From modular forms to multiple q-zeta values

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ALGEBRAIC AND ANALYTIC ASPECTS OF AUTOMORPHIC FORMS
ICTS Bangalore, March 24-27 2019

Why care about modular forms?

Modular forms can been seen either as

- functions with infinite symmetries
- q-series $\sum a_n q^n$ with arithmetical interesting coefficients

Modular forms span finite dimensional vector spaces and this allows for example "trivial" proofs of interesting identities.

$1 = 1^2 + 0^2 + 0^2 + 0^2 = (-1)^2 + 0^2 + 0^2 + 0^2$

Example

Let $n \geqslant 0$ and

$$r_4(n) = \#\{\, (a,b,c,d) \in \mathbb{Z}^4 \,|\, a^2+b^2+c^2+d^2 = n\},$$

then for the generating series

$$\sum_{n\geqslant 0} r_4(n)\, q^n = \Big(\sum_{i\in\mathbb{Z}} q^{i^2}\Big)^4 \overset{\heartsuit}{=} -\frac{1}{3}(E_2(q) - 4E_2(q^4)) = \sum_{n\geqslant 0}\Big(8\sum_{\substack{d|n\\ 4\nmid d}} d\Big) q^n$$

$$= 1 + 8q + 24q^2 + 32q^3 + 24q^4 + 48q^5 + 96q^6 + 64q^7 + \ldots$$

where $E_2(q)$ denotes the Eisenstein series of weight 2 (to be defined later).

For example we read of:

$$r_4(1) = \underline{8}$$
$$r_4(2019) = r_4(3\cdot 673) = 8(1+3+673+2019) = 21568.$$

What will we learn ?

We will study the ring of modular forms

$$M_* = \bigoplus_{k=0}^{\infty} M_k.$$

We will present proofs of the following characterisations:

(i) $M_* \cong \mathbb{C}[E_4, E_6]$ Whose thm is this ?
(ii) $M_k = \langle$ Hecke eigenforms $\rangle_{\mathbb{C}}$ Hecke
(iii) $M_k \oplus S_k \cong W_k$ Shimura-Eichler
(iv) $M_k = \langle \{E_k\} \cup \{E_a E_{k-a}\}\rangle_{\mathbb{C}}$ Kohnen-Zagier

If time permits, we relate modular forms to multiple zeta values and their q-analogues. We plan to indicate why (iv) should be seen in analogy to Eulers formula

$$\zeta(2k) = -\frac{B_{2k}}{2\cdot 2k}\frac{(2\pi i)^{2k}}{(2k-1)!}, \quad = \sum_{n\geq 1}\frac{1}{n^{2k}}$$

where the Bernoulli numbers are given by the series

$$\sum_{k=0}^{\infty}\frac{B_k}{k!}X^k = \frac{X}{e^X - 1}.$$

$\mathbb{Q}[\zeta(2)]$

Some selected references

$\sum_{\mathbb{Z}\setminus\{0\}} \frac{1}{n^3} = 0$ v.s. $\sum_{n>0} \frac{1}{n^3} = \zeta(3)$

(i) Modular forms:
Don Zagier: Utrecht lectures; Introduction to modular forms; The 1-2-3 of Modular Forms; Modular forms with rational periods. (online: https://people.mpim-bonn.mpg.de/zagier/)

Francois Martin & Emmanuel Royer: Formes modulaires et periodes. (online at docplayer.fr)

(many excellent books/surveys: Shimura, Diamond-Darmon-Taylor, Cornell-Silverman, ...)

(ii) Multiple Eisenstein series and multiple q-zeta functions:

Herbert Gangl & Masanobu Kaneko & Don Zagier: Double zeta values and modular forms. (online: https://people.mpim-bonn.mpg.de/zagier/)

Henrik Bachmann: Multiple Eisenstein series and q-analogues of multiple zeta values;
Henrik Bachmann & U.K.: A dimension conjecture for q-analogues of multiple zeta values.
(both available online: https://www.henrikbachmann.com)

(iii) Multiple Zeta values:

Jose Burgos-Gil & Javier Fresan: Multiple zeta values: from numbers to motives. (online: http://javier.fresan.perso.math.cnrs.fr/mzv.pdf)

