On Kudla’s Green function for signature (2,2)
Part II

By Rolf Berndt and Ulf Kühn

Abstract

Around 2000 Kudla presented conjectures about deep relations between arithmetic intersection theory, Eisenstein series and their derivatives, and special values of Rankin $L$–series. The aim of this text is to work out the details of an old unpublished draft on the second author’s attempt to prove these conjectures for the case of the product of two modular curves.

In part one we proved that the generating series of certain modified arithmetic special cycles is as predicted by Kudla’s conjectures a modular form with values in the first arithmetic Chow group. Here we pair this generating series with the square of the first arithmetic Chern class of the line bundle of modular forms. Up to previously known Faltings heights of Hecke correspondences only integrals of the Green functions $\Xi(m)$ over $X$ had to be computed. The resulting arithmetic intersection numbers turn out to be as predicted by Kudla to be strongly related to the Fourier coefficients of the derivative of the classical real analytic Eisenstein series $E_2(\tau, s)$.

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1 Introduction

As in the first part of our work we consider the natural models of product of modular curves $X = X(1) \times X(1)$ and its Hecke correspondence $T(N)$ over the integers $\mathbb{Z}$. We had introduced the modified arithmetic special cycles

$$\hat{Z}_\rho(m) := \langle T(m), \tilde{\Xi}_\rho(v, z, m) \rangle \in \widehat{CH}^1(X)$$

and a modified Kudla generating series

$$\hat{\phi}_{K, \rho} = \sum \hat{Z}_\rho(m)q^m. \tag{1.0.1}$$

We proved that $\hat{\phi}_{K, \rho}$ is a modular with coefficients in $\widehat{CH}^1(X)$.

Hence, for all linear maps $L : \widehat{CH}^1(X) \to \mathbb{R}$ the series $L(\hat{Z}_\rho(m))q^m$ is a (non-holomorphic) $\mathbb{R}$—valued modular form in the usual sense. Now we denote by $\hat{c}_1(\mathcal{L})$ the first arithmetic Chern class of the line bundle of modular forms $\mathcal{L}(12, 12)$ of bi-weight $(12, 12)$ equipped with the Petersson metric. Then we choose the linear map $L(-) = \hat{c}_1(\mathcal{L})^2(-)$ and prove the following result.

**Main Theorem** (modified Kudla conjecture). We have an identity of modular forms

$$\hat{c}_1(\mathcal{L})^2 \cdot \hat{\phi}_{K, \rho} = E_2'(\tau, 1) + f_\rho(\tau) \tag{1.0.2}$$

with the derivative of a non-holomorphic Eisenstein series $E_2(\tau, s)$ with respect to $s \in \mathbb{C}$ and a certain modular form $f_\rho(\tau)$ and.

This result complements the work of Kudla, Rapoport and Yang in the $O(1, 2)$ case and provides a first confirmation of Kudla’s conjecture in dimension 2.

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Note this is an arithmetic cycle in the arithmetic Chow group with loglog-growth in the sense of

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1 version May 30, 2012

2 Note this is an arithmetic cycle in the arithmetic Chow group with loglog-growth in the sense of
The above Eisenstein series equals
\[ E_2(\tau, s) := -12\psi(s)E_2(\tau, s), \]
where \( \psi \) is a meromorphic function with
\[ \psi(s) = -1 + 4\left( \frac{\zeta'(1)}{\zeta(1)} + \frac{1}{2} \right)(s - 1) + O((s - 1)^2) \]
and the expansion of \( E_2(\tau, s) = (1/(2\pi i))\partial_\tau E^*(\tau, s) \) (c.f. \( E_{2.0.2} \)) is determined as follows.

**Theorem.** The coefficients of Fourier expansion
\[ E_2(\tau, 1) = \sum_{m \in \mathbb{Z}} a(v, 1, m)q^m \]
and those of the the derivative of \( E_2(\tau, s) \) with respect to \( s \)
\[ E'_2(\tau, 1) = \sum_{m \in \mathbb{Z}} a'(v, 1, m)q^m \]
are given by
\[ a(v, 1, m) = \begin{cases} \sigma(m) = \sum_{d|m} d & \text{for } m > 0 \\ 0 & \text{for } m < 0 \end{cases} \]
and by
\[ a'(v, 1, m) = \begin{cases} \sigma(m)(1/(4\pi mv) + \sigma'(m)/\sigma/m)) & \text{for } m > 0 \\ \sigma(m)(\text{Ei}(-4\pi|mv|) + 1/(4\pi|mv|)e^{-4\pi|mv|}) & \text{for } m < 0 \end{cases} \]
where with
\[ \sigma^*_s(m) := |m|^s \sum_{d|m} d^{-2s} \]
we abbreviate \( \sigma'(m)/\sigma(m) := \sigma^*_s(m)/\sigma^*_s(m) \).

We also will calculate the constant terms \( a(v, 1, 0) \) and \( a'(v, 1, 0) \) below (c.f. Theorem \( E_{2.1.2} \)) although we won’t need them for this work.

**1.1. Remark.** The Fourier expansion \( E_2(\tau, s) = \sum a(v, s, m)q^m \) goes through the multiplication by \( \psi \) to \( E_2(\tau, s) = \sum A(v, s, m)q^m \), where the
first terms of the Taylor expansion at $s = 1$ of the Fourier coefficients are given by

$$A(v, 1, m) = 12a(v, 1, m),$$

$$A'(v, 1, m) = -48\left(\frac{\zeta'(-1)}{\zeta(-1)} + \frac{1}{2}\right)a(v, 1, m) + 12a'(v, 1, m).$$

The main steps in the proof of our main theorem are calculating and comparing both sides of termwise. Arakelov theory ([BKK] Proposition 7.56) gives us for $m \neq 0$ the relation

$$\hat{c}_1(\mathcal{L})^2 \cdot \hat{Z}_\rho(m) = ht_{\mathcal{T}}(T(m)) + \int_X \tilde{\Xi}_\rho(v, z, m)c_1(\mathcal{L})^2.$$

Observe that $c_1(\mathcal{L})$ is proportional to the hyperbolic measure and, as evaluated later (Remark 5.2), one has for the volume element

$$c_1(\mathcal{L})^2 = \left(\frac{18}{\pi^2}\right)d\mu(z) = \left(\frac{18}{\pi^2}\right)\frac{dx_1dy_1}{y_1^2} \frac{dx_2dy_2}{y_2^2}.$$

Now from Theorem 7.61 in [BKK] p.81 we already know

**Proposition.** The Faltings height of $T(m)$ is given by

$$ht_{\mathcal{T}}(T(m)) = \begin{cases} 0 & \text{if } m < 0 \\ 24^2((\sigma(m)((1/2)\zeta(-1) + \zeta'(-1)) \\ + \sum_{d|m}(\frac{d\log d}{24} - \frac{\sigma(m)\log m}{48})) & \text{if } m > 0. \end{cases}$$

Therefore we need only to study the integrals

$$\int_X \tilde{\Xi}_\rho(v, z, m)c_1(\mathcal{L})^2 = \int_X \Xi(v, z, m)c_1(\mathcal{L})^2 + \int_X \rho(z)\tilde{\Xi}(v, z, m)c_1(\mathcal{L})^2.$$

For the integrals $\int_X \rho(z)\tilde{\Xi}(v, z, m)c_1(\mathcal{L})^2$ we first recall from Proposition 4.3 in Part I that by adding an appropriate zeroth coefficient the $q$-series

$$\tilde{\Xi}^+(\tau, z) = \tilde{\Xi}^+(v, z, 0) + \sum_{m \neq 0} \tilde{\Xi}(v, z, m)q^m$$

is a modular form with respect to $\text{SL}(2, \mathbb{Z})$. Thus, the existence of those integrals implies our first result:
Theorem A. There exists a non-holomorphic modular form $f_\rho(\tau)$ of weight 2 for $\text{SL}_2(\mathbb{Z})$ such that

$$f_\rho(\tau) = \int_X \rho(z) \tilde{\Xi}^+(\tau, z)c_1(\mathcal{L})^2.$$ 

1.2. Remark. We observe that the existence of the integral of $\tilde{\Xi}_\rho$ is guaranteed by Arakelov theory. Then the existence of the integral of $\int_X \rho\tilde{\Xi}c_1(\mathcal{L})^2$ implies the existence of the integral of the Kudla Green function $\Xi$.

Using $O(2,2)$-theory we are able to calculate the remaining integrals (see Theorem 4.2).

Theorem B. We have

$$\int_X \Xi(v, z, m)c_1(\mathcal{L})^2$$

(1.2.1) $\begin{cases} 12\sigma_1(m)(1/(4\pi mv)) & \text{for } m > 0 \\ 12\sigma_1(m)((1/(2\pi|m|v))e^{-4\pi|m|v} + \text{Ei}(-4\pi|m|v)) & \text{for } m < 0. \end{cases}$

It is now a pleasant exercise to relate these arithmetic intersection numbers for $m \neq 0$ to the Fourier coefficients of the Eisenstein series as in our Main Theorem (see Theorem 2.2). Now, since we already know that the right hand side of our main theorem is a non-holomorphic modular form for $\text{Sl}_2(\mathbb{Z})$ of weight 2, the remaining arithmetic intersection number for $m = 0$ must equal the coefficients of the modular form of the right hand side.

As already stated in Part I, our treatment of this topic owes a lot to discussions with J.Bruinier, J. Funke and S. Kudla. And this time we even got some local help by hints from H. Brückner and J. Michaliček. We thank them all.

