Generalized arithmetic intersection numbers

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Abstract. We present an arithmetic intersection theory for hermitian line bundles on arithmetic surfaces, where the metrics are allowed to have logarithmic singularities at a finite set of points. Using this theory we show that the generalized arithmetic self-intersection number of the line bundle of modular forms equipped with its canonical metric equals $\zeta_{\mathbb{Q}}(-1)+2\cdot\zeta'_{\mathbb{Q}}(-1)$ up to a trivial factor; here, $\zeta_{\mathbb{Q}}(s)$ denotes the Riemann zeta function.

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Introduction. Arakelov geometry is an arithmetic intersection theory for varieties, which are defined over the ring of integers of a number field. Of general interest is the determination of arithmetic intersection numbers of hermitian vector bundles over these varieties, cf. [So2]. We restrict ourselves to arithmetic intersection numbers of metrized line bundles over arithmetic surfaces associated to modular curves. Due to a series of papers by A. Abbes, P. Michel, E. Ullmo , cf. [AU], [MU], or by a recent paper of J. Jorgenson and J. Kramer, cf. [JK], an upper and lower bound for the arithmetic self-intersection number of the dualizing sheaf equipped with the Arakelov metric for the modular curves $X_0(N)$ is known. In this article we present an arithmetic intersection theory for hermitian line bundles on arithmetic surfaces, where the metrics are allowed to have logarithmic singularities at a finite set of points. Using this theory we show that the corresponding generalized arithmetic self-intersection number of the line bundle of modular forms equipped with its canonical metric equals $\zeta_{\mathbb{Q}}(-1)+2\cdot\zeta'_{\mathbb{Q}}(-1)$ up to a trivial factor; here, $\zeta_{\mathbb{Q}}(s)$ denotes the Riemann zeta function.

This article is organized as follows; after a short review of arithmetic intersection theory on arithmetic surfaces in the first two sections, we generalize this theory in the third section by allowing that the hermitian metrics under consideration become logarithmically singular. After a review of the theory of modular curves and modular forms in the fourth section, we compute the generalized arithmetic self-intersection number of the line bundle of modular forms equipped with the Petersson metric in the sixth section. In the last section we show that our generalized arithmetic intersection product is compatible with the one proposed by J. Bost in [Bo1].

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1 Review intersection theory on arithmetic surfaces

1.1. Preliminaries. In this article an *arithmetic surface* $\mathcal{X}/\operatorname{Spec}\mathbb{Z}$ is an integral, 2-dimensional, regular scheme defined over $\operatorname{Spec}\mathbb{Z}$ together with a flat, projective morphism

$$\pi : \mathcal{X} \longrightarrow \operatorname{Spec}\mathbb{Z}.$$

As we consider in this article only arithmetic surfaces $\mathcal{X}/\operatorname{Spec}\mathbb{Z}$, we will sometimes denote them by $\mathcal{X}$ to ease notation.

The ring of regular functions on $\mathcal{X}$ is the ring of integers $\mathcal{O}_E$ for a number field E, and there exists a canonical, projective, flat morphism $\mathcal{X} \longrightarrow \operatorname{Spec}\mathcal{O}_E$, having geometrically connected fibres. For all embeddings $\sigma : E \longrightarrow \mathbb{C}$ the complex curve $\mathcal{X}_\sigma := \mathcal{X} \times_\sigma \operatorname{Spec}\mathbb{C}$ is projective, smooth and connected. We note that the scheme $\mathcal{X} \longrightarrow \operatorname{Spec}\mathcal{O}_E$ is an arithmetic surface in the sense of the definition in [So1], p. 328. We point out that we do not assume $\mathcal{X}$ to be semi-stable as in [Fa], p. 389, but in contrast to [Bo1], p. 261, we assume $\mathcal{X}$ to be regular.

We denote the complex valued points $\mathcal{X}(\mathbb{C})$ by $\mathcal{X}_\infty$; this is a compact, 1-dimensional, complex manifold, which may have several connected components. Actually we have the following decomposition

$$\mathcal{X}_\infty = \coprod_{\sigma : E \to \mathbb{C}} \mathcal{X}_\sigma(\mathbb{C}).$$

We note that the complex conjugation F_∞ acts on $\mathcal{X}_\infty$.

1.2. Definition. Let $\pi : \mathcal{X} \longrightarrow \operatorname{Spec}\mathbb{Z}$ be an arithmetic surface. Every divisor $D \subset \mathcal{X}$ posseses a decomposition $D = D_{\text{ver}} + D_{\text{hor}}$; here D_{ver} is a finite linear combination of irreducible components of the closed fibres of $\mathcal{X}$ over $\operatorname{Spec}\mathbb{Z}$, and D_{hor} is a finite linear combination of irreducible horizontal divisors. Each of the latter ones is isomorphic to the spectrum of a ring of integers of a finite extension of $\mathbb{Q}$. For a divisor D on $\mathcal{X}$ let $\mathcal{O}_D$ be the structure sheaf of D. For any two divisors D_1 and D_2 on $\mathcal{X}$ the coherent sheaves $\operatorname{Tor}_j^{\mathcal{O}_\mathcal{X}}(\mathcal{O}_{D_1}, \mathcal{O}_{D_2})$ $(j = 0, 1)$ are defined by the following long exact sequence:

$$\begin{aligned} 0 \;=\; & \operatorname{Tor}_1^{\mathcal{O}_\mathcal{X}}(\mathcal{O}_\mathcal{X}, \mathcal{O}_{D_2}) \longrightarrow \operatorname{Tor}_1^{\mathcal{O}_\mathcal{X}}(\mathcal{O}_{D_1}, \mathcal{O}_{D_2}) \longrightarrow \mathcal{O}_\mathcal{X}(-D_1) \otimes_{\mathcal{O}_\mathcal{X}} \mathcal{O}_{D_2} \\ & \longrightarrow \mathcal{O}_\mathcal{X} \otimes_{\mathcal{O}_\mathcal{X}} \mathcal{O}_{D_2} \longrightarrow \operatorname{Tor}_0^{\mathcal{O}_\mathcal{X}}(\mathcal{O}_{D_1}, \mathcal{O}_{D_2}) = \mathcal{O}_{D_1} \otimes_{\mathcal{O}_\mathcal{X}} \mathcal{O}_{D_2} \longrightarrow 0. \end{aligned}$$

Let now $D_1, D_2 \subset \mathcal{X}$ be two effective divisors having no horizontal component in common, i.e., $D_{1,\mathrm{hor}} \cap D_{2,\mathrm{hor}} \cap \mathcal{X}_\infty = \emptyset$, then the *intersection number at the finite places* $(D_1, D_2)_{\mathrm{fin}}$ *of* D_1 *and* D_2 is defined by

$$(D_1, D_2)_{\mathrm{fin}} := \sum_{i,j=0}^{1} (-1)^{i+j} \log \# \left(R^i \pi_* \left(\mathrm{Tor}_j^{\mathcal{O}_\mathcal{X}} (\mathcal{O}_{D_1}, \mathcal{O}_{D_2}) \right) \right); \tag{1}$$

here the $R^i\pi_*$ are the derived functors of the direct image π_* (see, e.g., [De], p. 3).

By the symmetry (resp. bilinearity) of the tensor product and its derived functors Tor the intersection number at the finite places is symmetric and bilinear. We extend the intersection number at the finite places by linearity to arbitrary divisors.

1.3. Proposition. *Let* $D_{\mathrm{ver}} \subseteq \mathcal{X}_{\mathbb{F}_p} = \mathcal{X} \times_{\mathrm{Spec}\,\mathbb{Z}} \mathrm{Spec}\,\mathbb{F}_p$ *and* D *be effective divisors on* $\mathcal{X}$.

(a) *Then* $(D_{\mathrm{ver}}, D)_{\mathrm{fin}} = \deg_{D_{\mathrm{ver}}}(\mathcal{O}(D)\,|_{D_{\mathrm{ver}}}) \cdot \log(p)$.

(b) *If* D_{ver} *and* D *intersect in finitely many points, then* $(D_{\mathrm{ver}}, D)_{\mathrm{fin}} \geq 0$. *Equality holds iff* D_{ver} *and* D *are disjoint.*

(c) *For all principal divisors* D *(*D *need not be effective) on* $\mathcal{X}$ *one has* $(D_{\mathrm{ver}}, D)_{\mathrm{fin}} = 0$. *Especially, we have* $(D_{\mathrm{ver}}, \mathcal{X}_{\mathbb{F}_p})_{\mathrm{fin}} = 0$.

Proof. See, e.g., [De], p. 3. □

1.4. Proposition. *Let* $D_{\mathrm{hor}} \subset \mathcal{X}$ *be an irreducible, horizontal divisor,* K *be the number field whose ring of integers* $\mathcal{O}_K$ *satisfies* $D_{\mathrm{hor}} \cong \mathrm{Spec}\,\mathcal{O}_K$, *and* $k(\mathcal{X})$ *be function field of* $\mathcal{X}$. *Then, for all rational functions* $r \in k(\mathcal{X})^*$, *whose horizontal part is distinct to* D_{hor}, *we have*

$$(D_{\mathrm{hor}}, \mathrm{div}(r))_{\mathrm{fin}} = \log \left(\mathrm{Nm}_{K/\mathbb{Q}}(r|_K) \right);$$

here $\mathrm{Nm}_{K/\mathbb{Q}}(r|_K)$ *denotes the norm of* K *to* $\mathbb{Q}$ *of the rational function* r *restricted to* $K \subseteq k(\mathcal{X})$.

Proof. See, e.g., [La], p. 56. □

2 Arithmetic intersection numbers for hermitian line bundles

2.1. Definitions. A *hermitian line bundle* $\overline{\mathcal{L}} = (\mathcal{L}, h)$ is a line bundle $\mathcal{L}$ on $\mathcal{X}$ together with a smooth, hermitian metric h, invariant under F_∞, on the complex line bundle $\mathcal{L}_\infty := \mathcal{L} \otimes_{\mathbb{Z}} \mathbb{C}$ on $\mathcal{X}_\infty$; the norm corresponding to h is denoted by $\|\cdot\|$. Two hermitian line bundles $\overline{\mathcal{L}}, \overline{\mathcal{M}}$ on $\mathcal{X}$ are *isomorphic*, if there is an isometry

$$\overline{\mathcal{L}} \otimes \overline{\mathcal{M}}^{-1} \cong (\mathcal{O}_\mathcal{X}, |\cdot|);$$

here $|\cdot|$ denotes the usual absolute value. The *arithmetic Picard group* $\widehat{\mathrm{Pic}}(\mathcal{X})$ is the group of isomorphy classes of hermitian line bundles $\overline{\mathcal{L}}$ on $\mathcal{X}$, the group structure being given by the tensor product.

2.2. Intersection numbers at the finite places. Let $\overline{\mathcal{L}}$ and $\overline{\mathcal{M}}$ be two hermitian line bundles on an arithmetic surface $\mathcal{X}$. Assume that l und m are non-trivial, regular sections of $\mathcal{L}$ and $\mathcal{M}$ (resp.), whose divisors $\mathrm{div}(l)$ and $\mathrm{div}(m)$ have no horizontal component in common. Then, we define the *intersection number at the finite places* $(l.m)_{\mathrm{fin}}$ *of* l *and* m by the formula

$$(l.m)_{\mathrm{fin}} := (\mathrm{div}(l), \mathrm{div}(m))_{\mathrm{fin}}\,. \tag{2}$$

2.3. Intersection numbers at the infinite places. Let $\overline{\mathcal{L}}$ and $\overline{\mathcal{M}}$ be two hermitian line bundles on an arithmetic surface $\mathcal{X}/\mathbb{Z}$. Assume that $\mathcal{L}$ and $\mathcal{M}$ have non-trivial, global sections l and m (resp.). These induce global sections of $\mathcal{L}_\infty$ and $\mathcal{M}_\infty$, which we denote by abuse of notation again by l and m (resp.). Furthermore, assume that the divisor $\mathrm{div}(l)$ of l has no points in common with the divisor $\mathrm{div}(m)$ of m on $\mathcal{X}_\infty$. Writing $\mathrm{div}(l) = \sum_\alpha p_\alpha P_\alpha$ ($p_\alpha \in \mathbb{Z}$, $P_\alpha \in \mathcal{X}_\infty$), we then set

$$(\log\|m\|)[\mathrm{div}(l)] := \sum_\alpha p_\alpha \log\|m(P_\alpha)\|. \tag{3}$$

We define the *intersection number at the infinite places* $(l.m)_\infty$ *of* l *and* m by the formula

$$(l.m)_\infty := -(\log\|m\|)[\mathrm{div}(l)] - \int_{\mathcal{X}_\infty} \log\|l\| \cdot c_1(\overline{\mathcal{M}}), \tag{4}$$

where the first Chern form $c_1(\overline{\mathcal{L}}) \in H^{1,1}(\mathcal{X}_\infty, \mathbb{R})$ of $\overline{\mathcal{L}}$ is, away from the divisor of l on $\mathcal{X}_\infty$, given by

$$c_1(\overline{\mathcal{L}}) = dd^c(-\log\|l(\cdot)\|^2).$$

Note, as the function $\log\|l(\cdot)\|^2$ is locally integrable on $\mathcal{X}_\infty$, the intersection number at the infinite places is well-defined.