Modular group

Definition

We denote the upper complex halfplane by

$$\mathbb{H} = \{\tau \in \mathbb{C} \mid \mathrm{Im}(\tau) > 0\}.$$

We use the notation $\tau = x + iy$, with $x, y \in \mathbb{R}$

We have an action

$$\mathrm{SL}_2(\mathbb{Z}) \times \mathbb{H} \to \mathbb{H}$$
$$\left(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right), \tau\right) \mapsto \frac{a\tau + b}{c\tau + d},$$

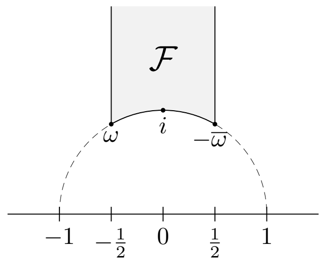
We call $\Gamma(1) = \mathrm{SL}_2(\mathbb{Z})/\pm 1$ the modular group (of Level 1).

Fundamental domain

Proposition (Fundamental domain for $\Gamma(1)$)

Let
$$\mathcal{F} = \{\tau \in \mathbb{H} \mid -\frac{1}{2} \leqslant \mathrm{Re}(\tau) \leqslant \frac{1}{2} \text{ and } |\tau| \geqslant 1\}$$
then we have

- for any $\tau \in \mathbb{H}$ there exist a $\gamma \in \Gamma(1)$ such that $\gamma.\tau \in \mathcal{F}$.
- if τ and $\gamma.\tau$ are elements of $\mathcal{F}$, then $\tau, \gamma.\tau \in \partial\mathcal{F}$.



Idea of proof: First we observe that the matrices

$$S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \text{ and } T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

generate $\mathrm{SL}_2(\mathbb{Z})$. We have

$$T\tau = \tau + 1 \quad \text{and} \quad S\tau = \frac{-1}{\tau}.$$

The following diagram shows how the fundamental domain $\mathcal{F}$ is translated by different matrices in $\mathrm{SL}_2(\mathbb{Z})$.

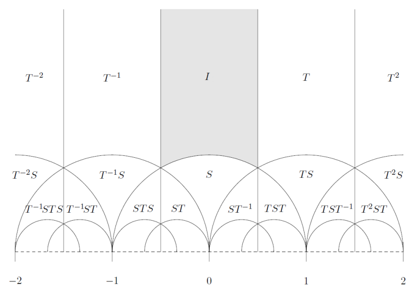
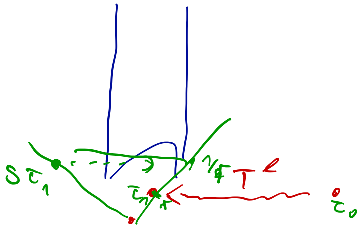


Figure : Translations of the fundamental domain $\mathcal{F}$.

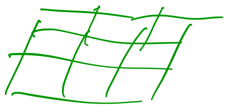
Given τ_0 there is some power T^{l_0} of the translation T such that

$$\tau_1 := T^{l_0}\tau_0 \in \left\{\tau \in \mathbb{H} \,\middle|\, -\frac{1}{2} \leqslant \operatorname{Im}\tau \leqslant \frac{1}{2}\right\}$$

Now using S we increase the imaginary part of τ_1 and continue the above procedure untill we end in $\mathcal{F}$.
It easy to see that the vertical boundary is identified via the translation T. Analogously the lower part is identified with S. Observe that the corners and i have non-trivial stabilisers. $\square$



Aside: Modular curve



Remark

In fact there are natural bijections

$$\Gamma(1)\backslash\mathbb{H} \xrightarrow{\sim} \mathcal{L}/\mathbb{C}^* \xrightarrow{\sim} \{\text{ complex elliptic curves }\}/\text{ isomorphisms}$$

$$\tau \qquad\qquad \Lambda_\tau = \mathbb{Z}\tau + \mathbb{Z} \qquad\qquad E_\tau = \mathbb{C}/\Lambda_\tau$$

Here $\mathcal{L}$ denotes the set of all lattices in $\mathbb{C}$ on which $\mathbb{C}^*$ acts by scaling. The second bijection uses the complex uniformisation of elliptic curves, i.e., for any elliptic curve $E: Y^2 = X^3 + AX + B$, there exists a lattice Λ and a complex isomorphism

$$\mathbb{C}/\Lambda \to E: Y^2 = X^3 + AX + B$$
$$z \mapsto \big(\mathfrak{p}(z,\Lambda), \mathfrak{p}'(z,\Lambda)\big)$$