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\footnote{We had spent much effort to calculate this identity directly, but we had not been able to do so and would be thankful for any helpful hints. Some traces of our efforts will be shown in the Appendix.}
2 Eisenstein series and its derivatives

We take over classical material from Zagier’s article [Za] p.32f. For \( \tau = u + iv \in \mathbb{H} \) and \( s \in \mathbb{C} \) with \( \text{Re} \ s > 1 \) one has the analytic Eisenstein series

\[
E(\tau, s) := (1/2) \sum'_{c,d} \frac{v^s}{c\tau + d}^{|2s} = v^s \zeta(2s) + v^s \sum_{c \in \mathbb{N}} \sum_{d \in \mathbb{Z}} \left| c\tau + d \right|^{-2s}
\]

resp. in normalized version

\[
E^*(\tau, s) := \pi^{-s} \Gamma(s) E(\tau, s).
\]

With

\[
\zeta^*(s) := \pi^{-s/2} \Gamma(s/2) \zeta(s)
\]

Zagier states the following Fourier development

\[
E^*(\tau, s) = v^s \zeta^*(2s) + v^{1-s} \zeta^*(2s - 1) + 2v^{1/2} \sum_{n \in \mathbb{Z}, n \neq 0} \sigma_{s-(1/2)}(\left| n \right|) K_{s-(1/2)}(2\pi |n| v) e^{2\pi i nu}
\]

where

\[
\sigma_{\nu}(n) := |n|^{\nu} \sum_{d|n} d^{-2\nu} = \sigma_{-\nu}(n)
\]

is an entire function in \( \nu \) and the \( K \)-Bessel function

\[
K_{\nu}(t) := \int_0^\infty e^{-t \cosh u} \cosh(\nu u) du = K_{-\nu}(t)
\]

is entire in \( \nu \) and exponentially small in \( t \) as \( t \to \infty \). For the sake of completeness we rededuce this Fourier development at the end in an appendix.

We introduce

2.1. Definition.

\[
E_2(\tau, s) := (1/(2\pi i)) \partial_{\tau} E^*(\tau, s) = (-1/(4\pi)) (\partial_v + i\partial_u) E^*(\tau, s)
\]

and want to study its Taylor expansion at \( s = 1 \). More precisely, we slightly extend the Theorem in the Introduction.
2.2. Theorem. We have

\[ E_2(\tau, 1) = \sum_{m \in \mathbb{Z}} a(v, 1, m)q^m \]  

and, denoting by \( E'_2(\tau, s) \) the derivative of \( E_2(\tau, s) \) with respect to \( s \), we get

\[ E'_2(\tau, 1) = \sum_{m \in \mathbb{Z}} a'(v, 1, m)q^m \]

with

\[ a(v, 1, m) = \begin{cases} 
\sigma(m) = \sum_{d|m} d & \text{for } m > 0 \\
-1/24 + 1/(8\pi v) & \text{for } m = 0 \\
0 & \text{for } m < 0
\end{cases} \]

\[ a'(v, 1, m) = \begin{cases} 
\sigma(m)(1/(4\pi mv) + \sigma'(m)/\sigma(m)) & \text{for } m > 0 \\
-(1/24)(24\zeta'(-1) - \gamma - 1 + \log(4\pi v)) & \text{for } m = 0 \\
\sigma(m)(\text{Ei}(-4\pi|m|v) + 1/(4\pi|m|v)e^{-4\pi|m|v}) & \text{for } m < 0.
\end{cases} \]

Proof. We have (see for instance Iwaniec [Iw] p.205)

\[ K_{\nu}(t) := \int_0^\infty e^{-t \cosh u} \cosh(\nu u)du = \frac{\sqrt{\pi}(t/2)^\nu}{\Gamma(\nu + (1/2))} \int_1^\infty e^{-tr}(r^2 - 1)^{\nu-(1/2)}dr. \]

Hence, from \((2.0.3)\) we get

\[ E^*(\tau, s) = v^s\zeta^*(2s) + v^{1-s}\zeta^*(2s - 1) \]

\[ + \sum_{m \in \mathbb{Z}, m \neq 0} 2\sigma_{s-1/2}(|m|) \left( \frac{v}{\Gamma(s)\sqrt{|m|}} \right)^s \int_1^\infty e^{-2\pi|m|\nu} (r^2 - 1)^{s-1}dr e^{2\pi im\nu}. \]

We abbreviate

\[ c_0(v, s) := v^s\zeta^*(2s) + v^{1-s}\zeta^*(2s - 1), \]

\[ \mathcal{E}_0(v, s) := \partial_v c_0(v, s) = sv^{s-1}\zeta^*(2s) + (1 - s)v^{-s}\zeta^*(2s - 1), \]
and, for $m \neq 0$,

$$
c_m(v, s) := 2\sigma^*_s(1/2)(|m|) \left( \frac{(v|m|\pi)^s}{\Gamma(s)\sqrt{|m|}} \right),
$$

$$
I_m(v, s) := \int_1^\infty e^{-2\pi|m|vr}(r^2 - 1)^{s-1}dr,
$$

$$
J_m(v, s) := \int_1^\infty e^{-2\pi|m|vr}(r^2 - 1)^{s-1}rdr,
$$

and get

$$
E_2(\tau, s) = -(1/(4\pi))(\tilde{c}_0(v, s) + \sum_m (s/v - 2\pi m)c_m(v, s)I_m(v, s)
- 2\pi|m|c_m(v, s)J_m(v, s))e(mu)
$$

i.e.

$$
E_2(\tau, s) = -\frac{1}{4\pi}(\tilde{c}_0(v, s)
+ \sum_{m>0} ((s/v)c_m(v, s)I_m(v, s) - 2\pi|m|c_m(v, s)(I_m(v, s) + J_m(v, s)))e(mu)
$$

$$
+ \sum_{m<0} ((s/v)c_m(v, s)I_m(v, s) + 2\pi|m|c_m(v, s)(I_m(v, s) - J_m(v, s)))e(mu) )
$$

Using Maple, one can determine from here the first two terms of the Taylor expansion of each coefficient of $e(mu)$ and hence get the claims in the Theorem. For those who don’t like Maple, we give a direct proof in an appendix. \(\square\)

### 2.3. Remark

The sigmas in this calculations are those from the paper by Zagier

$$
\sigma_s^*(n) := |n|^s \sum_{d|n, d>0} d^{-2s} = \sigma^*_s(n).
$$

Hence one has

$$
\sqrt{m} \sigma_{1/2}^*(m) = \sum_{d|m} d = \sigma(m).
$$
We set

\[\sigma'(m)/\sigma(m) := \sigma^{*'}(m)_{1/2}/\sigma_{1/2}^{*}(m)\]

\[= (\sigma(m) \log m - 2 \sum_{d|m} d \log d)/\sigma(m).\]  

As an immediate consequence to our Theorem, for the coefficients of the modified Eisenstein series \(E'_2(\tau, 1)\), we get

**2.4. Corollary.** One has

\[A'(v, 1, m) = -12 \begin{cases} \sigma(m)(4(\zeta'(-1)/\zeta(-1) + 1/2) - 1/(4\pi mv) \\
+\sigma^{*'}_{1/2}(m)/\sigma_{1/2}^{*}(m)) \quad \text{for } m > 0 \\
3\zeta'(-1) - (1/8) + (\gamma/24) + (1/24)\log(4\pi v) \\
+ (1/8\pi v)(-48\zeta'(-1) - \gamma + 2 + \log(4\pi v)) \quad \text{for } m = 0 \\
\quad \sigma(|m|)(\text{Ei}(-4\pi|m|v) + 1/(4\pi|m|v)e^{-4\pi|m|v}) \quad \text{for } m < 0 \end{cases}\]

**3 Boundary function integral**

In Section 2 in Part I we introduced a partition of the unity \(\rho\) with respect to the boundary \(D\) and the boundary function \(\tilde{\Xi}^+(\tau, z)\)

\[\tilde{\Xi}^+(\tau, z) = \sum_m \tilde{\Xi}(v, z, m)q^m - (1/2v)t(s + 1/s)\]

\[\tilde{\Xi}(v, z, m) = (1/2) \sum_{-bc=m} \xi(v, z; b, c)\]

with \(t = \sqrt{y_1y_2}, s = \sqrt{y_1/y_2}\) (unfortunately we here have the same letter as the one denoting the variable in the zeta and Eisenstein series but the kind reader will know to make the difference) and

\[\xi(v, z, ; b, c) = (t/\sqrt{v}) (B(v, s; b, c) - I(v, s; b, c))\]

\[B(v, s; b, c) = \int_1^\infty e^{-\pi v(b/s+cs)^2} r^{-3/2} dr\]

\[I(v, s; b, c) = \begin{cases} 4\pi\sqrt{v} \ min(|bs^{-1}|, |cs|)) \quad \text{if } -bc > 0 \\
0 \quad \text{if } -bc \leq 0. \end{cases}\]
And in Proposition 4.3 of Part I we proved the modularity of $\hat{\Xi}^+(\tau, z)$ as a function in $\tau$. From there we come to the following result:

3.1. Theorem. There exists a modular form $f_\rho$ such that

$$f_\rho(\tau) = \int_X \rho(z) \hat{\Xi}^+(\tau, z))d\mu.$$  \hspace{1cm} (3.1.1)