2.4. Proposition. *The following identities hold for* $(l.m)_\infty$*:*

$$(l.m)_\infty =$$
$$-(\log\|l\|)[\mathrm{div}(m)] - (\log\|m\|)[\mathrm{div}(l)] - 2\int_{\mathcal{X}_\infty} d(\log\|l\|) \wedge d^c(\log\|m\|), \tag{5}$$

$$(l.m)_\infty =$$
$$\int_{\mathcal{X}_\infty} \Big(-\log\|l\| \cdot c_1(\overline{\mathcal{M}}) - \log\|m\| \cdot c_1(\overline{\mathcal{L}}) + 2\cdot d(\log\|l\|) \wedge d^c(\log\|m\|)\Big). \tag{6}$$

Proof. Let $B_\varepsilon(P)$ denote the open disk of radius ε centered at P. Setting

$$\mathcal{X}_\varepsilon := \mathcal{X}_\infty \setminus \left(\bigcup_{P_\alpha \in \mathrm{div}(l)} B_\varepsilon(P_\alpha) \quad \cup \bigcup_{Q_\beta \in \mathrm{div}(m)} B_\varepsilon(Q_\beta) \right),$$

we have

$$\int_{\mathcal{X}_\infty} -\log \|l\| \cdot c_1(\overline{\mathcal{M}}) = \lim_{\varepsilon \to 0} \int_{\mathcal{X}_\varepsilon} \log \|l\| \cdot dd^c(\log \|m\|^2).$$

By means of Stokes' theorem, we obtain

$$\int_{\mathcal{X}_\varepsilon} \log \|l\| \cdot dd^c(\log \|m\|^2) = -\sum_{\alpha} \int_{\partial B_\varepsilon(P_\alpha)} \log \|l\| \cdot d^c(\log \|m\|^2) -$$
$$\sum_{\beta} \int_{\partial B_\varepsilon(Q_\beta)} \log \|l\| \cdot d^c(\log \|m\|^2) - 2 \int_{\mathcal{X}_\varepsilon} d(\log \|l\|) \wedge d^c(\log \|m\|).$$

An application of the residue theorem and letting ε go to zero then implies

$$-\int_{\mathcal{X}_\infty} \log \|l\| \cdot c_1(\overline{\mathcal{M}}) = -(\log \|l\|)[\mathrm{div}(m)] - 2 \int_{\mathcal{X}_\infty} d(\log \|l\|) \wedge d^c(\log \|m\|). \tag{7}$$

Substituting (7) into (4) yields (5). As the last integral in (7) is symmetric in l and m, we derive the claimed formula (6) by interchanging l and m, solving for $-(\log \|m\|)[\mathrm{div}(l)]$ and substituting the result into (4). □

2.5. Arithmetic intersection numbers. Let $\overline{\mathcal{L}}, \overline{\mathcal{M}}$ be two hermitian line bundles on $\mathcal{X}$ and l, m (resp.) be non-trivial, global sections, whose divisors on $\mathcal{X}_\infty$ have no points in common. Then, the *arithmetic intersection number* $\overline{\mathcal{L}}.\overline{\mathcal{M}}$ *of* $\overline{\mathcal{L}}$ *and* $\overline{\mathcal{M}}$ is given by

$$\overline{\mathcal{L}}.\overline{\mathcal{M}} := (l.m)_{\mathrm{fin}} + (l.m)_\infty; \tag{8}$$

here $(l.m)_{\mathrm{fin}}$ and $(l.m)_\infty$ are defined by formula (2) and (4) (resp.). The *arithmetic self-intersection number of* $\overline{\mathcal{L}}$ is given by $\overline{\mathcal{L}}.\overline{\mathcal{L}}$.

2.6. Theorem. (Arakelov [Ar]). *Formula (8) induces a bilinear, symmetric pairing*

$$\widehat{\mathrm{Pic}}(\mathcal{X}) \times \widehat{\mathrm{Pic}}(\mathcal{X}) \longrightarrow \mathbb{R}.$$

Proof. We only sketch the main ideas as given in [So1]. For more details, cf. [La], [SABK].

(a) Symmetry: As $(l.m)_{\mathrm{fin}} = (m.l)_{\mathrm{fin}}$, it suffices to show that $(l.m)_\infty = (m.l)_\infty$. But, since the integral in formula (6) of Proposition 2.4 is symmetric in l and m, it follows that $(l.m)_\infty = (m.l)_\infty$.

(b) Bilinearity: Let $\overline{\mathcal{L}'}$ be a third hermitian line bundle on $\mathcal{X}$ and l' a non-trivial, global section of $\mathcal{L}'$, whose divisor on $\mathcal{X}_\infty$ has no points in common with $\mathrm{div}(m)$. Then, we have the equality

$$(l \otimes l'.m)_{\mathrm{fin}} = (\mathrm{div}(l) + \mathrm{div}(l'), \mathrm{div}(m))_{\mathrm{fin}} = (l.m)_{\mathrm{fin}} + (l'.m)_{\mathrm{fin}}.$$

On the other hand, it is easy to see from formula (6) in Proposition 2.4 that $(l\otimes l'.m)_\infty = (l.m)_\infty + (l'.m)_\infty$ holds, from which the claimed bilinearity follows.

We note that because of the bilinearity of the arithmetic intersection numbers, we are allowed from now on to work with rational sections l and m instead of global ones.

(c) Well-definedness: Let $\overline{\mathcal{M}} = (\mathcal{O}_\mathcal{X}, |\cdot|)$ be the trivial hermitian line bundle and m a rational section of $\mathcal{M}$. It suffices to show that

$$(l.m)_{\mathrm{fin}} + (l.m)_\infty = 0$$

holds for rational sections l of $\mathcal{L}$, whose divisor is horizontal; in addition, we may assume that $\mathrm{div}(l)$ is irreducible and defined over the number field K. Now, let r be the restriction of the rational function m to $\mathrm{div}(l)$. Since $c_1(\overline{\mathcal{M}}) = 0$, we then derive from formula (4)

$$(l.m)_\infty = -(\log|m|)[\mathrm{div}(l)] = -\sum_{\sigma:K\hookrightarrow\mathbb{C}} \log|\sigma(r)|,$$

where the sum on the right hand side is taken over all equivalence classes of real and complex embeddings of K. On the other hand, we have

$$(l.m)_{\mathrm{fin}} = \log\left(\mathrm{N}_{K/\mathbb{Q}}(r)\right).$$

The well-definedness of the pairing (8) now follows using the product formula for the number field K. $\square$

2.7. Remark. In his fundamental paper [Ar], S.J. Arakelov considered the subgroup $\widehat{\mathrm{Pic}}(\mathcal{X},\mu) \subseteq \widehat{\mathrm{Pic}}(\mathcal{X})$ generated by the isomorphy classes of hermitian line bundles $\overline{\mathcal{L}} = \overline{\mathcal{O}(D)}$ satisfying the two conditions

$$c_1(\overline{\mathcal{L}}) = \deg(\mathcal{L})\cdot\mu \quad , \quad \int_{\mathcal{X}_\infty} \log\|1_D\|\cdot\mu = 0$$

for a fixed, normalized volume form μ on $\mathcal{X}_\infty$; here 1_D denotes the canonical section of $\mathcal{L} = \mathcal{O}(D)$. If $\{\omega_1,\dots,\omega_g\}$ denotes an orthonormal basis of $H^0(\mathcal{X}_\infty,\Omega^1_{\mathcal{X}_\infty})$, one can take for example

$$\mu = \frac{1}{g}\sum_{j=1}^{g}\omega_j\wedge\overline{\omega}_j.$$

For more details of this aspect we refer to [Fa] and [MB].

3 Arithmetic intersection numbers for line bundles with logarithmically singular, hermitian metrics

3.1. Definition. Let $\mathcal{X}/\mathbb{Z}$ be an arithmetic surface and $\mathcal{X}_\infty = \mathcal{X}(\mathbb{C})$ as in section 1.1; for a finite set $\mathcal{S}$ of points $S_1,\dots,S_r$ of $\mathcal{X}_\infty$ denote by $\mathcal{Y}_\infty$ the open Riemann surface $\mathcal{X}_\infty\setminus\mathcal{S}$.

For $S_j \in \mathcal{S}$ and $\varepsilon > 0$, denote by $B_\varepsilon(S_j) \subseteq \mathcal{X}_\infty$ the open disk of radius ε centered at S_j; let t be a local parameter at S_j $(j = 1, ..., r)$. For a line bundle $\mathcal{L}$ on $\mathcal{X}$, a singular metric h on $\mathcal{L}_\infty$ is called *hermitian, logarithmically singular (with respect to $\mathcal{S}$)*, if the following two conditions hold:

(a) h is a smooth, hermitian metric on $\mathcal{L}_\infty$ restricted to $\mathcal{Y}_\infty$, invariant under F_∞;

(b) for each $S_j \in \mathcal{S}$ and any section l of $\mathcal{L}$, there exist a real number α and a positive, continuous function φ defined on $B_\varepsilon(S_j)$ and smooth away from the origin such that the equality

$$\|l(t)\| = -\log(|t|^2)^\alpha \cdot |t|^{\mathrm{ord}_{S_j}(l)} \cdot \varphi(t)$$

holds for all $t \in B_\varepsilon(S_j) \setminus \{0\}$; furthermore, there exist positive constants β and ρ such that the inequalities

$$\left|\frac{\partial\varphi(t)}{\partial t}\right| \leq \frac{\beta}{|t|^{1-\rho}}, \quad \left|\frac{\partial\varphi(t)}{\partial \bar{t}}\right| \leq \frac{\beta}{|t|^{1-\rho}}, \quad \left|\frac{\partial^2\varphi(t)}{\partial t\, \partial \bar{t}}\right| \leq \frac{\beta}{|t|^{2-\rho}}$$

hold for all $t \in B_\varepsilon(S_j) \setminus \{0\}$.

We call a line bundle $\mathcal{L}$ on $\mathcal{X}$ equipped with a logarithmically singular metric h a *hermitian, logarithmically singular line bundle* and denote it as before by $\overline{\mathcal{L}} = (\mathcal{L}, h)$. To indicate the dependence of the quantities α (resp. φ, β, ρ) on $l, \overline{\mathcal{L}}$ and $S_j \in \mathcal{S}$, we write instead $\alpha_{\overline{\mathcal{L}},j}$ (resp. $\varphi_{\overline{\mathcal{L}},j}, \beta_{\overline{\mathcal{L}},j}, \rho_{\overline{\mathcal{L}},j}$).

3.2. Remarks. (a) The last inequality in Definition 3.1 ensures the local integrability of the first Chern form $c_1(\overline{\mathcal{L}})$ on the open disks $B_\varepsilon(S_j)$, hence on the whole of $\mathcal{X}_\infty$; away from $\mathrm{div}(l)$ and $\mathcal{S}$ the first Chern form is given by $c_1(\overline{\mathcal{L}}) = -dd^c(\log \|l\|^2)$.

(b) The above definition extends Mumford's notion of *good metrics* (see [Mu]), also referred to as metrics with logarithmic singularities along $\mathcal{S}$ (see [CS]); these metrics are natural examples of L_1^2-metrics (cf. 7.1).