Aside: Modular curve

Remark

The action of $\Gamma(1)$ on $\mathbb{H}$ extends to a proper discontinous action

$$\Gamma(1) \times \Big(\mathbb{H} \cup \mathbb{P}^1(\mathbb{Q})\Big) \to \mathbb{H} \cup \mathbb{P}^1(\mathbb{Q}).$$

Then by abstract topology and straightforward calculations we get an isomorphism

$$X(1) := \Gamma(1)\backslash\Big(\mathbb{H} \cup \mathbb{P}^1(\mathbb{Q})\Big) \to \mathbb{P}^1(\mathbb{C})$$

in such a way that

$$q := \exp(2\pi\tau)$$

is the local parameter at $\infty \in \mathbb{P}^1(\mathbb{C})$.

Modular forms

Definition

A holomorphic function $f: \mathbb{H} \to \mathbb{C}$ is called a modular form of weight k if

(i) $f\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^k f(\tau)$ for all $\tau \in \mathbb{H}$ and all $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$

(ii) $f(\tau) = \sum_{n \geqslant 0} a_n\, q^n$, for $q = \exp(2\pi i\tau)$

"infinite symmetry

If f is a modular form of weight k we have by (i)

$$f(\tau+1) = f(T\tau) = f(\tau),$$
$$f(-1/\tau) = F(S\tau) = \tau^k f(\tau),$$

Thus any function in (i) has a Fourier expansion $\sum_{n\in\mathbb{Z}} a_n\, q^n$. The additional requirement in (ii) is called

"f is holomorphic at infinity".

Since $-I \in \mathrm{SL}_2(\mathbb{Z})$, a modular form of weight k satisfies $f(\tau) = (-1)^k f(\tau)$. This shows that there are no non-trivial modular forms of odd weight.

Definition

We denote the vector space of modular forms of weight k by M_k. A modular form $f = \sum_{n\geqslant 0} a_n q^n$ with $a_0 = 0$ is called cusp form (or parabolic form) and the vector space of cusp forms is denoted by S_k

Proposition

Modular forms with different weights are linearly independent over $\mathbb{C}$.

Proof: Suppose we have nonzero modular forms $f_1, f_2, \ldots, f_m$ with respective weights $k_1 < k_2 < \cdots < k_m$, such that they admit a nontrivial linear relation

$$\alpha_1 f_1(\tau) + \alpha_2 f_2(\tau) + \cdots + \alpha_m f_m(\tau) = 0 \tag{1}$$

for all $\tau \in \mathbb{H}$ and $\alpha_j \neq 0$ for $j = 1, \ldots, m$. Replacing τ by $S(\tau)$ and using the modularity, i.e. $f_j(S(\tau)) = \tau^{k_j} f_j(\tau)$, we obtain

$$\alpha_1 \tau^{k_1} f_1(\tau) + \alpha_2 \tau^{k_2} f_2(\tau) + \cdots + \alpha_m \tau^{k_m} f_m(\tau) = 0 \qquad (1')$$

for all $\tau \in \mathbb{H}$. With Fourier expansions $f_j(\tau) = \sum_{n=0}^{\infty} a_n^{(j)} q^n$ where $q = e^{2\pi i \tau}$, this is equivalent to

$$\sum_{n=0}^{\infty} \left(\alpha_1 \tau^{k_1} a_n^{(1)} + \alpha_2 \tau^{k_2} a_n^{(2)} + \cdots + \alpha_m \tau^{k_m} a_n^{(m)} \right) e^{2\pi i n \tau} = 0\,.$$

Now we consider this equation for $\tau = iy$ $(y > 0)$, then

$$\sum_{n=0}^{\infty} \left(\alpha_1 (iy)^{k_1} a_n^{(1)} + \alpha_2 (iy)^{k_2} a_n^{(2)} + \cdots + \alpha_m (iy)^{k_m} a_n^{(m)} \right) e^{-2\pi n y} = 0\,. \tag{2}$$