Proof. As we have modularity in $\tau$ of $\hat{\Xi}^+$ and since the integral does not affect the $\tau-$variable, the modularity follows as soon as we checked the existence of the integrals. For $m = 0$ this is evaluated in the Proposition below and for $m \neq 0$ that follows from the three lemmata below as in these we have integrals of type

$$\int_X tF(s)d\mu = \int_X tF(s)x_1x_2dsdt/st$$

and, as the integrand does not depend on $x_1, x_2$, once the $s$-integration is done, one has a finite value as

$$\int_{t>0} dt/t^2 < \infty.$$  

□

3.2. Lemma. For $m = -bc < 0$ and

$$\tau(m) := \#\{d > 0 : d \mid |m|\}$$

one has

$$\sum_{-bc=m} \int_0^\infty B(v, s; b, c)ds/s \leq 2\tau(m)(1/(2\sqrt{|m|v})e^{-4\pi|m|v} + 2\pi \sqrt{|m|v} Ei(-4\pi |m|v))$$

Proof. Replacing $s$ by $s\sqrt{c/b}$ we get

$$\int_0^\infty B(v, s; b, c)ds/s = \int_0^\infty \int_1^\infty e^{-\pi v(b/s+c)^2}dr/r^{3/2}ds/s$$

$$= \int_0^\infty \int_1^\infty e^{-\pi v|m|((1/s)^2+s^2)^2}dr/r^{3/2}ds/s$$
and with $s = e^\varphi$ and $\cosh \varphi = 1 + \varphi^2/2 + \ldots$ we estimate

$$\int_0^\infty B(v, s; b, c) ds/s = \int_1^\infty \int_{-\infty}^\infty e^{-2\pi v|m|\varphi} \cosh 2\varphi e^{-2\pi |m|vr} dr/r^{3/2}$$

$$\leq \int_1^\infty \int_{-\infty}^\infty e^{-4\pi |m|r\varphi^2} d\varphi e^{-4\pi |m|vr} dr/r^{3/2}$$

$$= 1/(2\sqrt{|m|v}) \int_1^\infty e^{-4\pi |m|vr} dr/r^2$$

$$= 1/(2\sqrt{|m|v}) e^{-4\pi |m|v} + 2\pi \sqrt{|m|v} \text{Ei}(-4\pi |m|v)).$$

□

3.3. Lemma. For $m = -bc > 0$ one has

$$\sum_{-bc = m} \int_0^\infty B(v, s; b, c) ds/s \leq 2\tau(m)(1/(2\sqrt{|m|v}))$$

Proof. Replacing $b$ by $-b$ one has $\sum_{-bc = m} = 2\sum_{b, c > 0, bc = m}$ and again $s$ by $s\sqrt{c/b}$ we get this time

$$\int_0^\infty B(v, s; b, c) ds/s = \int_0^\infty \int_1^\infty e^{-\pi v(b/s + cs)^2} dr/r^{3/2} ds/s$$

$$= \int_0^\infty \int_1^\infty e^{-\pi v((1/s)^2 + s^2) - 2} dr/r^{3/2} ds/s$$

and with $s = e^\varphi$

$$\int_0^\infty B(v, s; b, c) ds/s = \int_1^\infty \int_{-\infty}^\infty e^{-2\pi v|m|s|cosh 2\varphi} e^{2\pi mv\varphi} dr/r^{3/2}$$

$$\leq \int_1^\infty \int_{-\infty}^\infty e^{-4\pi |m|r\varphi^2} d\varphi dr/r^{3/2}$$

$$= 1/(2\sqrt{mv}) \int_1^\infty dr/r^2$$

$$= 1/(2\sqrt{mv}).$$

□

3.4. Lemma. For $m = -bc > 0$ one has

$$\sum_{-bc = m} \int_0^\infty \min(|b/s|, |cs|) ds/s = 4\tau(m)\sqrt{m}$$
Proof. Replacing $s$ by $s\sqrt{|c|/|b|}$ we get
\[
\sum_{-bc=m} \int_0^\infty \min(|b/s|, |cs|) ds/s = \sum_{-bc=m} \sqrt{m} \int_0^\infty \min(1/s, s) ds/s
\]
\[
= 2\sqrt{m} \tau(m) \left( \int_0^1 ds/s + \int_1^\infty ds/s^2 \right)
\]
\[
= 2\sqrt{m} \tau(m) \cdot 2.
\]
\[\square\]

3.5. Proposition. For $m = bc = 0$ one has
\[
\int_X \hat{\Xi}^+(v, z, 0) d\mu < \infty
\]
Proof. From the Remark 2.23 from Part I we get
\[
2 \cdot \hat{\Xi}(v, z, 0) = (t/\sqrt{v}) \left( \sum_{b \neq 0} B(v, s; b, 0) + \sum_{c \neq 0} B(v, s; 0, c) + B(v, s; 0, 0) \right)
\]
\[
= -2t/\sqrt{v} + t(s + 1/s)(1/v + (2/\pi)\zeta(2))
\]
\[
- (2t/\pi)((1/s) \sum_{b \in \mathbb{N}} e^{-\pi s^2 b^2/v} / b^2 + s \sum_{c \in \mathbb{N}} e^{-\pi c^2/(s^2 v)}/c^2)
\]
(3.5.1)
and from (4.2.6) of Part I
\[
2 \cdot \hat{\Xi}^+(v, z, 0) = 2 \cdot \hat{\Xi}(v, z, 0) - (1/v) t(s + 1/s)
\]
\[
= -2t/\sqrt{v} + t(s + 1/s)(2/\pi)\zeta(2))
\]
\[
- (2t/\pi)((1/s) \sum_{b \in \mathbb{N}} e^{-\pi s^2 b^2/v} / b^2 + s \sum_{c \in \mathbb{N}} e^{-\pi c^2/(s^2 v)}/c^2).
\]
(3.5.2)

Step 1. We start by integrating
\[
I' = \int_{K_1 < y_1, K_2 < y_2 < T} t dy_1 dy_2 / (y_2^2 y_1^2)
\]
\[
= \int_{K_2} \int_{K_1}^\infty dy_1 / y_1^{3/2} dy_2 / y_2^{3/2} = (4/\sqrt{K_1})(1/\sqrt{K_2} - 1/\sqrt{T}).
\]
and
\[
I_0 = \int_{K_1 < y_1, K_2 < y_2 < T} y_2 dy_1 dy_2 / (y_2^2 y_1^2)
\]
\[
= \int_{K_2} \int_{K_1}^\infty dy_1 / y_1^2 dy_2 / y_2
\]
\[
= \int_{K_2}^T (1/K_1) dy_2 / dy_2 = (1/K_1)(\log T - \log K_1).
\]
Step 2. For $b \neq 0$ we look at

\[
I_b := \left( \frac{1}{b^2} \right) \int_{K_2}^{T} y_2 \int_{K_1}^{\infty} e^{-\left( \frac{\pi}{v} \right) b^2 y_1/y_2} dy_1 dy_2 / (y_1 y_2)^2
\]

\[
= \left( \frac{1}{b^2} \right) \int_{K_2}^{T} \left( \left[ e^{-\left( \frac{\pi}{v} \right) b^2 y_1/y_2} \left( -1/y_1 \right) \right] \right)_{K_1}^{\infty}
\]

\[
- \int_{K_1}^{\infty} \left( \frac{\pi b^2 / y_2 v}{y_2} \right) e^{-\left( \frac{\pi}{v} \right) b^2 y_1/y_2} dy_1 dy_2 / y_2
\]

\[
= \left( \frac{1}{b^2} \right) \int_{K_2}^{T} \left( \left[ e^{-\left( \frac{\pi}{v} \right) b^2 K_1 / y_2} \left( 1/K_1 \right) \right] \right) - \int_{1}^{\infty} \left( \frac{\pi b^2 / (y_2 v)}{y_2} \right) e^{-\left( \frac{\pi}{v} \right) b^2 K_1 y_1/y_2} dy_1 dy_2 / y_2.
\]

We remind

\[
-\text{Ei} \left( -x \right) = \int_{1}^{\infty} e^{-xt} dt / t = -\gamma - \log |x| + x - x^2 / (2 \cdot 2!) + \ldots
\]

and have for $T \to \infty$ in the first term

\[
I_{b,1} = \left( \frac{1}{b^2 K_1} \right) \int_{K_2}^{T} e^{-\left( \frac{\pi}{v} \right) b^2 K_1 / y_2} dy_2 / y_2
\]

\[
= \left( \frac{1}{b^2 K_1} \right) \int_{1/T}^{1/K_2} e^{-\left( \frac{\pi}{v} \right) b^2 K_1 u} du / u
\]

\[
= \left( \frac{1}{b^2 K_1} \right) \int_{1}^{T/K_2} e^{-\left( \frac{\pi}{v} \right) b^2 K_1 u / T} du / u
\]

\[
\to \left( \frac{1}{b^2 K_1} \right) \int_{1}^{\infty} e^{-\left( \frac{\pi}{v} \right) b^2 K_u / T} du / u
\]

\[
= \left( \frac{1}{b^2 K_1} \right) \left( -\gamma - \log x + x + \ldots \right)
\]

where $x = \pi (b^2 / v) (K_1 / T)$. And for the second term in $I_b$

\[
I_{b,2} := \left( \frac{\pi}{v} \right) \int_{K_2}^{T} \left( \int_{1}^{\infty} e^{-\left( \frac{\pi}{v} \right) b^2 K_1 y_1/y_2} dy_1 / y_1 \right) dy_2 / y_2^2
\]

\[
= -\left( \frac{\pi}{v} \right) \int_{K_2}^{T} \text{Ei} \left( -\left( \frac{\pi}{v} \right) b^2 K_1 / y_2 \right) dy_2 / y_2^2
\]

\[
= -\left( \frac{\pi}{v} \right) \int_{1/K_2}^{1} \text{Ei} \left( -\left( \frac{\pi}{v} \right) b^2 K_1 u \right) du
\]
with \( \alpha = \pi b^2 K_1 / v \) we have