3.3. Proposition. *Let $\overline{\mathcal{L}} = (\mathcal{L}, h)$ be a hermitian, logarithmically singular line bundle on the arithmetic surface $\mathcal{X}/\mathbb{Z}$. Then, we have*

$$\deg(\mathcal{L}) = \int_{\mathcal{X}_\infty} c_1(\overline{\mathcal{L}}).$$

Proof. Let l be a rational section of $\mathcal{L}$. As the statement is true for smooth, hermitian metrics, we may assume that $\mathrm{div}(l) = \sum_\alpha p_\alpha P_\alpha$ with the property $\{P_\alpha\} \subseteq \mathcal{S}$ on $\mathcal{X}_\infty$. Then, using Stokes' theorem on $\mathcal{X}_\varepsilon := \mathcal{X}_\infty \setminus \bigcup_{j=1}^r B_\varepsilon(S_j)$, we find

$$\int_{\mathcal{X}_\infty} c_1(\overline{\mathcal{L}}) = -\lim_{\varepsilon\to 0} \int_{\mathcal{X}_\varepsilon} dd^c(\log \|l\|^2) = \lim_{\varepsilon\to 0} \sum_{j=1}^r \int_{\partial B_\varepsilon(S_j)} d^c(\log \|l\|^2).$$

Observing

$$\log \|l(t)\| = \mathrm{ord}_{S_j}(l) \cdot \log|t| + \alpha_{\overline{\mathcal{L}},j} \log(-\log|t|^2) + \log(\varphi_{\overline{\mathcal{L}},j}(t))$$

for $t \in B_\varepsilon(S_j) \setminus \{0\}$, the claim follows, since

$$\lim_{\varepsilon\to 0} \int_{\partial B_\varepsilon(S_j)} d^c(\log(-\log|t|^2)) = 0 \quad \text{and} \quad \lim_{\varepsilon\to 0} \int_{\partial B_\varepsilon(S_j)} d^c(\log(\varphi_{\overline{\mathcal{L}},j}(t))) = 0,$$

where in the latter case the growth conditions of $\varphi_{\overline{\mathcal{L}},j}$ and its derivatives given in Definition 3.1 are used. □

3.4. Definition. Two hermitian, logarithmically singular line bundles $\overline{\mathcal{L}}, \overline{\mathcal{M}}$ on $\mathcal{X}$ are *isomorphic*, if

$$\overline{\mathcal{L}} \otimes \overline{\mathcal{M}}^{-1} \cong (\mathcal{O}_{\mathcal{X}}, |\cdot|).$$

We denote the group of isomorphy classes of hermitian, logarithmically singular line bundles $\overline{\mathcal{L}}$ on $\mathcal{X}$ by $\widehat{\mathrm{Pic}}(\mathcal{X}, \mathcal{S})$ the group structure being given by the tensor product.

3.5. Remark. We have a canonical inclusion of $\widehat{\mathrm{Pic}}(\mathcal{X})$ into $\widehat{\mathrm{Pic}}(\mathcal{X}, \mathcal{S})$. Let $\overline{\mathcal{L}}, \overline{\mathcal{M}}$ be two hermitian, logarithmically singular line bundles on $\mathcal{X}$ and l, m rational sections of $\mathcal{L}, \mathcal{M}$ (resp.), whose divisors on $\mathcal{X}_\infty$ have no points in common. Then, the expressions (3) and (4), which we used to define the arithmetic intersection number, are no longer defined in general. Therefore, we make the following

3.6. Definition. With the above notations we define the *generalized intersection number at the infinite places* $\langle l.m\rangle_\infty$ *of* l *and* m by the formula

$$\begin{aligned}\langle l.m\rangle_\infty \; := \; & \int_{\mathcal{X}_\infty} \Big(-\log\|l\| \cdot c_1(\overline{\mathcal{M}}) - \log\|m\| \cdot c_1(\overline{\mathcal{L}}) + 2 \cdot d(\log\|l\|) \wedge d^c(\log\|m\|) \Big) + \\ & \sum_{j=1}^{r} (\mathrm{ord}_{S_j}(l) \cdot \alpha_{\overline{\mathcal{M}},j} + \mathrm{ord}_{S_j}(m) \cdot \alpha_{\overline{\mathcal{L}},j}). \end{aligned} \tag{9}$$

3.7. Remark. If $\mathcal{S} = \emptyset$, we have

$$\langle l.m\rangle_\infty = (l.m)_\infty$$

by formula (6) of Proposition 2.4.

3.8. Theorem. *The generalized intersection number at the infinite places (9) is well-defined.*

Proof. Let $\overline{\mathcal{L}}, \overline{\mathcal{M}}$ be two hermitian, logarithmically singular line bundles on $\mathcal{X}$ and l, m generically non-trivial, rational sections of $\mathcal{L}, \mathcal{M}$ (resp.), whose divisors on $\mathcal{X}_\infty$ have no points in common. For sufficiently small $\varepsilon > 0$ we set

$$\mathcal{X}_\varepsilon := \mathcal{X}_\infty \setminus \left(\bigcup_{j=1}^{r} B_\varepsilon(S_j) \right),$$

and

$$\begin{aligned}\langle l.m\rangle_\varepsilon \;&:=\; \int_{\mathcal{X}_\varepsilon}\Big(-\log\|l\|\cdot c_1(\overline{\mathcal{M}})-\log\|m\|\cdot c_1(\overline{\mathcal{L}})+2\cdot d(\log\|l\|)\wedge d^c(\log\|m\|)\Big)+\\ &\sum_{j=1}^{r}\Big(\mathrm{ord}_{S_j}(l)\cdot\alpha_{\overline{\mathcal{M}},j}+\mathrm{ord}_{S_j}(m)\cdot\alpha_{\overline{\mathcal{L}},j}\Big).\end{aligned}$$

As the restrictions of the metrics on $\mathcal{L}_\infty$ and $\mathcal{M}_\infty$ to $\mathcal{X}_\varepsilon$ are smooth, hermitian metrics, we know that $\langle l.m\rangle_\varepsilon$ is bounded. We have to show that the limit

$$\lim_{\varepsilon\to 0}\langle l.m\rangle_\varepsilon=\langle l.m\rangle_\infty$$

exists. It suffices to determine the ε-dependence of the integrals

$$\int_{A_{\varepsilon,\varepsilon_0}(S_j)}\Big(-\log\|l\|\cdot c_1(\overline{\mathcal{M}})-\log\|m\|\cdot c_1(\overline{\mathcal{L}})+2\cdot d(\log\|l\|)\wedge d^c(\log\|m\|)\Big); \qquad (10)$$

here ε_0 is fixed such that $0<\varepsilon<\varepsilon_0$ and $A_{\varepsilon,\varepsilon_0}(S_j)$ denotes the annullus

$$A_{\varepsilon,\varepsilon_0}(S_j)=\{t\in B_{\varepsilon_0}(S_j)\,|\,\varepsilon<|t|<\varepsilon_0\}$$

centered at S_j $(j=1,...,r)$. For that purpose we consider the following local expansions around the point $S_j\in\mathcal{S}$:

$$\begin{aligned}\log\|l(t)\| &= \mathrm{ord}_{S_j}(l)\cdot\log|t|+\alpha_{\overline{\mathcal{L}},j}\cdot\log(-\log|t|^2)+\log(\varphi_{\overline{\mathcal{L}},j}(t)),\\ \partial\log\|l(t)\| &= \left(\mathrm{ord}_{S_j}(l)\cdot\frac{1}{t}-\alpha_{\overline{\mathcal{L}},j}\cdot\frac{1}{t(\log|t|^2)}+\frac{\partial\varphi_{\overline{\mathcal{L}},j}(t)}{\partial t}\cdot\frac{1}{\varphi_{\overline{\mathcal{L}},j}(t)}\right)dt,\\ \bar\partial\log\|l(t)\| &= \left(\mathrm{ord}_{S_j}(l)\cdot\frac{1}{\bar t}-\alpha_{\overline{\mathcal{L}},j}\cdot\frac{1}{\bar t(\log|t|^2)}+\frac{\partial\varphi_{\overline{\mathcal{L}},j}(t)}{\partial\bar t}\cdot\frac{1}{\varphi_{\overline{\mathcal{L}},j}(t)}\right)d\bar t,\end{aligned}$$

and

$$c_1(\overline{\mathcal{L}})=\left(\frac{2\alpha_{\overline{\mathcal{L}},j}}{|t|^2(\log|t|^2)^2}+\left(\varphi_{\overline{\mathcal{L}},j}(t)\frac{\partial^2\varphi_{\overline{\mathcal{L}},j}(t)}{\partial t\partial\bar t}+\frac{\partial\varphi_{\overline{\mathcal{L}},j}(t)}{\partial t}\cdot\frac{\partial\varphi_{\overline{\mathcal{L}},j}(t)}{\partial\bar t}\right)\cdot\frac{1}{\varphi^2_{\overline{\mathcal{L}},j}(t)}\right)\frac{dt\wedge d\bar t}{\pi i}.$$

Analogeous formulae hold for $\log\|m(t)\|,\partial\log\|m(t)\|,\bar\partial\log\|m(t)\|$ and $c_1(\overline{\mathcal{M}})$. Note that

$$d(\log\|l\|)\wedge d^c(\log\|m\|)=\frac{-1}{4\pi i}\left(\partial(\log\|l\|)\wedge\bar\partial(\log\|m\|)+\partial(\log\|m\|)\wedge\bar\partial(\log\|l\|)\right). \quad (11)$$

Substituting all the above expansions into the integrals (10), we get by using the conditions on $\varphi_{\overline{\mathcal{L}},j}(t)$ and $\varphi_{\overline{\mathcal{M}},j}(t)$ that all the integrals in question are bounded of the form $O_{\varepsilon_0}(1)+O(\varepsilon)$. Note that the terms proportional to $1/\left(|t|^2(\log|t|^2)\right)$ cancel each other and $\mathrm{ord}_{S_j}(l)\cdot\mathrm{ord}_{S_j}(m)=0$. □

The following Lemma is the analogue of Proposition 2.4.

3.9. Lemma. *With above notations we have the equalities:*

$$\begin{aligned}
&\langle l.m\rangle_\infty = \\
&-(\log\|m\|)[\mathrm{div}(l)-\sum_{j=1}^r \mathrm{ord}_{S_j}(l)\cdot S_j]+\sum_{j=1}^r \mathrm{ord}_{S_j}(l)\Big(\alpha_{\overline{\mathcal{M}},j}-\log(\varphi_{\overline{\mathcal{M}},j}(0))\Big)- \\
&\lim_{\varepsilon\to 0}\left(\sum_{j=1}^r \mathrm{ord}_{S_j}(l)\cdot\alpha_{\overline{\mathcal{M}},j}\cdot\log(-\log\varepsilon^2)+\int_{\mathcal{X}_\varepsilon}\log\|l\|\cdot c_1(\overline{\mathcal{M}})\right), \qquad (12)
\end{aligned}$$

and

$$\begin{aligned}
&\langle l.m\rangle_\infty = \\
&-(\log\|l\|)[\mathrm{div}(m)-\sum_{j=1}^r \mathrm{ord}_{S_j}(m)\cdot S_j]-(\log\|m\|)[\mathrm{div}(l)-\sum_{j=1}^r \mathrm{ord}_{S_j}(l)\cdot S_j]- \\
&\sum_{j=1}^r\Big(\mathrm{ord}_{S_j}(l)\cdot\log(\varphi_{\overline{\mathcal{M}},j}(0))+\mathrm{ord}_{S_j}(m)\cdot\log(\varphi_{\overline{\mathcal{L}},j}(0))\Big)- \\
&\lim_{\varepsilon\to 0}\left(\sum_{j=1}^r \log(-\log\varepsilon^2)\Big(\mathrm{ord}_{S_j}(l)\cdot\alpha_{\overline{\mathcal{M}},j}+\mathrm{ord}_{S_j}(m)\cdot\alpha_{\overline{\mathcal{L}},j}\Big)+\right. \\
&\left.2\int_{\mathcal{X}_\varepsilon} d(\log\|l\|)\wedge d^c(\log\|m\|)\right). \qquad (13)
\end{aligned}$$

Proof. Let $\overline{\mathcal{L}},\overline{\mathcal{M}}$ be two hermitian, logarithmically singular line bundles on $\mathcal{X}$ and l,m rational sections of $\mathcal{L},\mathcal{M}$ (resp.), whose divisors on $\mathcal{X}_\infty$ have no points in common. As in section 2.3, we write

$$\mathrm{div}(l)=\sum_\alpha p_\alpha P_\alpha \quad , \quad \mathrm{div}(m)=\sum_\beta q_\beta Q_\beta;$$

furthermore, we set for $\varepsilon>0$

$$\mathcal{X}_\varepsilon := \mathcal{X}_\infty\setminus\left(\bigcup_{\alpha:P_\alpha\notin\mathcal{S}} B_\varepsilon(P_\alpha)\cup\bigcup_{\beta:Q_\beta\notin\mathcal{S}} B_\varepsilon(Q_\beta)\cup\bigcup_{j=1}^r B_\varepsilon(S_j)\right)$$

and

$$\begin{aligned}
\langle l.m\rangle_\varepsilon \ := \ &\int_{\mathcal{X}_\varepsilon}\Big(-\log\|l\|\cdot c_1(\overline{\mathcal{M}})-\log\|m\|\cdot c_1(\overline{\mathcal{L}})+2\cdot d(\log\|l\|)\wedge d^c(\log\|m\|)\Big)+ \\
&\sum_{j=1}^r\Big(\mathrm{ord}_{S_j}(l)\cdot\alpha_{\overline{\mathcal{M}},j}+\mathrm{ord}_{S_j}(m)\cdot\alpha_{\overline{\mathcal{L}},j}\Big).
\end{aligned}$$