For $n > 0$ and any $r \geqslant 0$ we have $\lim_{y\to\infty} y^r e^{-2\pi n y} = 0$. Now let N be the smallest integer, such that at least for one $1 \leqslant j \leqslant m$ we have $a_N^{(j)} \neq 0$. Dividing (2) by $e^{-2\pi N y}$ and taking the limit $y \to \infty$ we obtain

$$\lim_{y\to\infty} \alpha_1 (iy)^{k_1} a_N^{(1)} + \alpha_2 (iy)^{k_2} a_N^{(2)} + \cdots + \alpha_m (iy)^{k_m} a_N^{(m)} = 0\,.$$

But the left-hand side of this equation is the limit $y \to \infty$ of a non-constant polynomial in y, which can not be zero and therefore a relation of the form (1) can not exist. $\square$

Eisenstein series

Proposition

Addition and multiplication of holomorphic functions induces the structure of a graded ring

$$M^*(\mathrm{SL}_2(\mathbb{Z})) = \bigoplus_{k\geqslant 0} M_k(\mathrm{SL}_2(\mathbb{Z}))$$

on the set of all modular forms.

Definition (Eisensteinseries)

Let $k \in \mathbb{N}$ be even, then the Eisenstein series of weight k is defined by

$$G_k(\tau) = \frac{(k-1)!}{2(2\pi i)^k} {\sum_{m,n}}' \frac{1}{(mz+n)^k}, \qquad \tau \in \mathbb{H}.$$

Remark

- ${\sum'_{m,n}}$ abbreviates $\sum_{(m,n)\in\mathbb{Z}^2/(0,0)}$
- $\frac{(k-1)!}{2(2\pi i)^k}$ is a normalising factor (different for different authors!)

Theorem

The Eisenstein series $G_k(\tau)$ are modular forms of weight $k \geqslant 4$ and they have the Fourier expansion

$$G_k(\tau) = \frac{(k-1)!}{(2\pi i)^k} \zeta(k) + \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n,$$

where $\sigma_k(n) = \sum_{d|n} d^k$ is the divisor function of degree k, $q = e^{2\pi i z}$ and $\zeta(k) = \sum_{n\geqslant 1} \frac{1}{n^k}$ is the value of the Riemann zeta functions at k.

Remark

(i) By Euler's relation we actually have $G_k \in \mathbb{Q}[[q]]$, indeed we have

$$G_k(\tau) = -\frac{B_k}{2k} + \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n.$$

(ii) Another common notation for the Eisenstein series is given by

$$E_k(\tau) = 1 - \frac{2k}{B_k} \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n.$$

Idea of Proof:
It is straightforward to show that for $k \geqslant 4$ the series $G_k(\tau)$ converges.
Using the convergence of $G_k(\tau)$, we deduce

$$\begin{aligned}
G_k\left(\frac{a\tau+b}{c\tau+d}\right) &= \frac{(k-1)!}{2(2\pi i)^k} \sum_{m,n}{}' \frac{1}{\left(m\frac{a\tau+b}{c\tau+d}+n\right)^k} \\
&= (c\tau+d)^k \frac{(k-1)!}{2(2\pi i)^k} \sum_{m,n}{}' \frac{1}{(m(a\tau+b)+n(c\tau+d))^k} \\
&= (c\tau+d)^k \frac{(k-1)!}{(2\pi i)^k} \sum_{m,n}{}' \frac{1}{((ma+nc)z+mb+nd)^k}.
\end{aligned}$$

For $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ the set of all $(ma+nc, mb+nd)$ equals the set of all $(m,n) \in \mathbb{Z}^2 \backslash (0,0)$. We therefore have

$$G_k(\tau) = (c\tau+d)^{-k} G_k\left(\frac{a\tau+b}{c\tau+d}\right)$$

for all $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$, i.e. G_k transforms as a modular form of weight k for $\mathrm{SL}_2(\mathbb{Z})$.