\[
I_{b,2} = (\pi / v) \int_{K_2}^{T} \int_{K_1}^{\infty} e^{-\alpha y_1 / y_2} dy_1 / y_1 dy_2 / y_2^2 \\
= (\pi / v) \int_{1}^{\infty} \int_{1 / T}^{1 / K_2} [e^{-\alpha y_1 v} dy_1 / y_1 \\
= (\pi / (v \alpha)) \int_{1}^{\infty} [e^{-\alpha y_1 / T} - e^{-\alpha y_1 / K_2}] dy_1 / y_1^2 \\
= (1 / (b^2 K_1)) \int_{1}^{\infty} [e^{-\alpha y_1 / T} - e^{-\alpha y_1 / K_2}] dy_1 / y_1 \\
= (1 / (b^2 K_1)) (e^{-\alpha / T} - e^{-\alpha / K_2}) \\
- (\pi / v) \int_{1}^{\infty} [e^{-\alpha y_1 / T} - e^{-\alpha y_1 / K_2}] dy_1 / y_1 \\
= (1 / (b^2 K_1)) (e^{-\alpha / T} - e^{-\alpha / K_2}) \\
- (\pi / (v K_2)) \text{Ei}(-\pi (b^2 / v) K_1 / K_2) + (\pi / (v T)) \text{Ei}(-\pi (b^2 / v) K_1 / T)
\]

i.e. something finite for \( T \to \infty \) as the first terms are harmless and for the last one one has

\[
(\pi / (v T)) \text{Ei}(-\pi (b^2 / v) K_1 / T) = (\pi / (v T)) (-\gamma - \log(-\pi (b^2 / v) K_1 / T) \\
+ (-\pi (b^2 / v) K_1 / T) + \ldots
\]

with \((1 / T) \log T \to 0\).

**Step 3.** We remark that for \( T \to \infty \) \( I_0 / b^2 \) and \( I_b \) have the same singularity, namely \( 1 / (b^2 K_1) \log T \).

**Step 4.** The same way, we have the the same singularity coming from

\[
I_0' = \int_{K_2 < y_2} \int_{K_1 < y_1 < T} y_1 dy_1 dy_2 / (y_2^2 y_1^2) \\
= \int_{K_1}^{T} \int_{K_2}^{\infty} dy_2 / y_2^2 dy_1 / y_1 \\
= \int_{K_1}^{T} (1 / K_2) dy_1 / y_1 = (1 / K_2)(\log T - \log K_2).
\]
and
\[
I_c := (1/c^2) \int_{K_1}^{T} y_1 \int_{K_2}^{\infty} e^{-(\pi/v)c^2 y_2/y_1} dy_1 dy_2 / (y_1 y_2)^2 \\
= (1/c^2) \int_{K_1}^{T} \left[ e^{-(\pi/v)c^2 y_2/y_1} (-1/y_1) \right]_{K_2}^{\infty} \\
- \int_{K_2}^{\infty} (\pi c^2 / (y_1 v)) e^{-(\pi/v)c^2 y_2/y_1} dy_2 / y_1 dy_1 / y_1 \\
= (1/c^2) \int_{K_1}^{T} \left[ e^{-(\pi/v)c^2 K_2/y_1} (1/K_2) \right] \\
- \int_{1}^{\infty} (\pi c^2 / (y_1 v)) e^{-(\pi/v)c^2 K_2 y_2/y_1} dy_2 / y_2 dy_1 / y_1.
\]
Hence all together adds up to something finite. □

4 Kudla’s Green function integral for \( m \neq 0 \)

At first we remark that, as explained at the end of the Introduction, it follows immediatly from Theorem \[\text{ThGreenint}\] that for \( m \neq 0 \) the integrals in question exist.

We look at the Green function integral

**4.1. Definition.**
\[
I_m := \int_{\Gamma \setminus \delta \times \Gamma \setminus \delta} \Xi(v, z, m) d\mu(z) \\
= \int_{\Gamma \setminus \delta \times \Gamma \setminus \delta} (1/2) \sum_{M \in L_m} \xi(v, z, m) d\mu(z)
\]
and want to prove the following

**4.2. Theorem.** With \( \sigma(m) := \sum_{d|m} d \) one has
\[
I_m = \begin{cases} 
\sigma(|m|)(\pi/3)(1/(2v|m|))(e^{-4\pi v|m|} + 4\pi v|m|\text{Ei}(-4\pi|m|v)) & \text{for } m < 0 \\
\sigma(m)(\pi/3)(1/(2vm)) & \text{for } m > 0.
\end{cases}
\]
Proof. At first we assemble some tools. For $m \in \mathbb{N}$ and
\[
\Gamma = \text{SL}(2, \mathbb{Z}),
\]
\[
L_m = \{ M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{Z}); \det M = m \},
\]
\[
L_m^* = \{ M \in L_m; M \text{ primitive} \},
\]
one has the standard facts (see for instance Ogg’s book [Og] p.II-7 and IV-4)
\[
L_m = \bigcup_{ad = m, d > 0, a \mid d} \Gamma \begin{pmatrix} a \\ d \end{pmatrix} \Gamma
\]
\[
= \bigcup_{ad = m, d > 0, b \mod d} \Gamma \begin{pmatrix} a & b \\ d & \end{pmatrix}
\]
and
\[
L_m^* = \Gamma \begin{pmatrix} m \\ 1 \end{pmatrix} \Gamma = \sqcup_\alpha \Gamma \alpha
\]
with
\[
\alpha = \begin{pmatrix} a & b \\ d & \end{pmatrix}, \quad ad = m, d > 0, 0 \leq b < d, (a, b, d) = 1.
\]
One has
\[
[L_m^* : \Gamma] = m \prod_{p \mid m} (1 + (1/p)) =: \psi(m)
\]
and also
\[
[\Gamma : \Gamma_0(m)] = \psi(m).
\]
Moreover, we have
\[
[L_m : \Gamma] = \sum_{d \mid m} d = \sigma(m)
\]
and hence the formula (which is easily verified using the multiplicativity of \(\sigma\), see for instance Rankin [Ran] p. 285)
\[
\sigma(m) = \sum_{n^2 \mid m} \psi(m/n^2).
\]
\(\Gamma\) acts transitively by right multiplication on \(\Gamma \backslash L_m^*\) with isotropy group \(\Gamma_0(m)\) at the coset \(\Gamma \begin{pmatrix} m \\ 1 \end{pmatrix}\) (for example as in Knapp [Kn] p.256, Proposition 9.3). Hence, one has a bijection
\[
L_m^* \simeq \Gamma \begin{pmatrix} m \\ 1 \end{pmatrix} (\Gamma_0(m) \backslash \Gamma).
\]
It also is a standard fact that one has
\[
\text{vol}(\Gamma \backslash \mathbb{H}) = \int_{\Gamma \backslash \mathbb{H}} \frac{dxdy}{y^2} = \pi/3.
\] (4.2.6)

and for \(m > 0\) (e.g. [FB] p.375)
\[
\text{vol}(\Gamma_0(m) \backslash \mathbb{H}) = \psi(m)\pi/3.
\] (4.2.7)

After the preparation of these tools, we come to calculate the Green function integral
\[
I_m := \int_{\Gamma_0 \times \Gamma' \backslash \mathbb{H}} \Xi(v, z, M) d\mu(z)
\]
with
\[
d\mu(z) = d\mu(z_1)d\mu(z_2) = \prod_{j=1,2} \frac{dx_j dy_j}{y_j^2}.
\]

We do this in several steps.

**Step 1. The integral for squarefree positive \(m\)**

To simplify things, we start by treating the special case of squarefree \(m > 0\) where \(L_m^* = L_m\).

As \(\Gamma = \Gamma \times \Gamma'\) acts on \(L_m\) via \(M \mapsto \gamma_1 M^t \gamma_2 =: M^\gamma\), we have
\[
I_m = \int_{\Gamma \backslash \mathbb{H} \times \Gamma' \backslash \mathbb{H}} (1/2) \sum_{M \in L_m^*} \xi(v, z, M) d\mu(z)
\]
\[
= \int_{\Gamma \backslash \mathbb{H} \times \Gamma' \backslash \mathbb{H}} (1/2) \sum_{\gamma_1 \in \Gamma, \beta \in (\Gamma_0(m) \backslash \Gamma)} \xi(v, (z_1, z_2); \gamma_1 \begin{pmatrix} m & 1 \\ 1 & 1 \end{pmatrix}^t \beta) d\mu(z)
\]
\[
= \int_{\Gamma \backslash \mathbb{H} \times \Gamma' \backslash \mathbb{H}} (1/2) \sum_{\gamma_1 \in \Gamma, \beta \in (\Gamma_0(m) \backslash \Gamma)} \xi(v, (\gamma_1^{-1} z_1, \beta^{-1} z_2); \begin{pmatrix} m & 1 \\ 1 & 1 \end{pmatrix}) d\mu(z)
\]
where we use the homogeneity \(\xi(gz, M^\gamma) = \xi(z, M)\). Hence, one has
\[
I_m = (1/2) \int_{\mathbb{H} \times (\Gamma_0(m) \backslash \mathbb{H})} \xi(v, z, \begin{pmatrix} m & 1 \\ 1 & 1 \end{pmatrix}) d\mu(z)
\]
with
\[
\xi(v, z, \begin{pmatrix} m & 1 \\ 1 & 1 \end{pmatrix}) = \int_1^\infty e^{-2\pi v R(z, \begin{pmatrix} m & 1 \\ 1 & 1 \end{pmatrix})} du/\mu
\]
and
\[ R(z, \begin{pmatrix} m \\ 1 \end{pmatrix}) = (1/(2y_1y_2))|m + z_1z_2|^2. \]