We have already shown that the limit

$$\langle l.m\rangle_\infty = \lim_{\varepsilon\to 0}\langle l.m\rangle_\varepsilon$$

exists. In order to prove the Lemma, we first consider the integral

$$-\int_{\mathcal{X}_\varepsilon}\log\|l\|\cdot c_1(\overline{\mathcal{M}}).$$

Using Stokes' theorem, we derive as in the proof of Proposition 2.4 (cf.(7))

$$\begin{aligned}
&-\int_{\mathcal{X}_\varepsilon}\log\|l\|\cdot c_1(\overline{\mathcal{M}}) = -(\log\|l\|)[\mathrm{div}(m)-\sum_{j=1}^{r}\mathrm{ord}_{S_j}(m)\cdot S_j]- \\
&2\sum_{j=1}^{r}\int_{\partial B_\varepsilon(S_j)}\log\|l\|\cdot d^c(\log\|m\|)-2\int_{\mathcal{X}_\varepsilon}d(\log\|l\|)\wedge d^c(\log\|m\|)+O(\varepsilon).
\end{aligned}\tag{14}$$

We now determine the ε-dependence of the integrals over $\partial B_\varepsilon(S_j)$. To this end we use the local expansions around the points $S_j\in\mathcal{S}$:

$$\begin{aligned}
\log\|l(t)\| &= \mathrm{ord}_{S_j}(l)\cdot\log|t|+\alpha_{\overline{\mathcal{L}},j}\cdot\log(-\log|t|^2)+\log(\varphi_{\overline{\mathcal{L}},j}(t)),\\
\log\|m(t)\| &= \mathrm{ord}_{S_j}(m)\cdot\log|t|+\alpha_{\overline{\mathcal{M}},j}\cdot\log(-\log|t|^2)+\log(\varphi_{\overline{\mathcal{M}},j}(t)).
\end{aligned}$$

To perform the subsequent computations we use polar coordinates $t=re^{i\varphi}$ and take into account the local expansion

$$\begin{aligned}
d^c(\log\|l(re^{i\varphi})\|) &= \left(\frac{\mathrm{ord}_{S_j}(l)}{4\pi}+\frac{\alpha_{\overline{\mathcal{L}},j}}{4\pi\log r}+\frac{r}{4\pi}\frac{\partial\log(\varphi_{\overline{\mathcal{L}},j}(re^{i\varphi}))}{\partial r}\right)d\varphi- \\
&\quad \frac{1}{4\pi r}\frac{\partial\log(\varphi_{\overline{\mathcal{L}},j}(re^{i\varphi}))}{\partial\varphi}dr;
\end{aligned}\tag{15}$$

here we used the formula

$$d^c=\frac{r}{4\pi}\frac{\partial}{\partial r}d\varphi-\frac{1}{4\pi r}\frac{\partial}{\partial\varphi}dr.$$

If $l(S_j)\neq 0$, we find for the integral in question

$$\begin{aligned}
&\int_{\partial B_\varepsilon(S_j)}\log\|l\|\cdot d^c(\log\|m\|)= \\
&\frac{1}{2}\cdot\mathrm{ord}_{S_j}(m)\Big(\alpha_{\overline{\mathcal{L}},j}\cdot\log(-\log\varepsilon^2)+\log(\varphi_{\overline{\mathcal{L}},j}(0))\Big)+O(\varepsilon).
\end{aligned}$$

If, on the other hand, $m(S_j)\neq 0$, we find for the integral under consideration

$$\int_{\partial B_\varepsilon(S_j)}\log\|l\|\cdot d^c(\log\|m\|)=\frac{1}{2}\cdot\mathrm{ord}_{S_j}(l)\cdot\alpha_{\overline{\mathcal{M}},j}+O(\varepsilon).$$

Adding up, we find for the integral (14)

$$-\int_{\mathcal{X}_\varepsilon} \log\|l\| \cdot c_1(\overline{\mathcal{M}}) = -(\log\|l\|)[\operatorname{div}(m) - \sum_{j=1}^{r} \operatorname{ord}_{S_j}(m)\cdot S_j] -$$
$$\sum_{j=1}^{r}\Big(\operatorname{ord}_{S_j}(l)\cdot \alpha_{\overline{\mathcal{M}},j} + \operatorname{ord}_{S_j}(m)\Big(\alpha_{\overline{\mathcal{L}},j}\cdot \log(-\log\varepsilon^2) + \log(\varphi_{\overline{\mathcal{L}},j}(0))\Big)\Big) -$$
$$2\int_{\mathcal{X}_\varepsilon} d(\log\|l\|)\wedge d^c(\log\|m\|) + O(\varepsilon)\,. \tag{16}$$

An analogeous formula holds with $\overline{\mathcal{L}},\overline{\mathcal{M}}$ (resp. l,m) interchanged. Therefore, we obtain the formula

$$\langle l.m\rangle_\varepsilon =$$
$$-(\log\|l\|)[\operatorname{div}(m) - \sum_{j=1}^{r}\operatorname{ord}_{S_j}(m)\cdot S_j] - (\log\|m\|)[\operatorname{div}(l) - \sum_{j=1}^{r}\operatorname{ord}_{S_j}(l)\cdot S_j] -$$
$$\sum_{j=1}^{r}\Big(\operatorname{ord}_{S_j}(l)\cdot\log(\varphi_{\overline{\mathcal{M}},j}(0)) + \operatorname{ord}_{S_j}(m)\cdot\log(\varphi_{\overline{\mathcal{L}},j}(0)) + \log(-\log\varepsilon^2)\cdot$$
$$\Big(\operatorname{ord}_{S_j}(l)\cdot\alpha_{\overline{\mathcal{M}},j} + \operatorname{ord}_{S_j}(m)\cdot\alpha_{\overline{\mathcal{L}},j}\Big)\Big) - 2\int_{\mathcal{X}_\varepsilon} d(\log\|m\|)\wedge d^c(\log\|l\|) + O(\varepsilon). \tag{17}$$

The formulae (16) (with l and m interchanged) and (17) lead to the two formulae claimed for the generalized intersection number at the infinite places. $\square$

3.10. Generalized arithmetic intersection numbers. Let $\overline{\mathcal{L}},\overline{\mathcal{M}}$ be two hermitian, logarithmically singular line bundles on $\mathcal{X}$ and l,m (resp.) be non-trivial, global sections, whose divisors on $\mathcal{X}_\infty$ have no points in common. Then, the *generalized arithmetic intersection number* $\overline{\mathcal{L}}.\overline{\mathcal{M}}$ *of* $\overline{\mathcal{L}}$ *and* $\overline{\mathcal{M}}$ is given by

$$\overline{\mathcal{L}}.\overline{\mathcal{M}} := (l.m)_{\mathrm{fin}} + \langle l.m\rangle_\infty, \tag{18}$$

where $(l.m)_{\mathrm{fin}}$ and $\langle l.m\rangle_\infty$ are defined by formula (2) and (9) (resp.). The *generalized arithmetic self-intersection number of* $\overline{\mathcal{L}}$ is given by $\overline{\mathcal{L}}.\overline{\mathcal{L}}$.

3.11. Theorem. *Formula (18) induces a bilinear, symmetric pairing*

$$\widehat{\operatorname{Pic}}(\mathcal{X},\mathcal{S})\times\widehat{\operatorname{Pic}}(\mathcal{X},\mathcal{S}) \longrightarrow \mathbb{R}$$

extending the pairing of Arakelov.

Proof. We have to show the symmetry, bilinearity and well-definedness of the generalized arithmetic intersection number.

(a) Symmetry: The symmetry of $(l.m)_{\mathrm{fin}}$ is well-known and the symmetry of $\langle l.m\rangle_\infty$ follows by means of formula (13).

(b) Bilinearity: The bilinearity of $(l.m)_{\text{fin}}$ is again well-known and the bilinearity of $\langle l.m\rangle_\infty$ follows by means of formula (13).

(c) Well-definedness: Here the proof reduces to the proof of the corresponding fact in Theorem 2.6, since we derive for $\overline{\mathcal{M}} = (\mathcal{O}_\mathcal{X}, |\cdot|)$ by formula (12)

$$\langle l.m\rangle_\infty = -(\log|m|)[\operatorname{div}(l)],$$

noting that $c_1(\overline{\mathcal{M}}) = 0$ and $\alpha_{\overline{\mathcal{M}},j} = 0$. □

4 Review of modular curves and modular forms

4.1. Modular curves over $\mathbb{C}$. Let $\mathfrak{h} := \{\tau = x + iy \in \mathbb{C} \,|\, \operatorname{Im}\tau = y > 0\}$ denote the upper half plane. The full modular group $\mathrm{SL}_2(\mathbb{Z})$ acts properly discontinuously on $\mathfrak{h}$ via fractional linear transformations:

$$\tau \mapsto \frac{a\tau + b}{c\tau + d}; \quad \tau \in \mathfrak{h}, \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}).$$

Define $\Gamma(1) := \mathrm{PSL}_2(\mathbb{Z}) = \mathrm{SL}_2(\mathbb{Z})/\{\pm 1\}$. For any subgroup Γ of finite index in $\Gamma(1)$ the quotient $\Gamma \setminus \mathfrak{h}$ has the structure of a Riemann surface, which can be compactified by adding finitely many cusps. We denote this compact Riemann surface by $X(\Gamma)$; in fact, we have $X(\Gamma) = \Gamma \setminus (\mathfrak{h} \cup \mathbb{P}^1(\mathbb{Q}))$. Following [Se2], p. 71, we call $X(\Gamma)$ a (general) modular curve. For $\tau \in \mathfrak{h} \cup \mathbb{P}^1(\mathbb{Q})$ let

$$\Gamma_\tau := \{\gamma \in \Gamma \,|\, \gamma(\tau) = \tau\}$$

denote the stabilizer of τ in Γ and by $n_\tau := |\Gamma_\tau|$ its order. On $X(\Gamma)$ there are three types of points:

- *Ordinary points* P_{τ_0} corresponding to the orbits of $\tau_0 \in \mathfrak{h}$ with $n_{\tau_0} = 1$; a local parameter t of such a point is given by $t = \frac{\tau - \tau_0}{\tau - \overline{\tau}_0}$.
- *Elliptic fixed points* Q_{τ_0} corresponding to the orbits of $\tau_0 \in \mathfrak{h}$ with $n := n_{\tau_0} > 1$; a local parameter of such a point is given by $t = \left(\frac{\tau - \tau_0}{\tau - \overline{\tau}_0}\right)^n$.
- *Parabolic fixed points* or *cusps* S_{τ_0} corresponding to the orbits of $\tau_0 \in \mathbb{P}^1(\mathbb{Q})$; a local parameter t of such a point is given by $t = e^{2\pi i \gamma(\tau)/s}$, where s denotes the cusp width of S_{τ_0} and $\gamma \in \Gamma(1)$ is chosen such that $\gamma(\tau_0) = i\infty$.

There are only finitely many elliptic fixed points and cusps on $X(\Gamma)$. The compact Riemann surface $X(1) := X(\Gamma(1))$ is known to be isomorphic to $\mathbb{P}^1(\mathbb{C})$, the isomorphism being induced by the classical *j-invariant*, whose Fourier expansion at the unique cusp $S_{i\infty} \in X(1)$ is given by

$$j(\tau) = \frac{1}{t} + 744 + \sum_{n=1}^{\infty} a_n(j)\, t^n \quad (t = e^{2\pi i \tau}). \tag{19}$$

Therefore, the function field $K(X(1))$ of $X(1)$ turns out to be given by the field $\mathbb{C}(j)$ of rational functions in $j(\tau)$. For a subgroup Γ of finite index in $\Gamma(1)$ we finally have a finite, holomorphic map

$$\pi_\Gamma : X(\Gamma) \longrightarrow X(1)$$

of degree $\deg \pi_\Gamma = [\Gamma(1) : \Gamma]$; the function field $K(X(\Gamma))$ of $X(\Gamma)$ is therefore a finite, algebraic extension of $\mathbb{C}(j)$ of degree $[\Gamma(1) : \Gamma]$. For more details, cf. [Sh].