Lemma (Lipschitz-formula)

$z \in \mathbb{C} \setminus \mathbb{Z}$

$$\sum_{n\in\mathbb{Z}} \frac{1}{(z+n)^k} = \frac{(-2\pi i)^k}{(k-1)!} \sum_{r=1}^{\infty} r^{k-1} e^{2\pi i r z}.$$

The Lipschitz formula follows either from the Poisson summation or from the Taylor expansion of $\frac{\pi}{\tan(\pi x)}$ (Exercise!).
Now the Fourier expansion of G_k follows

"two τ's"

$$\begin{aligned}
G_k(\tau) &= \frac{(k-1)!}{2(2\pi i)^k} \sum_{n\neq 0} \frac{1}{n^k} + \sum_{m\neq 0} \frac{(k-1)!}{2(2\pi i)^k} \sum_{n\in\mathbb{Z}} \frac{1}{(mz+n)^k} \\
&= \frac{(k-1)!}{(2\pi i)^k} \sum_{n=1}^{\infty} \frac{1}{n^k} + \sum_{m=1}^{\infty} \sum_{r=1}^{\infty} r^{k-1} e^{2\pi i r m z} \\
&= \frac{(k-1)!}{(2\pi i)^k} \zeta(k) + \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n,
\end{aligned}$$

Lipschitz for $m\tau$ instead of z

where $\sigma_k(n) = \sum_{d|n} d^k$ is the divisor function of degree k, $q = e^{2\pi i z}$ and $\zeta(k) = \sum_{n\geqslant 1} \frac{1}{n^k}$ is the value of the Riemann zeta functions at k.

$\square$

$$\begin{aligned}
G_4(\tau) &= \frac{1}{240} + q + 9q^2 + 28q^3 + 73q^4 + 126q^5 + \cdots \\
G_6(\tau) &= -\frac{1}{504} + q + 33q^2 + 244q^3 + 1057q^4 + \cdots \\
G_8(\tau) &= \frac{1}{480} + q + 129q^2 + 2188q^3 + \cdots
\end{aligned}$$

Later we will prove the identity $120G_4^2 = G_8$ and this implies Hurwitz identity

$$\sigma_7(n) = \sigma_3(n) + 120 \sum_{m=1}^{n-1} \sigma_3(m)\sigma_3(n-m)$$

for all $n \geqslant 0$. For example we have

$$\sigma_7(3) = 2188 = 28 + 120(1\cdot 9 + 9 \cdot 1) = 28 + 120 \cdot 18 = 28 + 2160$$

Eisenstein series of weight 2

We define the Eisenstein series $G_2(\tau)$ by its Fourier series

$$G_2(\tau) = -\frac{1}{24} + q + 3q^2 + 4q^3 + 7q^4 + 6q^5 + 12q^6 + \cdots$$

In fact, for the weight 2 case one has to inforce convergence by means of the Hecke trick:

$$G_2^*(\tau) = -\frac{1}{8\pi^2} \lim_{s\to 0} \sum_{m,n}{}' \frac{1}{(c\tau+d)^2} \frac{y^s}{|c\tau+d|^{2s}}.$$

The function G_2^* transforms as a modular form of weight 2. It satisfies

$$G_2^*(\tau) = G_2(\tau) + \frac{8}{\pi y}.$$

Now, using $\mathrm{Im}\left(\frac{a\tau+b}{c\tau+d}\right) = \frac{\mathrm{Im}\,\tau}{|c\tau+d|^2}$, we deduce the functional equation w.r.t. $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$

$$G_2\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^2 G_2(\tau) - \frac{c(c\tau+d)}{4\pi i}.$$

"quasi-modular form"

G_k & q-series

We have

$$\sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n = \sum_{n=1}^{\infty} \frac{n^{k-1} q^n}{1-q^n}, \qquad \leftarrow 1-q^n$$

Another not so well-known identity is given by a q-average of negative polylogarithms. Let

$$\mathrm{Li}_{-s} = \sum_{n=1}^{\infty} n^s z^n = \frac{z P_s(z)}{(1-z)^{s+1}} \qquad s \in \mathbb{N}$$

where $P_s(z)$ are the Eulerian polynomials[1], then we obtain

$$\sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n = \sum_{n=1}^{\infty} \frac{q^n P_{k-1}(q^n)}{(1-q^n)^k} \qquad \leftarrow (1-q)^n$$