We simplify this by changing two times our coordinates.

i) The change
\[(z_1, z_2) \mapsto (z_1, mz_2)\]
leads to \(d\mu(z) \mapsto d\mu(z)\) and
\[R(z, \begin{pmatrix} m \\ 1 \end{pmatrix}) \mapsto mR(z, \begin{pmatrix} 1 \\ 1 \end{pmatrix}).\]

ii) For \(g_{z_2}\) with \(g_{z_2}(i) = z_2\) we take
\[z = (z_1, z_2) \mapsto g(z) = (g_{z_2}(z_1), g_{z_2}^{-1}(z_2)) =: (z_1', i)\]
and have
\[R(z, \begin{pmatrix} 1 \\ 1 \end{pmatrix}) = R(g(z), \begin{pmatrix} 1 \\ 1 \end{pmatrix})^g = R(z_1', i; \begin{pmatrix} 1 \\ 1 \end{pmatrix})^g = (1/(2y_1')) | 1 + iz_1' |^2\]
and finally
\[I_m = \frac{\text{vol}(\Gamma_0(m) \setminus \mathbb{H})(1/2)}{2} \int_{\mathbb{H}} \int_1^\infty e^{-\pi \text{vm} R(z_1', i; \begin{pmatrix} 1 \\ 1 \end{pmatrix})} du/u d\mu(z_1').\]

Thus one is reduced to a two-dimensional integral
\[I'_m = \frac{\text{vol}(\Gamma_0(m) \setminus \mathbb{H})(1/2)}{2} \int_{\mathbb{H}} \int_1^\infty e^{-\pi \text{vm}(1/(y_1))(1-y_1)^2+x_1'^2} du/u d\mu(z_1).\]

Using parts of the \(\text{SO}(1, 2)\)-theory as for instance in Brunier-Funke [BF], we change coordinates
\[\mathbb{R}^2 \longrightarrow \mathbb{H}, \ (r, \varphi) \mapsto (z = x + iy)\]
with
\[y = 1/(\cosh r - \sinh r \cos \varphi), \ x = -\sinh r \sin \varphi/(\cosh r - \sinh r \cos \varphi).\]
i.e.,
\[\begin{align*}
(x^2 + y^2 + 1)/(2y) &= \cosh r \\
(x^2 + y^2 - 1)/(2y) &= \sinh r \cos \varphi \\
-x/y &= \sinh r \sin \varphi.
\end{align*}\]
A small calculation shows that one has
\[ d\mu(z) = dx \wedge dy/y^2 = \sinh r \, dr \wedge d\varphi. \]

We get
\[ I'_m = \int_\mathbb{H} \int_1^\infty e^{-\pi v m (1/y)((1+x^2+y^2-2y)u)} du/u \, d\mu(z) \]
\[ = \int_0^{2\pi} \int_0^{\infty} \left( \int_1^\infty e^{-2\pi v m (\cosh r - 1)u} du/u \right) \sinh r \, r dr d\varphi \]
\[ = 2\pi \int_1^{\infty} \left( \int_1^{\infty} e^{-2\pi v m u(t-1)} du/u \right) \sinh r drd\varphi \]
\[ = 2\pi \int_1^{\infty} \left( \int_1^{\infty} e^{-2\pi v m u} du/u \right) \sinh r drd\varphi \]
\[ = 2\pi \int_1^{\infty} \left( \int_1^{\infty} e^{-2\pi v m u} du/u \right) \sinh r drd\varphi \]
\[ = (1/vm) \int_1^{\infty} u^{-2} du \]
\[ = (1/vm). \]

and hence
\[ (4.2.9) \quad I_m = \text{vol}(\Gamma_0(m) \setminus \mathbb{H})(1/2)(1/vm) = \sigma(m)\pi/(6vm). \]

**Step 2. The integral for squarefree negative \( m \)**

For negative \( m \) we need slight changes in the calculation of the integral \( I_m \) in the second part of our proof. At first one has
\[ L^*_m \simeq \Gamma \left( \begin{pmatrix} m \\ 1 \end{pmatrix} \right) (\Gamma_0(|m|) \setminus \Gamma). \]

Then the transformation
\[ (z_1, z_2) \mapsto (z_1, mz_2) \]
transforms \( \mathbb{H} \times \mathbb{H} \) to \( \mathbb{H} \times \mathbb{H} \). Hence, in the next step, we have to replace the
old $g_{z_2}$ by another one with $g_{z_2}(-i) = z_2$ and come to

$$I_m = \text{vol}(\Gamma_0(|m|) \setminus \mathbb{H})(1/2) \int_{\mathbb{H}} \left( \int_1^{\infty} e^{-2\pi v|m|} R(z_1', -i; (1 \ 1)) du \right) d\mu(z_1')$$

$$= \text{vol}(\Gamma_0(|m|) \setminus \mathbb{H})(1/2) \int_{\mathbb{H}} \left( \int_1^{\infty} e^{-\pi(1/v_1)((1 + v_1^2 + x_1^2)v|m|u)du} \right) d\mu(z_1)$$

$$= \text{vol}(\Gamma_0(|m|) \setminus \mathbb{H})(1/2) \int_{\mathbb{H}} \left( \int_1^{\infty} e^{-\pi(1/v_1)((1 + x_1^2 + y_1^2 + 2v_1)v|m|u)du} \right) d\mu(z_1)$$

$$= \text{vol}(\Gamma_0(|m|) \setminus \mathbb{H})(1/2) 2\pi \int_{1}^{\infty} \left( \int_{1}^{\infty} e^{-2\pi v|m|(\cosh r + 1)u} du \right) \sinh rdr$$

$$= \text{vol}(\Gamma_0(|m|) \setminus \mathbb{H})(1/2) 2\pi \int_{1}^{\infty} \left( \int_{1}^{\infty} e^{-2\pi v|m|(t+1)u} dt \right) du$$

$$= \text{vol}(\Gamma_0(|m|) \setminus \mathbb{H})(1/2)(-1/vm) \int_{1}^{\infty} e^{-4\pi v|m|u} - 4\pi \int_{1}^{\infty} e^{-4\pi v|m|u} du.$$  \hspace{1cm} (4.2.10)

Step 3. The integral for general $m \neq 0$

We use the results which we already have and for positive $m$ calculate

$$I_m = \int_{\mathbb{H} \setminus \Gamma \times \mathbb{H}} (1/2) \sum_{M \in L_m} \xi(z, M) d\mu(z)$$

$$= \int_{\mathbb{H} \setminus \Gamma \times \mathbb{H}} \sum_{n^2|m} (1/2) \sum_{M \in L^*_{m/n^2}} \xi(z, nM) d\mu(z)$$

$$= \sum_{n^2|m} \int_{\mathbb{H} \setminus \Gamma \times \mathbb{H}} (1/2) \sum_{M \in L^*_{m/n^2}} \xi(z, nM) d\mu(z).$$

With

$$L^*_{m/n^2} = \Gamma \left( m/n^2 \right) \left( \Gamma_0(m/n^2) \setminus \Gamma \right)$$

we get

$$I_m = (1/2) \sum_{n^2|m} \int_{\mathbb{H} \setminus \Gamma_0(m/n^2)} \xi(z_1, z_2; n \left( m/n^2 \right) \xi(m/n^2, 1)) d\mu(z).$$

One has

$$R(z, n \left( m/n^2 \right) \xi(m/n^2, 1)) = (1/2y_1y_2) | m/n + nz_1z_2 |^2$$
and changing the coordinates, as in the second part of our proof above, this time by \((z_1, z_2) \mapsto (1/2y_1y_2)m \mid 1 + z_1z_2 \mid^2 = mR(z, \begin{pmatrix} 1 \\ 1 \end{pmatrix})\)
and hence, as in the second part above, via \(z \mapsto g(z) = (z_1, i)\)

\[ I_m = (1/2) \sum_{n \mid m} \psi(m/n^2) I'_m = \sigma_1(m) \kappa I'_m \]

where we used the formulae (4.2.4) and (4.2.10) from the first part of the proof and put \(\kappa := \text{vol}(\Gamma \setminus \mathbb{H}) = \pi/3\).

For negative \(m\) one gets the same way, with \(m\) replaced by \(|m|\), analogously the formula (4.2.10) from above

\[ I_m = \sigma_1(|m|)(\pi/3)(1/v|m|)(e^{-4\pi v|m|} + 4\pi v|m|Ei(-4\pi v|m|v)). \]

\(\square\)

5 Proof of the Main Theorem

Now we relate the results obtained for \(m \neq 0\) in Theorem 4.2 for the Green function integral \(I_m\) to the Fourier coefficients of our modified Eisenstein series in Corollary 2.4.