4.2. The hyperbolic metric on $X(\Gamma)$. The volume form μ associated to the hyperbolic metric on $\mathfrak{h}$ is given by the $(1,1)$-form

$$\mu = \frac{dx \wedge dy}{y^2} = \frac{i}{2} \cdot \frac{d\tau \wedge d\bar{\tau}}{(\operatorname{Im} \tau)^2}.$$

Denote by μ_Γ the volume form induced by μ on $X(\Gamma)$. This $(1,1)$-form is singular and its singularities occur at the elliptic fixed points and at the cusps of $X(\Gamma)$. More precisely, we have:

- In a sufficiently small open disk $B_\varepsilon(Q_{\tau_0})$ around an elliptic fixed point Q_{τ_0} the restriction of μ_Γ to $B_\varepsilon(Q_{\tau_0})$ is given in the corresponding local parameter t by $(n = n_{\tau_0})$

$$\mu_\Gamma|_{B_\varepsilon(Q_{\tau_0})} = \frac{2i}{n^2} \cdot \frac{dt \wedge d\bar{t}}{|t|^{2-2/n}(1 - |t|^{2/n})^2}.$$

- In a sufficiently small open disk $B_\varepsilon(S_{\tau_0})$ around a cusp S_{τ_0} the restriction of μ_Γ to $B_\varepsilon(S_{\tau_0})$ is given in the corresponding local parameter t by

$$\mu_\Gamma|_{B_\varepsilon(S_{\tau_0})} = 2i \cdot \frac{dt \wedge d\bar{t}}{|t|^2 \log(|t|^2)^2}.$$

We note that the volume of $X(\Gamma)$ with respect to μ_Γ is well-defined and given by

$$\int_{X(\Gamma)} \mu_\Gamma = \int_{\mathcal{F}(\Gamma)} \mu = \frac{\pi}{3} \cdot [\Gamma(1) : \Gamma]; \tag{20}$$

here $\mathcal{F}(\Gamma)$ denotes a fundamental domain for the action of Γ on $\mathfrak{h}$.

4.3. Modular forms over $\mathbb{C}$. Let $M_2(\mathbb{Z})$ be the algebra of (2×2)-matrices with integer coefficients and k a positive integer. The slash operator $|_k$ induces an action of $M_2(\mathbb{Z})$ on holomorphic functions f on $\mathfrak{h}$ in the following way: For a given $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{Z})$ this action is given by

$$f(\tau) \mapsto (f|_k\gamma)(\tau) := f\left(\frac{a\tau + b}{c\tau + d}\right) \cdot (c\tau + d)^{-k} \cdot \det(\gamma).$$

A modular form f of weight $k \in 2\mathbb{N}$ with respect to Γ is a holomorphic function $f : \mathfrak{h} \longrightarrow \mathbb{C}$ satisfying the following two properties:

(a) $(f|_k\gamma)(\tau) = f(\tau)$ for all $\gamma \in \Gamma$;

(b) f is regular at the cups, i.e., for all cusps we have a Fourier expansion $f(\tau) = \sum_{n=0}^{\infty} a_n(f)\, t^n$ with t the corresponding local parameter.

We denote the $\mathbb{C}$-vector space of such functions by $M_k(\Gamma)$. The direct sum $\bigoplus_k M_k(\Gamma)$ is a commutative ring; the multiplication is induced by the multiplication of holomorphic functions and the grading is given by the weight. The inclusion $\Gamma \subseteq \Gamma(1)$ induces an inclusion $M_k(\Gamma(1)) \subset M_k(\Gamma)$. Conversely, we have

4.4. Proposition. *Let Γ be a subgroup of finite index $d = [\Gamma(1) : \Gamma]$ in $\Gamma(1)$. Every modular form $f \in M_k(\Gamma)$ is integral over $\bigoplus_k M_k(\Gamma(1))$.*

Proof. Each modular form $f \in M_k(\Gamma)$ is a root of the polynom

$$\prod_{\rho \in \Gamma \backslash \Gamma(1)} (X - f|_k\rho),$$

whose coefficients are the elementary symmetric functions in the $f|_k\rho$'s. An application of $|_k\gamma$ for $\gamma \in \Gamma(1)$ re-orders the $f|_k\rho$'s; therefore, the coefficients are modular forms for $\Gamma(1)$.

4.5. Example. The classical *discriminant* $\Delta(\tau)$ defined by the product expansion

$$\Delta(\tau) = t \prod_{n=1}^{\infty} (1 - t^n)^{24} \quad (t = e^{2\pi i \tau}), \tag{21}$$

is a modular form of weight $k = 12$ with respect to $\Gamma = \Gamma(1)$. We note that $\Delta(\tau) \neq 0$ for all $\tau \in \mathfrak{h}$ and $\Delta(i\infty) = 0$.

4.6. Definition. The functional equations satisfied by a modular form $f(\tau) \in M_k(\Gamma)$ show that the elements of $M_k(\Gamma)$ are global sections of a line bundle on $X(\Gamma)$, which we call the *line bundle of modular forms (of weight k with respect to Γ)* and which we denote by $\mathcal{M}_k(\Gamma)_\infty$.

4.7. Proposition. *Let $k \in \mathbb{N}$ and $12|k$. Then, there is an isomorphism*

$$\mathcal{M}_k(\Gamma)_\infty \cong \pi_\Gamma^* \mathcal{O}_{X(1)}(S_{i\infty})^{\otimes k/12}.$$

Proof. We have an isomorphism

$$M_k(\Gamma) \cong H^0(X(\Gamma), \mathcal{O}_{X(\Gamma)}(\pi_\Gamma^* S_{i\infty})^{\otimes k/12})$$

given by sending a modular form $f \in M_k(\Gamma)$ to the modular function $f/\Delta^{k/12} \in K(X(\Gamma))$. In fact, by the regularity of f we have the inequality

$$\operatorname{div}(f/\Delta^{k/12}) \geq -\frac{k}{12} \pi_\Gamma^* S_{i\infty}.$$

□

4.8. The Petersson metric on $\mathcal{M}_k(\Gamma)_\infty$. The *Petersson metric* $\|\cdot\|_{Pet}$ on the line bundle $\mathcal{M}_k(\Gamma)_\infty$ of modular forms is defined by the formula

$$\|f(\tau)\|^2_{Pet} := |f(\tau)|^2 (4\pi \operatorname{Im} \tau)^k;$$

here f is a rational section of $\mathcal{M}_k(\Gamma)_\infty$, e.g., $f(\tau)$ is a meromorphic modular form of weight k. For an explanation of the normalization factor we refer to 4.14.

4.9. Proposition. *The Petersson metric $\|\cdot\|_{Pet}$ on $\mathcal{M}_k(\Gamma)_\infty$ is logarithmically singular with respect to the set consisting of the elliptic fixed points and the cusps of $X(\Gamma)$.*

Proof. (a) Away from the set of elliptic fixed points and cusps of $X(\Gamma)$ the Petersson metric $\|\cdot\|_{Pet}$ is easily seen to be a smooth, hermitian metric, invariant under F_∞.

(b) For the set of elliptic fixed points and cusps of $X(\Gamma)$ we have the following local behaviour of $\|\cdot\|_{Pet}$:

- In a sufficiently small open disk $B_\varepsilon(Q_{\tau_0})$ around an elliptic fixed point Q_{τ_0} the restriction of $\|f\|_{Pet}$ to $B_\varepsilon(Q_{\tau_0})$ is given in the corresponding local parameter t by $(n = n_{\tau_0})$

$$\|f(t)\|_{Pet} = |f(t)| \cdot \left(\frac{1 - |t|^{2/n}}{|t^{1/n} - 1|^2} \cdot 4\pi \operatorname{Im} \tau_0 \right)^{k/2}.$$

- In a sufficiently small open disk $B_\varepsilon(S_{\tau_0})$ around a cusp S_{τ_0} the restriction of $\|f\|_{Pet}$ to $B_\varepsilon(S_{\tau_0})$ is given in the corresponding local parameter t by

$$\|f(t)\|_{Pet} = |f(t)| \cdot \left(-s \cdot \log |t|^2 \right)^{k/2}.$$

A comparison of these computations with Definition 3.1 now completes the proof of the proposition. □

4.10. Remark. By Remark 3.2 the first Chern form $c_1(\mathcal{M}_k(\Gamma)_\infty, \|\cdot\|_{Pet})$ of the line bundle of modular forms $\mathcal{M}_k(\Gamma)_\infty$ equipped with the Petersson metric $\|\cdot\|_{Pet}$ is integrable; namely, it is given by

$$c_1(\mathcal{M}_k(\Gamma)_\infty, \|\cdot\|_{Pet}) = \frac{k}{4\pi} \cdot \mu_\Gamma.$$

Furthermore, using Proposition 3.3 and formula (20) we recover

$$\deg(\mathcal{M}_k(\Gamma)_\infty) = \int_{X(\Gamma)} c_1(\mathcal{M}_k(\Gamma)_\infty, \|\cdot\|_{Pet}) = \int_{X(\Gamma)} \frac{k}{4\pi} \cdot \mu_\Gamma = \frac{k}{12} \cdot [\Gamma(1) : \Gamma].$$

4.11. Modular curves over $\mathbb{Z}$. The compact Riemann surface $X(1)$ is isomorphic to $\mathbb{P}^1(\mathbb{C})$. This isomorphism is induced by the ample line bundle $\mathcal{M}_{12}(\Gamma(1))_\infty$. Namely, we identify the canonical sections Z_0 and Z_1 of the hyperplane bundle $\mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(1)$ with $j \cdot \Delta, \Delta$ (resp.) and get

$$X(1) \cong \operatorname{Proj} \mathbb{C}[Z_0, Z_1] = \mathbb{P}^1_{\mathbb{C}}.$$

The *arithmetic surface* $\mathcal{X}(1)$ *associated to* $\Gamma(1)$, which extends $X(1)$ to an arithmetic surface over $\operatorname{Spec}\mathbb{Z}$, is therefore given by

$$\mathcal{X}(1) = \operatorname{Proj} \mathbb{Z}[Z_0, Z_1] = \mathbb{P}^1_{\mathbb{Z}}.$$

Let $\Gamma \subseteq \Gamma(1)$ be a subgroup of finite index in $\Gamma(1)$ and

$$\pi_\Gamma : X(\Gamma) \longrightarrow X(1)$$

the corresponding covering of compact Riemann surfaces, which is unramified outside the $\Gamma(1)$-orbits of $i, (1+i\sqrt{3})/2, i\infty \in \mathfrak{h} \cup \mathbb{P}^1(\mathbb{Q})$; these orbits correspond to $0, 1728, \infty \in \mathbb{P}^1(\mathbb{C})$ (resp.). Using this fact one can show that $X(\Gamma)$ has a smooth, projective model $X(\Gamma)_E$ defined over some number field $E/\mathbb{Q}$ of minimal degree. Note on the other hand that by the theorem of Belyi, every algebraic curve defined over some number field is a general modular curve (see, e.g., [Bo2], p. 99). The covering map π_Γ induces a morphism

$$\pi_{\Gamma,E} : X(\Gamma)_E \longrightarrow \mathbb{P}^1_{\mathbb{Q}}$$

of smooth, projective curves defined over E. The function field $k(X(\Gamma)_E)$ of $X(\Gamma)_E$ determines therefore a finite field extension of $k(\mathcal{X}(1)) = \mathbb{Q}(j)$ having degree $[\Gamma(1) : \Gamma] \cdot [E : \mathbb{Q}]$. We denote by $X(\Gamma)_{\mathbb{Z}}$ the normalization of $\mathcal{X}(1) = \mathbb{P}^1_{\mathbb{Z}}$ in $k(X(\Gamma)_E)$; this is a normal, integral, projective scheme (see [EGAII], chapter 6.3), which is flat over $\operatorname{Spec}\mathbb{Z}$ (see [Ha], chapter III, 9.1.3). The singularities of $X(\Gamma)_{\mathbb{Z}}$ are isolated since $X(\Gamma)_{\mathbb{Z}}$ is normal. By the theorem of Lichtenbaum these can be resolved (see [Li], p. 151). Let $\mathcal{X}(\Gamma) \longrightarrow X(\Gamma)_{\mathbb{Z}}$ be a desingularisation of $X(\Gamma)_{\mathbb{Z}}$. We call $\mathcal{X}(\Gamma)$ *an arithmetic surface associated to* Γ. We denote the induced morphism from $\mathcal{X}(\Gamma)$ to $\mathcal{X}(1)$ by $\pi_{\Gamma,\mathbb{Z}}$; its degree is $\deg(\pi_{\Gamma,\mathbb{Z}}) = [E : \mathbb{Q}] \cdot [\Gamma(1) : \Gamma]$. Finally, we note that every dominant morphism from $\mathcal{X}(\Gamma)$ to $\mathcal{X}(1)$ factorizes through $X(\Gamma)_{\mathbb{Z}}$ (see [EGAII], chapter 6.3), i.e., we have the following commutative diagram

$$\begin{tikzcd} \mathcal{X}(\Gamma) \arrow[r] \arrow[dr, "\pi_{\Gamma,\mathbb{Z}}"'] & X(\Gamma)_{\mathbb{Z}} \arrow[d] \\ & \mathcal{X}(1) \end{tikzcd} \tag{22}$$