Foata: Eulerian poly

[1] e.g. $P_0(t) = P_1(t) = t, P_2(t) = t^2 + t, P_3 = t^3 + 4t^2 + t, \ldots$

G_k is a q-zeta value

Gauss q-integers

Now using $P_s(1) = s!$ and

$$\lim_{q \to 1} \frac{(1-q^n)}{(1-q)^{\cancel{n}}} = \lim_{q \to 1} \left(1 + q + \ldots + q^{n-1}\right) = n$$

we get

$$\lim_{q \to 1} (1-q)^k G_k(q) = \lim_{q \to 1} \sum_{n=1}^{\infty} \frac{q^n P_{k-1}(q^n)}{\left(\frac{1-q^n}{(1-q)^{\cancel{n}}}\right)^k} = (k-1)!\zeta(k).$$

now its fine

"Jef ~~Back~~ formula"

Remark

There is a natural map

$$\{\text{modular forms f of weight } k\} \xrightarrow{\sim} \{F : \mathcal{L} \to \mathbb{C} \mid F(c\Lambda) = c^{-k} F(\Lambda),\ \forall c \in \mathbb{C}^*,\ \Lambda \in \mathcal{L}\}$$

given by

$$f\left(\frac{\omega_1}{\omega_2}\right) := \omega_2^{-k} F(\mathbb{Z}\omega_1 + \mathbb{Z}\omega_2)$$

The Eisenstein series $G_k(\tau)$ corresponds to the homogenous lattice function

$$G_k(\Lambda) = \sum_{\substack{\lambda \in \Lambda \\ \lambda \neq 0}} \frac{1}{\lambda^k}.$$

Remark: modular forms are sections of a line bundle on the modular curve

Discriminant

Ramanujan's τ fct

Definition

The discriminant (or Delta funtion) $\Delta(\tau)$ is defined by

$$\Delta(\tau) = q \prod_{n=1}^{\infty} (1-q^n)^{24}$$

where $\tau \in \mathbb{H}$ and $q = e^{2\pi i \tau}$. $\quad = \sum_{n \geq 1} \tau(n) q^n$

Theorem

$\Delta(\tau)$ is a modular form of weight 12.

Proof: We have

$$\frac{\mathrm{d}}{\mathrm{d}\,\tau} \log(\Delta(\tau)) = \frac{\Delta'(\tau)}{\Delta(\tau)} = \frac{\mathrm{d}}{\mathrm{d}\,\tau} \left(2\pi i \tau + 24 \sum_{n=1}^{\infty} \log(1-q^n)\right)$$

$$= 2\pi i \left(1 - 24 \sum_{n=1}^{\infty} \frac{nq^n}{1-q^n}\right) = -48\pi i \left(-\frac{1}{24} + \sum_{n=1}^{\infty} \left(\sum_{r|n} r\right) q^n\right)$$

$$= -48\pi i G_2(\tau).$$

Now by means of the transformation laws for G_2, we derive for $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$

$$\frac{1}{(c\tau+d)^2}\frac{\Delta'\left(\frac{a\tau+b}{c\tau+d}\right)}{\Delta\left(\frac{a\tau+b}{c\tau+d}\right)} = \frac{\Delta'(\tau)}{\Delta(\tau)} + 12\frac{c}{c\tau+d},$$

hence

$$\frac{\mathrm{d}}{\mathrm{d}\,\tau}\log\left(\Delta\left(\frac{a\tau+b}{c\tau+d}\right)\right) = \frac{\mathrm{d}}{\mathrm{d}\,\tau}\log\left(\Delta(\tau)(c\tau+d)^{12}\right)$$

In other words we have

$$\Delta\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^{12}\Delta(\tau)\cdot\kappa_\gamma$$

for some $\kappa_\gamma \in \mathbb{C}$. We now show that $\kappa_\gamma = 1$ for all $\gamma \in \mathrm{SL}_2(\mathbb{Z})$.
We have $\Delta(\tau) = \Delta(\tau+1)$, thus $\kappa_T = 1$. Furthermore for $S = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ we get

$$\Delta(i) = \Delta(Si) = \Delta\left(-\frac{1}{i}\right) = (-i)^{12}\Delta(i)\cdot\kappa_S$$

thus we obtain $\kappa_S = 1$. Since T and S generate $\mathrm{SL}_2(\mathbb{Z})$, we finished the proof. □

Theorem (Valence formula)

For a non-zero modular form f of weight k we have

$$v_\infty(f) + \frac{1}{2}v_i(f) + \frac{1}{3}v_\omega(f) + \sum_{\substack{p\in\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}\\ p\neq i,\omega}} v_p(f) = \frac{k}{12}.$$

Idea of proof: Use Cauchy's argument principle: You get the (order) of the zeros and poles of f in $\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}$ by integrating the logarithmic derivative f'/f around the boundary of the fundamental domain $\mathcal{F}$.