5.1. Theorem. For \(m \neq 0\), with \(kxcxZm\) one has

\[ \hat{c}_1(\Gamma)^2 \cdot \hat{Z}(m) = \text{ht}_\Gamma(T(m)) + \int_X \hat{z}(v, z, m)c_1(\overline{L})^2 \]

\[ = A'(v, 1, m). \]  

Proof. We already observed

\[ A'(v, 1, m) = -48\left(\frac{\zeta'(-1)}{\zeta(-1)} + \frac{1}{2}\right)a(v, 1, m) + 12a'(v, 1, m). \]
Hence, for \( m > 0 \), from Theorem 2.2 resp. Corollary 2.4 and the Remark 2.3 on the different sigmas one has

\[
A'(v, 1, m) = -48\sigma(m)\left(\frac{\zeta'(-1)}{\zeta(-1)} + \frac{1}{2}\right)
+ 12\sigma(m)(1/(4\pi mv) + \sigma^*(m)/\sigma^*_{1/2}(m))
= -48\sigma(m)\left(\frac{\zeta'(-1)}{\zeta(-1)} + \frac{1}{2}\right)
+ 12(\sigma(m)/(4\pi mv) + (\sigma(m) \log m - 2\sum d|m d \log d)).
\]

On the other hand, as already remarked in the introduction, from Theorem 7.62 in [BKK] and Theorem 4.2

\[
\hat{c}_1(\mathcal{L})^2 \cdot \hat{Z}(m) = \text{ht}_{\mathcal{E}}(T(m)) + c \int_X \overline{\xi}(v, z, m)d\mu
= 24^2(\sigma(m)((1/2)\zeta(-1) + \zeta'(-1)) + \sum_{d|m} \left(\frac{d \log d}{24} - \frac{\sigma(m) \log m}{48}\right))
+ c\sigma_1(m)(\pi/3)2\pi(1/(4\pi mv)).
\]

For \( m < 0 \) one has by Theorem 2.2

\[
A'(v, 1, m) = -(\text{Ei}(-4\pi|m|v) + e^{-4\pi|m|v}/(4\pi|m|v))\sigma_1(m)
\]
and again from Theorem 1.2

\[
\hat{c}_1(\mathcal{L})^2 \cdot \hat{Z}(m) = \text{ht}_{\mathcal{E}}(T(m)) + c \int_X \overline{\xi}(v, z, m)d\mu
= c\sigma_1(m)\pi^2(2/3)(\text{Ei}(-4\pi|m|v) + e^{-4\pi|m|v}/(4\pi|m|v)).
\]

In both cases we get the equality we claimed with \( c = 18/\pi^2 \).

\[\square\]

5.2. Remark. The constant \( c \) is explained by the fact that in the context of [BKK] and [BBK] one has

\[
\int c_1(\mathcal{L})^2 = c_1(\mathcal{L}) \cdot c_1(\mathcal{L}) = T(1) \cdot T(1) = 2
\]

with

\[
c_1(\mathcal{L}) = 12 \left(dx_1dy_1/(4\pi y_1^2) + dx_2dy_2/(4\pi y_2^2)\right)
\]
and
\[ c_1(\mathcal{L})^2 = 2 \cdot (12/4\pi)^2 (dx_1dy_1dx_2dy_2/(y_1y_2)^2) = (18/\pi^2) d\mu(z). \]

Finally, we have all the material for the

**Proof of the Main Theorem.** As we know the modularity of both sides of (1.0.2) the equality for \(m \neq 0\) following from Theorem 5.1 is sufficient (and, hence, gives the value for \(m = 0\) we up to now did not determine directly). \(\square\)

6 Remarks towards a direct calculation of the constant term

Though we don’t really need this, to strive for some completeness, we will make some remarks concerning the case \(m = 0\).

For \(m = 0\), as a consequence of the log-log-singularity of the metric on \(\mathcal{L}\), \(\text{ht}_{\mathcal{L}}(T(0))\) is not defined and by the same reason \( \int_X \Xi_{\rho}(v, z, 0) d\mu \) does not exist. Therefore instead of (1.1.2), we have to use the formula
\[
\hat{c}_1(\mathcal{L})^2 \cdot \hat{Z}_{\rho}(0) = \int_X \left( \Xi_{\rho}(v, z, 0) + (1/24 - 1/(8\pi v))g_0 \right) d\mu
\]
\[+ \tilde{c}(\zeta'(-1)/\zeta(-1) + 1/2), \tag{6.0.1}\]
where
\[ g_0 := -\log \| \Delta(z_1)\Delta(z_2) \|^2. \tag{6.0.2} \]

This formula comes out as here one has
\[ \hat{Z}_{\rho}(0) = (T(0), \Xi_{\rho}) \]
\[ \hat{T}(0) = (T(0), c_0 g_0), \quad c_0 := -1/24 + 1/8\pi v \]
and
\[ \hat{c}_1(\mathcal{L})^2 \cdot \hat{Z}_{\rho}(0) = \hat{c}_1(\mathcal{L})^2 \cdot (\hat{T}(0) + (\hat{Z}_{\rho}(0) - \hat{T}(0)) \]
\[ = \hat{c}_1(\mathcal{L})^2 \cdot \hat{T}(0) + \hat{c}_1(\mathcal{L})^2 \cdot (\hat{Z}_{\rho}(0) - \hat{T}(0)) \tag{6.0.3} \]
where the first summand is known to be ([BKK] Theorem 7.61)
\[ \hat{c}_1(\mathcal{L})^2 \cdot \hat{T}(0) = \hat{c}_1(\mathcal{L})^2 \cdot c_0 \hat{c}_1(\mathcal{L}) \]
\[ = \hat{c}(\zeta'(-1)/\zeta(-1) + 1/2), \quad \hat{c} = -(1/2)12^2 c_0 \]
and (using the formulae (3.3.1), (3.1.1), and (3.0.5) from Part I) the last one gives the integral above in the formula (6.0.4)
\[
\hat{c}_1(\mathcal{Z})^2 \cdot (\hat{Z}_\rho(0) - (\hat{T}(0))) = \int_X (\Xi(v, z, 0) - c_0 g_0) c_1(\mathcal{Z})^2
\]

We already fixed
\[
f_\rho(0) = \int_X \rho(z) \Xi^+(v, z, 0) d\mu.
\]
Thus, if one wants to avoid the reasoning from the proof above, for a direct proof of the \(m = 0\) case it remains to show that \((\pi^2/18) A'(v, z, 0)\) has the same value as
\[
\int_X (\Xi^+(v, z, 0) - (1/24) g_0) + (1/24 - 1/(8\pi v)) g_0 d\mu.
\]
Observe, if we split the integrals, then we would get two divergent integrals where for the integral over \(g_0\) the relevant terms in an asymptotic expansion at the boundary had been calculated already in [K]. We shall report on this and some up to now unsuccessful proposals concerning the determination of the \(m = 0\) case in a final appendix.

Appendix

A Some \(\text{SO}(1, 2)\) Theory

In Step 1 of our proof of Theorem 4.2 we reduced things from \(\text{SO}(2, 2)\) theory to \(\text{SO}(1, 2)\) theory. To give some background, here we denote
\[
V := \{X = \begin{pmatrix} x_1 & x_2 \\ x_3 & -x_1 \end{pmatrix} \in M_2(\mathbb{R})\}
\]
with
\[
(X, Y) := -\text{tr}XY, \quad q(X) := (1/2)(X, X) = \det X = -x_1^2 - x_2x_3.
\]
$G(\mathbb{R}) = \text{SL}(2, \mathbb{R})$ acts on $V$ by conjugation

$$g.X := gXg^{-1}$$

and for

$$z = x + iy, g_z = \begin{pmatrix} y^{1/2} & xy^{-1/2} \\ y^{-1/2} & x \end{pmatrix}, X_0 = \begin{pmatrix} -1 & 1 \\ -1 & x \end{pmatrix}$$

one has

$$g_z .X_0 = \left( 1/y \right) \begin{pmatrix} -x & x^2 + y^2 \\ -1 & x \end{pmatrix} =: X(z)$$

and $q(X(z)) = 1, g.X(z) = X(g(z))$ for $g \in G(\mathbb{R})$,

$$(X, X(z)) = -(1/y)(x_3(x^2 + y^2) - 2x_1x - x_2), \text{ if } x_3 \neq 0.$$
and realize $D_1$ as
\[ D_1 \simeq \{ v \in V; (v, v) = 1, (v, v_1) > 0 \}, \]
i.e., by the hyperboloid
\[ y_1^2 - y_2^2 - y_3^2 = 1, y_1 > 0. \]
One has the parametrization
\[ \psi : (0, \infty) \times S^1 \rightarrow D_1, \ (r, w) \mapsto (\cosh r, \sinh r w) \]
where $w = (\cos \varphi, \sin \varphi)$, i.e.,
\[ (r, \varphi) \mapsto (y_1 = \cosh r, y_2 = \sinh r \cos \varphi, y_3 = \sinh r \sin \varphi). \]
We compare this to the coordinization above and get
\[
\begin{align*}
y_1(z) &= \frac{(x^2 + y^2 + 1)}{(2y)} = \cosh r \\
y_2(z) &= \frac{(x^2 + y^2 - 1)}{(2y)} = \sinh r \cos \varphi \\
y_3(z) &= -\frac{x}{y} = \sinh r \sin \varphi
\end{align*}
\]
and
\[ y = 1/(\cosh r - \sinh r \cos \varphi), \ x = -\sinh r \sin \varphi/(\cosh r - \sinh r \cos \varphi). \]
A small calculation shows that one has
\[ d\mu(z) = dx \wedge dy/y^2 = \sinh r \ dr \wedge d\varphi. \]
And for $X_0 = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$ we find from the formulae above
\[
R(z, X_0) = (1/2)(X_0, X(z))^2 - (X_0, X_0) = (1/2)((1/y)(x^2 + y^2 + 1))^2 - 2 = 2((\cosh r)^2 - 1) = (\sinh r)^2.
\]
A.1. Remark. These expressions lead to the nice computation of the integral $I$ in the determination of the SO(2, 1)–Green function integral in Proposition 12.1 in [KRY] (for $m = 1$)
\[
(A.1.1) \quad I = \int_{D_1^+} \frac{-\text{Ei}(-2\pi R(z, v^{1/2}X_0))d\mu(z)}{J(4\pi v)} = J(4\pi v)
\]
where
\[ J(t) = \int_0^\infty e^{-tw}[(w + 1)^{1/2} - 1]dw/w. \]
B Fourier and Taylor expansion of the Eisenstein series

Here we assemble background material to the main part section on Eisenstein series. At first we rederive a well known result.