4.12. Modular forms over $\mathbb{Z}$. Let $\Gamma \subseteq \Gamma(1)$ be a subgroup of finite index in $\Gamma(1)$, $\mathcal{X}(\Gamma)$ an arithmetic surface associated to Γ and $\pi_{\Gamma,\mathbb{Z}} : \mathcal{X}(\Gamma) \longrightarrow \mathcal{X}(1)$ the corresponding morphism (see 4.11). The cusp $S_{i\infty} \in X(1)$ induces a section $s_\infty : \operatorname{Spec}\mathbb{Z} \longrightarrow \mathcal{X}(1)$, whose divisor we denote by D_∞. For $k \in \mathbb{N}, 12|k$, taking into account Proposition 4.7, we define the *line bundle of modular forms* $\mathcal{M}_k(\Gamma)$ *on* $\mathcal{X}(\Gamma)$ by

$$\mathcal{M}_k(\Gamma) := \pi^*_{\Gamma,\mathbb{Z}}\mathcal{O}_{\mathcal{X}(1)}(D_\infty)^{\otimes k/12}.$$

We note that by the construction of the models, we have the equality

$$X(\Gamma)_{\mathbb{Z}} = \operatorname{Proj}\left(\bigoplus_{k\geq 0} H^0(\mathcal{X}(\Gamma), \mathcal{M}_k(\Gamma))\right);$$

therefore, every section $l \in H^0(\mathcal{X}(\Gamma), \mathcal{M}_k(\Gamma))$ is integral over $\bigoplus_{k\geq 0} H^0(\mathcal{X}(1), \mathcal{M}_k(\Gamma(1)))$. Finally, we set $\overline{\mathcal{M}}_k(\Gamma) := (\mathcal{M}_k(\Gamma), \|\cdot\|_{Pet})$; this is a hermitian, logarithmically singular line bundle in the sense of Definition 3.1.

4.13. Remark. As we will show in the next section the above construction of arithmetic surfaces associated to modular curves is compatible with the usual one for congruence subgroups using a modular interpretation. Furthermore, we like to point out that we do not use the so-called q-expansion principle to define integral modular forms.

4.14. Moduli of elliptic curves. Let $\mathcal{X}(N)$ be the arithmetic surface representing the moduli problem 'generalized elliptic curve with a level N-structure'; for the precise definitions we refer to [DeRa], pp. 11, 68, 73. This moduli problem is representable over $\operatorname{Spec}\mathbb{Z}[\zeta_N]$ (see [DeRa], p. 78); here ζ_N is a primitive N-th root of unity. One can show that $\mathcal{X}(N)_\infty \cong \coprod_{j=1}^{\Phi(N)} X(\Gamma(N))$ (see [DeRa], p. 84); here $\Phi(N) = [\mathbb{Q}(\zeta_N) : \mathbb{Q}]$ and $\Gamma(N)$ equals the principal congruence subgroup of $\Gamma(1)$ of level N. There exists a universal generalized elliptic curve

$$\nu : \mathcal{E}(N) \longrightarrow \mathcal{X}(N),$$

which is an elliptic curve over the open, dense subscheme $\mathcal{U}$ consisting of the complement of the cusps and the supersingular points of $\mathcal{X}(N)$. The fibres of $\mathcal{E}(N)$ over the cusps are certain m-gons with $m|N$, obtained by blowing up the Tate curve. On $\mathcal{U}$ we have the classification map, which is given by the j-invariant, onto the coarse moduli space of elliptic curves $\mathbb{A}^1_{\mathbb{Z}}$. This map extends to a map

$$j_{\mathcal{E}(N)/\mathcal{X}(N)} : \mathcal{X}(N) \longrightarrow \mathbb{P}^1_{\mathbb{Z}}$$

and gives rise to an isomorphism (see [DeRa], pp. 11, 73)

$$\mathcal{X}(N) \cong \mathcal{X}(\Gamma(N)) = X(\Gamma(N))_{\mathbb{Z}}.$$

The regular part of $\mathcal{E}(N)$ has the structure of a commutative group scheme; we let $e : \mathcal{X}(N) \longrightarrow \mathcal{E}(N)$ be the zero section and $\Omega^1_{\mathcal{E}(N)/\mathcal{X}(N)}$ be the relative cotangent sheaf. Then, we set

$$\omega_{\mathcal{E}(N)/\mathcal{X}(N)} := e^*\Omega^1_{\mathcal{E}(N)/\mathcal{X}(N)}.$$

This gives a canonical isomorphism (see [DeRa], p. 161)

$$j^*_{\mathcal{E}(N)/\mathcal{X}(N)}\mathcal{O}_{\mathcal{X}(1)}(D_\infty) \cong \omega^{\otimes 12}_{\mathcal{E}(N)/\mathcal{X}(N)}$$

and therefore a canonical section of $\omega^{\otimes 12}_{\mathcal{E}(N)/\mathcal{X}(N)}$, which we denote by Δ. On $\mathcal{U}$ we also have an isomorphism

$$\omega_{\mathcal{E}(N)/\mathcal{X}(N)}|_{\mathcal{U}} \cong \nu_*\Omega^1_{\mathcal{E}(N)/\mathcal{X}(N)}|_{\mathcal{U}}.$$

Therefore, $\omega_{\mathcal{E}(N)_\infty/\mathcal{X}(N)_\infty}$ has a natural metric; namely, for all $x \in \mathcal{X}(N)_\infty \backslash \{j^*_{\mathcal{E}(N)/\mathcal{X}(N)}S_{i\infty}\}$ and all $\alpha \in \omega_{\mathcal{E}(N)_\infty/\mathcal{X}(N)_\infty,x} \cong H^0(\mathcal{E}(N)_{\infty,x}, \Omega^1_{\mathcal{E}(N)_{\infty,x}})$ this metric is defined by

$$\|\alpha\|^2_{\mathrm{L}^2} := \frac{i}{2\pi}\int_{\mathcal{E}(N)_{\infty,x}} \alpha \wedge \overline{\alpha}.$$

This metric extends to a singular metric on $\mathcal{X}(N)$, which is a hermitian, logarithmically singular metric. On the Tate curve $\mathbb{G}_m(\mathbb{C})/q^{\mathbb{Z}}$ ($q = e^{2\pi i\tau}$), which describes $\mathcal{E}(N)_\infty$ close the cusps, the canonical section Δ is given by

$$\Delta = q\prod_{n=1}^{\infty}(1-q^n)^{24} \cdot \left(\frac{dw}{w}\right)^{\otimes 12};$$

here $\frac{dw}{w}$ denotes the natural differential on $\mathbb{G}_m(\mathbb{C})$. Finally, we have

$$\begin{aligned}
\|\Delta^{1/12}\|^2_{\mathrm{L}^2} &= \frac{i}{2\pi} \cdot \int_{\mathbb{G}_m(\mathbb{C})/q^{\mathbb{Z}}} |\Delta^{1/12}|^2 \cdot \frac{dw}{w} \wedge \frac{d\bar{w}}{\bar{w}} \\
&= \frac{1}{\pi} \cdot |\Delta|^{1/6} \cdot \int_{|q|}^{1} \frac{2\pi r \cdot dr}{r^2} \\
&= |\Delta|^{1/6} \cdot (-2\log|q|)
\end{aligned}$$

(see [Bo], pp. 8,10). Hence, we get

$$\|\Delta^{1/12}\|^2_{\mathrm{L}^2} = |\Delta|^{1/6} \cdot (4\pi \operatorname{Im}\tau).$$

Moreover, for $k \in \mathbb{N}, 12|k$, we derive an isomorphism

$$(\omega^{\otimes k}_{\mathcal{E}(N)/\mathcal{X}(N)}, \|\cdot\|_{\mathrm{L}^2}) \cong (\mathcal{M}_k(\Gamma(N)), \|\cdot\|_{Pet})$$

of hermitian, logarithmically singular line bundles. Finally, we point out that this isomorphism also gives a definition of $\overline{\mathcal{M}}_k(\Gamma(N))$ for arbitrary $k \in \mathbb{N}$. The preceding isomorphism of hermitian, logarithmically singular line bundles justifies our choice of the normalization factor in 4.8.

5 The real analytic Eisenstein series

5.1. Preliminaries. Let $E(\tau,s)$ be the real analytic Eisenstein series, which for $s\in\mathbb{C}$ with $\mathrm{Re}(s)>1$ is given by

$$E(\tau,s) \;=\; \sum_{g\in\Gamma_\infty\backslash\Gamma(1)} \mathrm{Im}\,(g\,\tau)^s = \frac{1}{2}\sum_{\substack{c,d\in\mathbb{Z}\\(c,d)=1}} \frac{y^s}{|c\tau+d|^{2s}} \qquad \left(\Gamma_\infty = \begin{pmatrix} 1 & \mathbb{Z}\\ 0 & 1\end{pmatrix}\right)$$

(see, e.g., [Za2], p. 415). As a function of τ it is $\Gamma(1)$-invariant; furthermore, as a function of s it has a meromorphic continuation to $\mathbb{C}$ with a pole in $s=1$ with residue $3/\pi$ independently of τ. The Fourier expansion of $E(\tau,s)$ is shown to be

$$E(\tau,s) \;=\; y^s+\varphi(s)y^{1-s}+\frac{2\pi^s}{\Gamma(s)\zeta_{\mathbb{Q}}(2s)}\sum_{m\neq 0}|m|^{s-1/2}\sigma_{1-2s}(m)y^{1/2}K_{s-1/2}(2\pi|m|y)e^{2\pi i m x};$$

here $K_{s-1/2}$ is a Bessel function, $\sigma_s(m)=\sum_{d|m}d^s$ is the generalized divisor function and

$$\varphi(s)=\frac{\Gamma(\frac{1}{2})\Gamma(s-\frac{1}{2})\zeta_{\mathbb{Q}}(2s-1)}{\Gamma(s)\zeta_{\mathbb{Q}}(2s)} = \frac{3}{\pi}(s-1)^{-1}+\frac{3}{\pi}\left(2-2\log(4\pi)-24\zeta'_{\mathbb{Q}}(-1)\right)+O(s-1).$$

5.2. Proposition. *(Kronecker limit formula). Let $\Delta(\tau)\in M_{12}(\Gamma(1))$ be the discriminant. Then, we have*

$$-\log\left(|\Delta(\tau)|^2y^{12}\right) = 4\pi\lim_{s\to 1}\left(E(\tau,s)-\varphi(s)\right).$$

Proof. The proof follows easily from [Za1], p. 267. □

5.3. Lemma. *Let $\mathcal{F}_{y\le T}$ be the truncated fundamental domain*

$$\mathcal{F}_{y\le T} = \left\{\tau=x+iy\in\mathfrak{h}\,|\,1\le|\tau|,\,-\frac{1}{2}<x\le\frac{1}{2},\,y<T\right\}.$$

Then, we have for $T\gg 0$

$$\int_{\mathcal{F}_{y\le T}} -\log\left(|\Delta(\tau)|^2y^{12}\right)\frac{dxdy}{4\pi y^2} \;=\; -2+\log(4\pi)+24\zeta'_{\mathbb{Q}}(-1)+\log(4\pi T)+O\left(\frac{\log T}{T}\right).$$