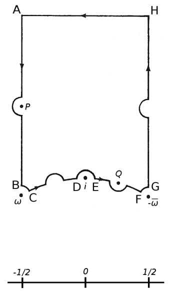


Figure : The contour of integration

Corollary

Let $k \in \mathbb{N}$ be an integer. Then

i) $M_0 = \mathbb{C}$,
ii) If $k = 2$, $k < 0$ or if k is odd then $M_k = 0$.
iii) If $k \in \{4, 6, 8, 10, 14\}$, then $M_k = \mathbb{C}E_k$.
iv) If $k < 12$ or $k = 14$ then $S_k = 0$.
v) $S_{12} = \mathbb{C}\Delta$ and if $k > 12$ then $S_k = \Delta\cdot M_{k-12}$.
vi) If $k \geqslant 4$ then $M_k = \mathbb{C}E_k \oplus S_k$.

Proof:

i) Only constants have no zero.
ii) We already know this for odd k and there is no solution for the others.
iii) When $k \in \{4, 6, 8, 10, 14\}$, then the only solutions to the valence formula are:

$k = 4$: $v_\omega(f) = 1$ and all other $v_p(f) = 0$.
$k = 6$: $v_i(f) = 1$ and all other $v_p(f) = 0$.
$k = 8$: $v_\omega(f) = 2$ and all other $v_p(f) = 0$.
$k = 10$: $v_\omega(f) = v_i(f) = 1$ and all other $v_p(f) = 0$.
$k = 14$: $v_\omega(f) = 2$, $v_i(f) = 1$ and all other $v_p(f) = 0$.

Now, if $f_1, f_2 \in M_k$ for any such k, then $\frac{f_1}{f_2}$ is a modular form of weight 0, which by i) must be constant. Therefore f_1 and f_2 are proportional to $E_k \in M_k$.

Corollary

Let $k \in \mathbb{N}$ be an integer. Then

i) $M_0 = \mathbb{C}$,
ii) If $k = 2$, $k < 0$ or if k is odd then $M_k = 0$.
iii) If $k \in \{4, 6, 8, 10, 14\}$, then $M_k = \mathbb{C}E_k$.
iv) If $k < 12$ or $k = 14$ then $S_k = 0$.
v) $S_{12} = \mathbb{C}\Delta$ and if $k > 12$ then $S_k = \Delta \cdot M_{k-12}$.
vi) If $k \geqslant 4$ then $M_k = \mathbb{C}E_k \oplus S_k$.

Proof:

iv) If $f \in S_k$ we have $v_\infty(f) > 0$, which is impossible for $k < 12$ or $k = 14$.
v We know that $v_\infty(\Delta) = 1$ and by the valenc formula this must be the only zero of Δ. Therefore for any $f \in S_k$ the function $\frac{f}{\Delta}$ is a modular form of weight $k - 12$.
vi) We can substract the constant term of any modular form $f \in M_k$ by adding a suitable multiple of E_k. $\square$

Theorem (Dimension formula)

For an even positiver integer k we have

$$\dim_{\mathbb{C}} M_k = \begin{cases} \lfloor \frac{k}{12} \rfloor + 1 & , \quad k \not\equiv 2 \mod 12 \\ \lfloor \frac{k}{12} \rfloor & , \quad k \equiv 2 \mod 12 \end{cases}.$$

Proof: This will now follow by induction on k from the results in previous Proposition. For $k < 12$ the above dimension formula is already proven. Combing the results of previous Proposition we have

$$M_{k+12} = \mathbb{C}E_{k+12} \oplus \Delta \cdot M_k$$

and since $\lfloor \frac{k}{12} \rfloor + 1 = \lfloor \frac{k+12}{12} \rfloor$ the statement follows inductively.

k	0	2	4	6	8	10	12	14	16	18	20	22	24	26	28	30	32	34	36
$\dim_{\mathbb{C}} M_k$	1	0	1	1	1	1	2	1	2	2	2	2	3	2	3	3	3	3	4

Figure : Dimension of M_k for even $0 \leqslant k \leqslant 36$.