B.1. Proposition. One has the following Fourier development

\[ E^*(\tau, s) = v^s \zeta^s(2s) + v^{1-s} \zeta^s(2s - 1) + 2v^{1/2} \sum_{n \in \mathbb{Z}, n \neq 0} \sigma_{s-(1/2)}(\| n \|) K_{s-(1/2)}(2\pi |nv|) e^{2\pi i nu}. \]

Proof. \( K_\nu(t) \) is related to the function

\[ k_\nu(t) := \int_{\mathbb{R}} e^{-itx}(x^2 + 1)^{-\nu} \, dx \]

as one has

\[ k_\nu(t) = \begin{cases} 2\sqrt{\pi} \frac{|t|^{s-(1/2)}}{\Gamma(s)} K_{s-(1/2)}(|t|), & t \neq 0 \\ \frac{\sqrt{\pi}}{\Gamma(s)} \Gamma(2s - 1), & t = 0. \end{cases} \]

This relation comes out as follows. One applies the Mellin transform

\[ \phi(s) \mapsto M\phi(s) = \int_0^\infty \phi(t)t^{s-1} \, dt \]

to

\[ \phi(t) = e^{-\lambda t} \]

to get

\[ M\phi(s) = \int_0^\infty e^{-\lambda t} t^{s-1} \, dt = \lambda^{-s} \Gamma(s) \]

and for

\[ \phi_a(t) = \int_{\mathbb{R}} e^{-iax} e^{-(x^2+1)t} \, dx \]

one has using the previous result and the definition of \( k_\nu(t) \)

\[ M\phi_a(s) = \int_0^\infty \int_{\mathbb{R}} e^{-iax} e^{-(x^2+1)t} \, dx \, dt = \Gamma(s)k_\nu(a). \]

As usual, we may transform

\[ \phi_a(t) = \int_{\mathbb{R}} e^{-iax} e^{-(x^2+1)t} \, dx = \sqrt{\pi/t} e^{-t-a^2/(4t)}. \]
Hence the last formula goes to

\[ k_s(a) = \frac{1}{\Gamma(s)}M_{\phi_a}(s) = \frac{1}{\Gamma(s)}\sqrt{\pi} \int_0^\infty e^{-t-a^2/(4t)} t^{s-3/2} dt. \]

Obviously, for \( a = 0 \) one has

\[ k_s(a) = \frac{\sqrt{\pi}}{\Gamma(s)} \Gamma(2s - 1). \]

For \( a > 0 \) we put \( t =: (1/2)ae^u \) and get

\[ \int_{a/2}^\infty e^{-t-a^2/(4t)} t^{s-3/2} dt = \left(\frac{a}{2}\right)^{s-1/2} \int_0^\infty e^{-a \cosh u} e^{u(s-(1/2))} du. \]

In the same way, for \( t =: (1/2)ae^{-u} \) we get

\[ \int_0^{a/2} e^{-t-a^2/(4t)} t^{s-3/2} dt = \left(\frac{a}{2}\right)^{s-1/2} \int_0^\infty e^{-a \cosh u} e^{-u(s-(1/2))} du \]

and finally

\[ \int_0^\infty e^{-t-a^2/(4t)} t^{s-3/2} dt = 2\left(\frac{a}{2}\right)^{s-1/2} \int_0^\infty e^{-a \cosh u} \cosh(-u(s - (1/2))) du \]

which immediately leads to the relation between the \( K- \) and the \( k- \) function announced above. Now the Fourier expansion of the Eisenstein series is not difficult: One applies Poisson summation

\[ \sum_{n \in \mathbb{Z}} \varphi(x + n) = \sum_{\nu \in \mathbb{Z}} \left( \int_\mathbb{R} \varphi(t) e^{-2\pi ivt} dt \right) e^{-2\pi ivx} \]

for \( s > 1 \) and \( y > 0 \) to \( \varphi(x) := |x + iy|^{-s} \) and we get, putting in the integral \( t/y =: x, \)

\[ \sum_{n \in \mathbb{Z}} |x + iy + n|^{-s} = \sum_{\nu \in \mathbb{Z}} \left( \int_\mathbb{R} |t + iy|^{-s} e^{-2\pi ivt} dt \right) e^{-2\pi ivx} \]

\[ = \sum_{\nu \in \mathbb{Z}} \left( y^{1-s} \left( \int_\mathbb{R} (x^2 + 1)^{-s/2} e^{-2\pi ivy} dx \right) \right) e^{-2\pi ivx} \]

\[ = \sum_{\nu \in \mathbb{Z}} y^{1-s} k_{s/2}(2\pi \nu y) e^{-2\pi ivx}. \]

Changing \( s \) to \( 2s \) and using the relation between the \( k- \) and the \( K- \) function, we get
\[
\sum_{n \in \mathbb{Z}} |x + iy|^{-2s} = y^{1 - 2s} \frac{\sqrt{\pi}}{\Gamma(s)} \Gamma(s - 1/2) \\
+ y^{1 - 2s} \frac{2\sqrt{\pi}}{\Gamma(s)} \sum_{\nu \in \mathbb{Z}, \nu \neq 0} (\pi|\nu|y)^{s - 1/2} K_{s-1/2}(2\pi|\nu|y)e^{-2\pi i \nu x}.
\]

Replacing \(x = mu, y = mv\) and going back to the definition of the Eisenstein series Zagier’s expansion given above comes out

\[E^*(\tau, s) = v^s \zeta^*(2s) + v^{1-s} \zeta^*(2s - 1) + 2v^{1/2} \sum_{n \in \mathbb{Z}, n \neq 0} \sigma^*_{s - (1/2)}(|n|) K_{s - (1/2)}(2\pi |n| v)e^{2\pi inu}.\]

\[\Box\]

Now, here we add the direct proof of Theorem 2.2 where in the main part we relied on Maple calculations.

**B.2. Theorem.** We have

\begin{equation}
E_2(\tau, 1) = \sum_{m \in \mathbb{Z}} a(v, 1, m) q^m
\end{equation}

and, denoting by \(E'_2(\tau, s)\) the derivative of \(E_2(\tau, s)\) with respect to \(s\), we get

\begin{equation}
E'_2(\tau, 1) = \sum_{m \in \mathbb{Z}} a'(v, 1, m) q^m
\end{equation}

with

\[a(v, 1, m) = \begin{cases} 
\sigma_1(m) & \text{for } m > 0 \\
-1/24 + 1/(8\pi v) & \text{for } m = 0 \\
0 & \text{for } m < 0
\end{cases}\]

\[a'(v, 1, m) = \begin{cases} 
\sigma_1(m)(1/(4\pi mv) + \sigma'/\sigma) & \text{for } m > 0 \\
-(1/24)(24\zeta'(-1) + \gamma - 1 + \log(4\pi v)) \\
-(1/(8\pi v))(-\gamma + \log(4\pi v)) & \text{for } m = 0 \\
\sigma_1(m)(\text{Ei}(-4\pi |m| v) + 1/(4\pi |m| v)e^{-4\pi |m| v}) & \text{for } m < 0
\end{cases}\]
where with $\sigma^*$ as in (2.0.4)

\[
\sigma := \sigma^*_{1/2}(m), \quad \sigma' := \sigma'^*_{1/2}(m).
\]

**Proof.** 1. We have (see for instance Iwaniec [Iw] p.205)

\[
K_{\nu}(t) := \int_0^\infty e^{-t\cosh u} \cosh(\nu u) du
= \sqrt{\pi} (t/2)^{\nu} \int_1^\infty e^{-tr^2 - 1)^{\nu-(1/2)}} dr.
\]

Hence, from (EisFourdev 2.0.3) we get

\[
E^*(\tau, s) = v^s\zeta^*(2s) + v^{1-s}\zeta^*(2s - 1)
+ \sum_{m \neq 0} 2\sigma^*_{-(1/2)}(|m|) \frac{(v|m|\pi)^s}{\Gamma(s)\sqrt{|m|}} \int_1^\infty e^{-2\pi|m|\nu(r^2 - 1)^{s-1} dr} e^{2\pi imu}.
\]

As in the rudimentary proof of Theorem ThderEiss 2.2 we abbreviate

\[
c_0(v, s) := v^s\zeta^*(2s) + v^{1-s}\zeta^*(2s - 1),
\]

\[
\bar{c}_0(v, s) := \partial_v c_0(v, s) = sv^{s-1}\zeta^*(2s) + (1-s)v^{-s}\zeta^*(2s - 1),
\]

and, for $m \neq 0$,

\[
\alpha := 2\pi|m|v,
\]

\[
c_m(v, s) := 2\sigma^*_{-(1/2)}(|m|) \frac{(v|m|\pi)^s}{\Gamma(s)\sqrt{|m|}},
\]

\[
I_m(v, s) := \int_1^\infty e^{-2\pi|m|\nu(r^2 - 1)^{s-1} dr},
\]

\[
J_m(v, s) := \int_1^\infty e^{-2\pi|m|\nu(r^2 - 1)^{s-1} r dr},
\]

and get

\[
E_2(\tau, s) = -(1/(4\pi))(\bar{c}_0(v, s) + \sum_m ((s/v - 2\pi m)c_m(v, s)I_m(v, s)
- 2\pi|m|c_m(v, s)J_m(v, s)) e(mu)
\]
\[ E_2(\tau, s) = -\frac{1}{4\pi}(c_0(v, s) + \sum_{m>0}((s/v)c_m(v, s)I_m(v, s) - 2\pi|m|c_m(v, s)(I_m(v, s) + J_m(v, s))e(mu)\]

\[ + \sum_{m<0}((s/v)c_m(v, s)I_m(v, s) + 2\pi|m|c_m(v, s)(I_m(v, s) - J_m(v, s))e(mu)). \]

\[ (B.2.5) \]

\[ E'_{2s}(\tau, s) = -\frac{1}{4\pi}(c'_0(v, s) + \sum_{m>0}((1/v)c_m + (s/v)c'_m)I_m - 2\pi|m|c'_m(I_m + J_m)
\]

\[ + (s/v)c_m'I_m - 2\pi|m|c_m'(I_m' + J_m')e(mu)\]

\[ + \sum_{m<0}((1/v)c_m + (s/v)c'_m)I_m + 2\pi|m|c'_m(I_m - J_m)
\]

\[ + (s/v)c_m'I_m + 2\pi|m|c_m'(I_m' - J_m')e(mu)). \]

\[ (B.2.6) \]