Proof. By the Kronecker limit formula we have

$$\begin{aligned}\int_{\mathcal{F}_{y\le T}} -\log(|\Delta(\tau)|^2y^{12})\frac{dxdy}{4\pi y^2} &= \int_{\mathcal{F}_{y\le T}}\lim_{s\to1}\left(E(\tau,s)-\varphi(s)\right)\frac{dxdy}{y^2}\\ &= \lim_{s\to 1}\left(\int_{\mathcal{F}_{y\le T}}E(\tau,s)\frac{dxdy}{y^2}-\int_{\mathcal{F}_{y\le T}}\varphi(s)\frac{dxdy}{y^2}\right).\end{aligned}$$

Using a refinement of the Rankin Selberg method we get for first integral (see, e.g., [Za2], formula (33), p. 426)

$$\int_{\mathcal{F}_{y\leq T}} E(\tau,s)\frac{dxdy}{y^2} = \frac{T^{s-1}}{s-1} - \varphi(s)\frac{T^{-s}}{s}.$$

For the second integral we easily get

$$\int_{\mathcal{F}_{y\leq T}} \varphi(s)\frac{dxdy}{y^2} = \varphi(s)\left(\frac{\pi}{3}-\frac{1}{T}\right).$$

Using the Laurent expansion of $\varphi(s)$ we get

$$\int_{\mathcal{F}_{y\leq T}} -\log\left(|\Delta(\tau)|^2 y^{12}\right)\frac{dxdy}{4\pi y^2} =$$

$$\lim_{s\to 1}\left(-2+2\log(4\pi)+24\zeta'_{\mathbb{Q}}(-1)+\log(T)+O\left(\frac{\log T}{T}\right)+O(s-1)\right) =$$

$$-2+2\log(4\pi)+24\zeta'_{\mathbb{Q}}(-1)+\log(T)+O\left(\frac{\log T}{T}\right).$$

The claim follows by reordering the above terms. □

5.4. Corollary. *With the notations as in 5.3 we have for $T \gg 0$*

$$\int_{\mathcal{F}_{y\leq T}} \log\|\Delta(\tau)\|_{Pet}\frac{dxdy}{4\pi y^2} = 1-12\zeta'_{\mathbb{Q}}(-1)-\frac{1}{2}\log(4\pi T)+O\left(\frac{\log T}{T}\right).$$

6 The generalized arithmetic self-intersection of the line bundle of modular forms

6.1. Theorem. *Let $k\in\mathbb{N}, 12|k$ and denote by $\zeta_{\mathbb{Q}}(s)$ the Riemann zeta function. Then, the generalized arithmetic self-intersection number of the line bundle of modular forms $\overline{\mathcal{M}}_k(\Gamma(1))$ on $\mathcal{X}(1)$ equipped with the Petersson metric is given by*

$$\overline{\mathcal{M}}_k(\Gamma(1)).\overline{\mathcal{M}}_k(\Gamma(1)) = k^2\left(\frac{1}{2}\zeta_{\mathbb{Q}}(-1)+\zeta'_{\mathbb{Q}}(-1)\right).$$

Proof. Let l be the global section of $\mathcal{M}_{12}(\Gamma(1))$, which corresponds to the discriminant $\Delta\in M_{12}(\Gamma(1))$ and let m be the section of $\mathcal{M}_{12}(\Gamma(1))$, which corresponds to $j\cdot\Delta\in M_{12}(\Gamma(1))$. It is obvious that $(l.m)_{\mathrm{fin}}=0$. Therefore, we have for these sections

$$\overline{\mathcal{M}}_{12}(\Gamma(1)).\overline{\mathcal{M}}_{12}(\Gamma(1)) = \langle l.m\rangle_\infty.$$

We will use formula (12) of Lemma 3.9 for the calculation of $\langle l.m\rangle_\infty$. The set of points on $X(1)$, which are singular for the Petersson metric is given by the cusp $S_1=S_{i\infty}$ and the elliptic fixed points S_2, S_3, which corrspond to the $\Gamma(1)$-orbits of $i, (1+\sqrt{-3})/2 \in \mathfrak{h}$ (resp.); hence, $X(1)_\varepsilon$ is given by $X(1)_\varepsilon = X(1) \setminus \bigcup_{j=1}^{3} B_\varepsilon(S_j)$, where $B_\varepsilon(S_j)$ are ε-neighbourhoods as in 3.1. Since the cusp width s of the cusp $S_{i\infty}$ is one, the local expansion of the section m at $S_{i\infty}$ is given by

$$\|m(t)\|_{Pet} = |j(t) \cdot \Delta(t)| \cdot \left(\log|t|^2\right)^6 \ \ (t = e^{2\pi i \tau}).$$

Therefore, we have $\alpha_{\overline{\mathcal{M}}_{12},1} = 6$ and $\varphi_{\overline{\mathcal{M}}_{12},1} = |j(t) \cdot \Delta(t)|$ for this section. For the other two points we have $\alpha_{\overline{\mathcal{M}}_{12},2} = \alpha_{\overline{\mathcal{M}}_{12},3} = 0$. With the use of formula (12) we then get

$$\begin{aligned}
&\langle l.m \rangle_\infty = \\
&-(\log\|m\|_{Pet})[\operatorname{div}(l) - \sum_{j=1}^{3} \operatorname{ord}_{S_j}(l) \cdot S_j] + \sum_{j=1}^{3} \operatorname{ord}_{S_j}(l) \Big(\alpha_{\overline{\mathcal{M}}_{12},j} - \log(\varphi_{\overline{\mathcal{M}}_{12},j}(0)) \Big) - \\
&\lim_{\varepsilon\to 0} \left(\sum_{j=1}^{3} \operatorname{ord}_{S_j}(l) \cdot \alpha_{\overline{\mathcal{M}}_{12},j} \cdot \log(-\log\varepsilon^2) + \int_{X(1)_\varepsilon} \log\|l\|_{Pet} \cdot c_1(\overline{\mathcal{M}}_{12}) \right).
\end{aligned}$$

As $\operatorname{ord}_{S_1}(l) = 1$ and $\operatorname{ord}_{S_2}(l) = \operatorname{ord}_{S_3}(l) = 0$ this equation simplifies to

$$\begin{aligned}
\langle l.m \rangle_\infty &= 0 + 1 \cdot \Big(6 - \log(1) \Big) - \\
&\quad \lim_{\varepsilon\to 0} \left(1 \cdot 6 \cdot \log(-\log\varepsilon^2) + \int_{X(1)_\varepsilon} \log\|\triangle\|_{Pet} \cdot c_1(\overline{\mathcal{M}}_{12}) \right).
\end{aligned}$$

Hence, it remains to calculate the integral over $X(1)_\varepsilon$. Letting $\varepsilon = e^{-2\pi T}$ for large $T \in \mathbb{R}$ we then find

$$\langle l.m \rangle_\infty = 6 - \lim_{T\to\infty} \left(6\log(4\pi T) + \int_{\mathcal{F}_{y\leq T}} \log\|\Delta(\tau)\|_{Pet} \cdot \frac{12}{4\pi} \cdot \frac{dxdy}{y^2} \right). \tag{23}$$

Applying Corollary 5.4 to the integral in (23) we get

$$\begin{aligned}
\langle l.m \rangle_\infty &= 6 - \lim_{T\to\infty} \left(6\log(4\pi T) + \int_{\mathcal{F}_{y\leq T}} \log\|\Delta(\tau)\|_{Pet} \cdot \frac{12}{4\pi} \cdot \frac{dxdy}{y^2} \right) \\
&= 6 - \lim_{T\to\infty} \left(6\log(4\pi T) - 6\log(4\pi T) + 12 - 144\zeta'_{\mathbb{Q}}(-1) + O\left(\frac{\log T}{T}\right) \right) \\
&= 144 \cdot \left(\frac{1}{2}\zeta_{\mathbb{Q}}(-1) + \zeta'_{\mathbb{Q}}(-1) \right).
\end{aligned}$$

By the bilinearity of the generalized arithmetic intersection number we get the claimed formula for all $k \in \mathbb{N}$, $12|k$. □

6.2. Corollary. *Let Γ be a subgroup of finite index in $\Gamma(1)$, $\mathcal{X}(\Gamma)$ an arithmetic surface associated to Γ, $\pi_{\Gamma,\mathbb{Z}} : \mathcal{X}(\Gamma) \longrightarrow \mathcal{X}(1)$ the corresponding morphism and $d = \deg(\pi_{\Gamma,\mathbb{Z}})$ its degree (see 4.11). Furthermore, let $k \in \mathbb{N}$ be such that $\mathcal{M}_k(\Gamma)$ is defined (see 4.12 or*

4.14). Then, we have the following formula for the generalized arithmetic self-intersection number of the line bundle of modular forms $\overline{\mathcal{M}}_k(\Gamma)$ on $\mathcal{X}(\Gamma)$ equipped with the Petersson metric

$$\overline{\mathcal{M}}_k(\Gamma).\overline{\mathcal{M}}_k(\Gamma) = k^2 \cdot d \cdot \left(\frac{1}{2}\zeta_{\mathbb{Q}}(-1) + \zeta'_{\mathbb{Q}}(-1)\right).$$

Proof. By the bilinearity of the generalized arithmetic intersection product it suffices to prove the statement for $k \in \mathbb{N}, 12|k, k \gg 0$. We will choose the sections $l, m \in H^0(\mathcal{X}(1), \mathcal{M}_k(\Gamma(1)))$ such that their divisors $\mathrm{div}(l), \mathrm{div}(m) \subset \mathcal{X}(1)$ are disjoint and do not contain the singular points of $X(\Gamma)_{\mathbb{Z}}$ with respect to the factorization (22); such sections exist for $k \in \mathbb{N}$ being large enough. We now set $\tilde{l} := \pi^*_{\Gamma,\mathbb{Z}}(l)$ and $\tilde{m} := \pi^*_{\Gamma,\mathbb{Z}}(m)$. Note that by construction we have $\tilde{l}, \tilde{m} \in H^0(\mathcal{X}(\Gamma), \mathcal{M}_k(\Gamma))$; furthermore, these sections do not meet at any finite fibre, from which we derive

$$(\tilde{l}.\tilde{m})_{\mathrm{fin}} = 0.$$

The sections l and $\tilde{l}$ correspond to the same modular form because of the inclusion $M_k(\Gamma(1)) \subseteq M_k(\Gamma)$; analogeously this holds for m and $\tilde{m}$. The generalized arithmetic intersection number at the infinite places of $\tilde{l}$ and $\tilde{m}$ is therefore given by

$$\langle \tilde{l}.\tilde{m}\rangle_\infty = d \cdot \langle l.m\rangle_\infty.$$

The corollary now follows by the well-definedness of the generalized arithmetic intersection number and Theorem 6.1. □

6.3. Remarks. (a) We point out that the commutativity of the generalized arithmetic intersection number at the infinite places and the functional equation of the Riemann zeta function $\zeta_{\mathbb{Q}}(s)$ imply the modular Jensen's formula (cf. [Ro]); namely, for a given modular form $f \in M_k(\Gamma(1))$ with $f(i\infty) = 1$ the following formula holds

$$\begin{aligned}&\frac{3}{\pi}\int_{\mathcal{F}(\Gamma(1))} \log(|f(\tau)|(\mathrm{Im}\,\tau)^{k/2}) \cdot \frac{dx\,dy}{y^2} = \\ &-\sum_{\tau \in \mathcal{F}(\Gamma(1))} \mathrm{ord}_\tau(f) \cdot \log(|\Delta(\tau)|(\mathrm{Im}\,\tau)^6) - k\left(\frac{1}{2} + \log(2) - \gamma + \zeta'_{\mathbb{Q}}/\zeta_{\mathbb{Q}}(2)\right), \quad (24)\end{aligned}$$

where γ is Euler's constant. This formula provides an alternative proof of Theorem 6.1 (see [Kü2]).

(b) We point out that J.-B. Bost has obtained independently the same results by different methods. We thank him for providing us a manuscript containing his calculations.

(c) We recall that by Remark 4.10 the (geometric) degree of $\mathcal{M}_k(\Gamma(1))$ is given by $-k \cdot \zeta_{\mathbb{Q}}(-1)$. The main result of this section shows that the generalized arithmetic self-intersection number, i.e. the 'arithmetic degree', of $\overline{\mathcal{M}}_k(\Gamma(1))$ contains as the interesting part the derivative $\zeta_{\mathbb{Q}}(-1)$ of the Riemann zeta function at $s = -1$. Therefore, one may ask if this result generalizes to give rise to an arithmetic Klingen-Siegel formula.

