j \widetilde{\mathrm{Li}}_j(z) + \sum_{j=1}^{b} \lambda_{b,a}^j \widetilde{\mathrm{Li}}_j(z)\,,$$

where the coefficient $\lambda_{a,b}^j \in \mathbb{Q}$ for $1 \le j \le a$ is given by

$$\lambda_{a,b}^j = (-1)^{b-1}\binom{a+b-j-1}{a-j}\frac{B_{a+b-j}}{(a+b-j)!}\,,$$

with the Bernoulli numbers B_n.

This settles the proof of the claimed algebra structure for products of depth one elements. Now by means of the above Lemma one can deduce the general case similar as for the stuffle product of MZV's. □

Multiple divisor functions - Derivation

Theorem (Bachmann-K.)

The operator $\mathrm{d} = q\frac{d}{dq}$ is a derivation on $\mathcal{MD}$.

Examples:

$$\mathrm{d}[1] = [3] + \frac{1}{2}[2] - [2,1]\,,$$

$$\mathrm{d}[2] = [4] + 2[3] - \frac{1}{6}[2] - 4[3,1]\,,$$

$$\mathrm{d}[2] = 2[4] + [3] + \frac{1}{6}[2] - 2[2,2] - 2[3,1]\,,$$

$$\mathrm{d}[1,1] = [3,1] + \frac{3}{2}[2,1] + \frac{1}{2}[1,2] + [1,3] - 2[2,1,1] - [1,2,1]\,.$$

The second and third equation lead to the first linear relation between multiple divisor functions in weight 4:

$$[4] = 2[2,2] - 2[3,1] + [3] - \frac{1}{3}[2]\,.$$

Multiple divisor functions - Connections to MZV

For $k \in \mathbb{N}$ consider the map $Z_k : \mathrm{Fil}_k^{\mathrm{W}}(\mathcal{MD}) \to \mathbb{R} \cup \{\pm\infty\}$ given by

$$Z_k(f) = \lim_{q\to 1}(1-q)^k f(q)\,.$$

Theorem (B.-K., arXiv.NT:1309.3920)

(i) For $s_1 > 1$ and $s_1 + \cdots + s_l = k$ it is

$$Z_k([s_1,\ldots,s_l]) = \zeta(s_1,\ldots,s_l)\,.$$

(ii) If $s_1 + \cdots + s_l < k$ then $Z_k([s_1,\ldots,s_l]) = 0$.

(iii) ~~For any $f \in \mathrm{Fil}_{k-2}^{\mathrm{W}}(\mathcal{MD})$~~ we have $Z_k(\mathrm{d}(f)) = 0$, if f of weight $\leq k-2$

(iv) If $f \in \mathrm{Fil}_k^{\mathrm{W}}(\mathcal{MD})$ is a cusp form for $\mathrm{SL}_2(\mathbb{Z})$, then $Z_k(f) = 0$.

Elements in the kernel of Z_k give rise to relations between MZV. In particular since $0 \in \ker Z_k$, any linear relation between multiple divisor functions in $\mathrm{Fil}_k^{\mathrm{W}}(\mathcal{MD})$ gives an element in the kernel.

Multiple divisor functions - Connections to MZV

We also rediscover exotic relations related to cusp forms, e.g. the cusp form $\Delta = q\prod_{n>0}(1-q^n)^{24}$ can be written as

$$\frac{1}{2^6\cdot 5\cdot 691}\Delta = 168[5,7] + 150[7,5] + 28[9,3] + \frac{1}{1408}[2] - \frac{83}{14400}[4] + \frac{187}{6048}[6] - \frac{7}{120}[8] - \frac{5197}{691}[12]\,.$$

Letting Z_{12} act on both sides one obtains the relation

$$\frac{5197}{691}\zeta(12) = 168\zeta(5,7) + 150\zeta(7,5) + 28\zeta(9,3)\,.$$

These type of relations can also be explained via the theory of period polynomials (Gangl, Kaneko, Zagier) or via a motivic interpretation (Pollack, Schneps, Baumard).

Multiple divisor functions - Summary

- Multiple divisor functions are formal power series in q with coefficient in $\mathbb{Q}$ coming from the calculation of the Fourier expansion of multiple Eisenstein series.
- The space spanned by all multiple divisor functions form an differential algebra which contains the algebra of (quasi-) modular forms.
- A connection to multiple zeta values is given by the map Z_k whose kernel contains all relations between multiple zeta values of weight k.
- Some questions and open problems:
  - (i) Is there a modular/geometric/motivic interpretation of the multiple divisor functions ?
  - (ii) Dimensions of the graded parts ? Basis ?
  - (iii) Is there an analogue of the Broadhurst-Kreimer conjecture ? Algebra generators ?
  - (iv) What is the kernel of Z_k ?