The above result is compatible to the the general philosophy [So2] that arithmetic intersection numbers are derivatives of zeta functions; moreover it is remarkable that the generalized arithmetic self-intersection number in question is proportional to the first term of the Gillet-Soulé R-genus, although the arithmetic Riemann-Roch theorem can not be applied in this context.

7 Compatibility with the theory of L_1^2-Green's currents

In this last section we recall the theory of L_1^2-Green's currents, which was developped by J.-B. Bost (see [Bo], [Bo1]) and A. Moriwaki (see [Mo]), and show that our generalized arithmetic intersection numbers are compatible with this theory.

7.1. L_1^2-Green's currents. Let M be a Riemann surface and $U \subseteq M$ an open subset of M. The vector space $L_1^2(U)_{\mathrm{loc}}$ is generated by generalized complex valued funtions Φ such that Φ and $d\Phi$ are locally L^2 (i.e., square integrable). This space has the property that for any $\Phi \in L_1^2(U)_{\mathrm{loc}}$ we have $e^{\Phi} \in L^1(U)$ (see [Bo], p. 1). We note that the function $(z \mapsto \log\log|z|^{-1})$ is in $L_1^2(B_\varepsilon(0))_{\mathrm{loc}}$ where $B_\varepsilon(0) = \{z \in \mathbb{C}|\, |z| < \varepsilon < 1\}$. Let $D = \sum_j n_j P_j$ be a divisor on M. A *Green's current g_D for D* is an F_∞-invariant current satisfying the differential equation

$$dd^c g_D + \delta_D = \omega_D;$$

here δ_D is the Dirac current for D and ω_D is a smooth form. An *L_1^2-Green's current g_{D,L_1^2} for D* is the sum of a Green's current g_D for D and a generalized function $\Phi \in L_1^2(M)_{\mathrm{loc}}$, i.e., we have

$$g_{D,L_1^2} = g_D + \Phi. \tag{25}$$

It is clear that these definitions extend in the usual way to 1-dimensional complex manifolds, which need not be connected.

7.2. The extended arithmetic Chow group $\widehat{CH}^1(\mathcal{X}, L_1^2)$. Let $\mathcal{X}$ be an arithmetic surface, $\widehat{Z}^1(\mathcal{X}, L_1^2)$ be the set of pairs (D, g_{D,L_1^2}) consisting of a divisor D on $\mathcal{X}$ and a L_1^2-Green's current g_{D,L_1^2} for the divisor induced on $\mathcal{X}_\infty$ and $\widehat{R}^1(\mathcal{X})$ be the set of pairs $(\mathrm{div}(f), -\log|f|^2)$, where $f \in k(\mathcal{X})^*$ is a rational function. The *first extended arithmetic Chow group* $\widehat{CH}^1(\mathcal{X}, L_1^2)$ is defined by

$$\widehat{CH}^1(\mathcal{X}, L_1^2) = \widehat{Z}^1(\mathcal{X}, L_1^2)\Big/\widehat{R}^1(\mathcal{X}).$$

If $(D_1, g_{D_1,L_1^2}), (D_2, g_{D_2,L_1^2}) \in \widehat{Z}^1(\mathcal{X}, L_1^2)$ are such that the divisors on $\mathcal{X}_\infty$ induced by D_1, D_2 are disjoint, then their arithmetic intersection product is defined by

$$(D_1, g_{D_1,L_1^2}).(D_2, g_{D_2,L_1^2}) :=$$

$$(D_1,D_2)_{\mathrm{fin}}+\frac{1}{2}\int_{\mathcal{X}_\infty}\left(g_1\delta_2+g_2\omega_1+\Phi_1\omega_2+\Phi_2\omega_1+\frac{1}{2\pi i}\partial\Phi_1\bar{\partial}\Phi_2\right); \qquad (26)$$

here $(D_1,D_2)_{\mathrm{fin}}$ is as in formula (1), $g_{D_j,L_1^2}=g_j+\Phi_j$, where $g_j=g_{D_j}$ is a Green's current for D_j, $\delta_j=\delta_{D_j}$ and $\omega_j=dd^cg_j+\delta_j$ $(j=1,2)$. The arithmetic intersection product (26) induces a bilinear, symmetric pairing

$$\widehat{CH}^1(\mathcal{X},L_1^2)\times\widehat{CH}^1(\mathcal{X},L_1^2)\longrightarrow\mathbb{R}$$

(see [Bo1], p. 274).

7.3. Let now $\mathcal{L}$ be a line bundle on $\mathcal{X}$. An almost everywhere defined metric $\|\cdot\|$ on the holomorphic line bundle $\mathcal{L}_\infty$ on $\mathcal{X}_\infty$ induced by $\mathcal{L}$ is L_1^2-regular, if there exists a smooth metric $\|\cdot\|_0$ on $\mathcal{L}_\infty$ such that the equation

$$\|\cdot\|=e^{\Phi}\cdot\|\cdot\|_0$$

with $\Phi\in L_1^2(\mathcal{X}_\infty)_{\mathrm{loc}}$ holds (see [Bo], p. 1). We denote by $\overline{\mathcal{L}}=(\mathcal{L},\|\cdot\|_{L_1^2})$ the pair consisting of a line bundle $\mathcal{L}$ and an L_1^2-regular hermitian metric $\|\cdot\|_{L_1^2}$ on $\mathcal{L}_\infty$. Every such pair determines via

$$\widehat{c}_1(\overline{\mathcal{L}}):=[\mathrm{div}(l),-\log\|l\|^2_{L_1^2}] \qquad (27)$$

a well-defined class in $\widehat{CH}^1(\mathcal{X},L_1^2)$, which does not depend on the choice of the non-trivial, rational section l of $\mathcal{L}$ (see [Bo], p. 2). Hermitian, logarithmically singular metrics, as defined in 3.1, are L_1^2-regular, and by (27) we then obtain an inclusion

$$\widehat{\mathrm{Pic}}(\mathcal{X},\mathcal{S})\hookrightarrow\widehat{CH}^1(\mathcal{X},L_1^2).$$

7.4. Proposition. *The natural inclusion $\widehat{\mathrm{Pic}}(\mathcal{X},\mathcal{S})\hookrightarrow\widehat{CH}^1(\mathcal{X},L_1^2)$ is compatible with the generalized arithmetic intersection number (18) and the arithmetic intersection product (26).*

Proof. Let $\overline{\mathcal{L}}$ and $\overline{\mathcal{M}}$ be two logarithmically singular, hermitian line bundles on an arithmetic surface $\mathcal{X}$. We assume that $\mathcal{L}$ and $\mathcal{M}$ have non-trival global sections l and m (resp.). These induce global sections of $\mathcal{L}_\infty$ and $\mathcal{M}_\infty$, which we also denote by l and m (resp.). Furthermore, we assume that on $\mathcal{X}_\infty$ the divisor $\mathrm{div}(l)$ of l has no points in common with the divisor $\mathrm{div}(m)$ of m on $\mathcal{X}_\infty$. To perform the subsequent calculations we choose decompositions

$$-\log\|l\|^2=g_1+\Phi_1,\qquad -\log\|m\|^2=g_2+\Phi_2, \qquad (28)$$

where g_1 (resp. g_2) is a Green's current for $\mathrm{div}(l)$ (resp. $\mathrm{div}(m)$). We have to show that the two quantities

$$\overline{\mathcal{L}}.\overline{\mathcal{M}} \;=\; (l.m)_{\mathrm{fin}}+\langle l.m\rangle_\infty, \qquad (29)$$

$$\begin{aligned}\widehat{c}_1(\overline{\mathcal{L}}).\widehat{c}_1(\overline{\mathcal{M}}) &= (\mathrm{div}(l),\mathrm{div}(m))_{\mathrm{fin}} + \\ &\frac{1}{2}\int_{\mathcal{X}_\infty}\left(g_1\delta_2+g_2\omega_1+\Phi_1\omega_2+\Phi_2\omega_1+\frac{1}{2\pi i}\partial\Phi_1\bar\partial\Phi_2\right) \qquad (30)\end{aligned}$$

coincide. It suffices to compare the contributions at the infinite place as the intersection numbers at the finite places are the same. By means of (28) we get, using (9),

$$\begin{aligned}\langle l.m\rangle_\infty &= \frac{1}{2}\int_{\mathcal{X}_\infty}\Big((g_1+\Phi_1)\cdot(\omega_2+c_1(\Phi_2))+(g_2+\Phi_2)\cdot(\omega_1+c_1(\Phi_1))+ \\ &\qquad d(g_1+\Phi_1)\cdot d^c(g_2+\Phi_2)\Big)+\sum_{j=1}^r\left(\mathrm{ord}_{S_j}(l)\cdot\alpha_{\overline{\mathcal{M}},j}+\mathrm{ord}_{S_j}(m)\cdot\alpha_{\overline{\mathcal{L}},j}\right) \\ &= \frac{1}{2}\int_{\mathcal{X}_\infty}\Big(g_1\omega_2+g_2\omega_1+dg_1d^cg_2+\Phi_1\omega_2+\Phi_2\omega_1+\Phi_1c_1(\Phi_2)+\Phi_2c_1(\Phi_1)+ \\ &\qquad d\Phi_1d^c\Phi_2+g_1c_1(\Phi_2)+dg_1d^c\Phi_2+g_2c_1(\Phi_1)+d\Phi_1d^cg_2\Big)+ \\ &\qquad \sum_{j=1}^r\left(\mathrm{ord}_{S_j}(l)\cdot\alpha_{\overline{\mathcal{M}},j}+\mathrm{ord}_{S_j}(m)\cdot\alpha_{\overline{\mathcal{L}},j}\right) \\ &= \frac{1}{2}\int_{\mathcal{X}_\infty}\Big(g_1\omega_2+g_2\omega_1+dg_1d^cg_2+\Phi_1\omega_2+\Phi_2\omega_1+\frac{1}{2\pi i}\partial\Phi_1\bar\partial\Phi_2+ \\ &\qquad d(g_1d^c\Phi_2)+d(g_2d^c\Phi_1)\Big)+\sum_{j=1}^r\left(\mathrm{ord}_{S_j}(l)\cdot\alpha_{\overline{\mathcal{M}},j}+\mathrm{ord}_{S_j}(m)\cdot\alpha_{\overline{\mathcal{L}},j}\right);\end{aligned}$$

here we have used the identities

$$\begin{aligned}&\int_{\mathcal{X}_\infty}\Phi_2c_1(\Phi_1)=\int_{\mathcal{X}_\infty}\Phi_2dd^c\Phi_1=\int_{\mathcal{X}_\infty}\frac{1}{2\pi i}\partial\Phi_1\bar\partial\Phi_2= \\ &\int_{\mathcal{X}_\infty}\frac{1}{2\pi i}\partial\Phi_2\bar\partial\Phi_1=\int_{\mathcal{X}_\infty}\Phi_1dd^c\Phi_2=\int_{\mathcal{X}_\infty}\Phi_1c_1(\Phi_2)\end{aligned}$$

(see [Bo1], p. 254 and the definition of d^c). As in formula (16) we now have

$$\frac{1}{2}\int_{\mathcal{X}_\infty}d\left(g_1d^c\Phi_2\right)=-\sum_{j=1}^r\mathrm{ord}_{S_j}(l)\cdot\alpha_{\overline{\mathcal{M}},j};$$

analogeously we determine the integral over $d\left(g_2d^c\Phi_1\right)$. Finally, we get as in the proof of (6) in Proposition 2.4

$$\begin{aligned}\langle l.m\rangle_\infty &= \frac{1}{2}\int_{\mathcal{X}_\infty}\left(g_1\omega_2+g_2\omega_1+dg_1d^cg_2+\Phi_1\omega_2+\Phi_2\omega_1+\frac{1}{2\pi i}\partial\Phi_1\bar\partial\Phi_2\right) \\ &= \frac{1}{2}\left(g_1[\mathrm{div}(m)]+\int_{\mathcal{X}_\infty}g_2\omega_1+\int_{\mathcal{X}_\infty}\left(\Phi_1\omega_2+\Phi_2\omega_1+\frac{1}{2\pi i}\partial\Phi_1\bar\partial\Phi_2\right)\right) \\ &= \frac{1}{2}\int_{\mathcal{X}_\infty}\left(g_1\delta_2+g_2\omega_1+\Phi_1\omega_2+\Phi_2\omega_1+\frac{1}{2\pi i}\partial\Phi_1\bar\partial\Phi_2\right).\end{aligned}$$

The claim now follows. □

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