From

\[ c_m(v, s) := 2\sigma^s_{s-(1/2)}(\mid m \mid \frac{(v \mid m \mid \pi)^s}{\Gamma(s)\sqrt{|m|}}) \]

we get

\[ c_m(v, 1) = \sigma^s_{1/2}(\mid m \mid \alpha/\sqrt{|m|}) \]

and with \( \sigma_s := \sigma^s_s(\mid m \mid) \)

\[ c'_m(v, s) = ((\sigma'_{s-(1/2)}/\sigma_{s-1/2}) + \log(\alpha/2) - (\Gamma'(s)/\Gamma(s)))c_m(v, s). \]

Using \( \Gamma'(1) = -\gamma, \gamma \) the Euler constant, we have

\[ c'_m(v, 1) = ((\sigma'_{1/2}/\sigma_{1/2}) + \log(\alpha/2 + \gamma)c_m(v, 1). \]

Now, we can write for \( s = 1 \)

\[ E'_2(\tau, 1) = -\frac{1}{(4\pi)}(c'_0(v, 1)
\]

\[ + \sum_{m>0}((1/v)c_m((1 + (\sigma'_{1/2}/\sigma_{1/2}) + \log(\alpha/2 + \gamma))I_m
\]

\[ - \alpha((\sigma'_{1/2}/\sigma_{1/2}) + \log(\alpha/2 + \gamma))(I_m + J_m)
\]

\[ + I'_m - \alpha(I'_m + J'_m))e(mu)\]

\[ + \sum_{m<0}((1/v)c_m((1 + (\sigma'_{1/2}/\sigma_{1/2}) + \log(\alpha/2 + \gamma))I_m
\]

\[ + \alpha((\sigma'_{1/2}/\sigma_{1/2}) + \log(\alpha/2 + \gamma))(I_m - J_m)
\]

\[ + I'_m + \alpha(I'_m - J'_m))e(mu). \]
One has to determine the values at $s = 1$ of the functions in this relation: From the definitions one has

\[ I_m(v, 1) = e^{-\alpha(1/\alpha)}, \]
\[ J_m(v, 1) = e^{-\alpha((1/\alpha) + (1/\alpha^2))}, \]
i.e.

\[ I_m(v, 1) - J_m(v, 1) = -e^{-\alpha(1/\alpha^2)}, \]
\[ I_m(v, 1) + J_m(v, 1) = e^{-\alpha((2/\alpha) + (1/\alpha^2))}, \]
and, hence, via (B.2.5) immediately the formulae for the $a(v, 1, m)$, $m \neq 0$, in the theorem.

3. For the other terms we will use the well known relation

\[ \Gamma'(1) = \int_0^\infty e^{-t} \log t \, dt = -\gamma \]
and its consequence

\[ \int_0^\infty e^{-at} \log t \, dt = -(1/\alpha)(\gamma + \log \alpha). \]
Moreover, one has using partial integration

\[ \int_0^\infty e^{-at} \log t \, dt = \frac{1}{\alpha}((1/\alpha) - (1/\alpha)(\gamma + \log \alpha)), \]
and with $-\text{Ei}(-s) = \int_1^\infty e^{-st} dt/t$

\[ \int_2^\infty e^{-at} \log t \, dt = \frac{1}{\alpha}(-\text{Ei}(-2\alpha) + e^{-2\alpha} \log 2) \]
\[ \int_2^\infty e^{-at} t \log t \, dt = \frac{1}{\alpha^2}(-\text{Ei}(-2\alpha) + e^{-2\alpha}(2\alpha \log 2 + 1 + \log 2)). \]
Hence we get

\[ I'_m(v, s) = \int_1^\infty e^{-ar}(r^2 - 1)^{s-1} \log(r^2 - 1) \, dr \]
\[ I'_m(v, 1) = \int_1^\infty e^{-ar} \log(r - 1) \, dr + \int_1^\infty e^{-ar} \log(r + 1) \, dr \]
\[ = e^{-\alpha} \int_0^\infty e^{-ar} \log r \, dr \]
\[ = e^{-\alpha(1/\alpha)}(-\text{Ei}(-\alpha)) - e^{-\alpha(1/\alpha)(\log(\alpha/2) + \gamma)} \]
and similarly

\[ J'_m(v, s) = \int_1^\infty e^{-ar}(r^2 - 1)^{s-1} r \log(r^2 - 1) \, dr \]
\[ J'_m(v, 1) = \int_1^\infty e^{-ar} r \log(r - 1) \, dr + \int_1^\infty e^{-ar} r \log(r + 1) \, dr \]
\[ = e^{-\alpha} \int_0^\infty e^{-ar} (r + 1) \log r \, dr \]
\[ = e^{-\alpha(\text{Ei}(-2\alpha))((1/\alpha) - (1/\alpha^2))} + e^{-\alpha(1/\alpha^2)(2 - (\log(\alpha/2) + \gamma)(1 + \alpha))}. \]
One has
\[ I'_m(v, 1) + J'_m(v, 1) = e^\alpha(-Ei(-2\alpha)(1/\alpha^2)) + e^{-\alpha}(1/\alpha^2)(2 - (\log(\alpha/2) + \gamma) - 2\alpha(\log(\alpha/2) + \gamma)) \]
\[ I'_m(v, 1) - J'_m(v, 1) = e^\alpha(Ei(-(2\alpha))(1/\alpha^2) - (2/\alpha)) - e^{-\alpha}(1/\alpha^2)(2 - (\log(\alpha/2) + \gamma)). \]

Hence, from (3.2.6), in the equation for \( E'_2(\tau, 1) \) the coefficient of \( c_m(v, 1)/v \) for \( m > 0 \) comes out as
\[ e^{-\alpha}(-2\sigma'/\sigma - (1/\alpha)) \]
and for \( m < 0 \) as
\[ e^{-\alpha}(-1/\alpha) - e^\alpha 2Ei(-(2\alpha)). \]

Remembering that for \( m > 0 \) with \( q = e(u + iv) \) one has \( q^m = e^{-\alpha}e(mu) \) and for \( m < 0 \) \( q^m = e^{\alpha}e(mu) \), we get with \( \sigma := \sigma_{1/2}^* |m| \)
\[ E'_2(\tau, 1) = -(1/4\pi)c'_0(v, 1) + \sum_{m > 0} (\sqrt{|m|}/(4\pi))(\sigma/(4\pi mv) + \sigma')q^m \]
\[ + \sum_{m < 0} (\sqrt{|m|}/(2\pi))\sigma(-Ei(-(2\alpha)) + e^{4\pi mv}/(4\pi mv))q^m. \]

One can simplify this a bit using the relation \( \sigma = \sigma_{1/2}^* = \sigma_{-1/2}^* \) and, with
\[ \sqrt{|m|}\sigma = \sigma_1 = \sum_{d|m} d. \]
have our claim for \( m \neq 0 \).

4. We still have to treat the case \( m = 0 \), i.e., starting with
\[ c_0(v, s) := v^s \zeta^s(2s) + v^{1-s} \zeta^s(2s - 1), \]
\[ \bar{c}_0(v, s) := \partial_v c_0(v, s) = sv^{s-1} \zeta^s(2s) + (1 - s)v^{-s} \zeta^s(2s - 1), \]
determine \( \bar{c}_0(v, 1) \) and \( \bar{c}_0'(v, 1) \).

We look at
\[ a_1(s) := sv^{s-1} \zeta^s(2s) = \zeta(1 - 2s)\Gamma(1/2 - s)\pi^{s-1/2}sv^{s-1} \]
and get, using standard material assembled in the Zeta Tool Remarks below,
\[ a_1(1) = -\zeta(-1)2\pi = \pi/6, \]
\[ a'_1(s) = -2\zeta'(-2s)/\zeta(1 - 2s) - \Gamma'(1/2 - s)/\Gamma(1/2 - s) + \log \pi + 1/s + \log v)a_1(s) \]
\[ a'_1(1) = -2\zeta'(-1)/\zeta(-1) - \Gamma'(-1/2)/\Gamma(-1/2) + 1 + \log(\pi v))a_1(1) \]
\[ = (\pi/6)(24\zeta'(-1) + \gamma - 1 + \log(4\pi v)). \]
Similarly, we take
\[ a_2(s) := (1 - s)v^{-s}\zeta'(2s - 1) = (1 - s)\zeta(2s - 1)\Gamma(s - 1/2)\pi^{1/2-s}v^{-s} \]
and for
\[ F(s) := (1 - s)\zeta(2s - 1) = (1 - s)(1/(2(s - 1)) + \gamma + \ldots) \]
get
\[ F(1) := -1/2, \quad F'(1) = -\gamma, \]
while for
\[ G(s) := \Gamma(s - 1/2)\pi^{1/2-s}v^{-s} \]
one has
\[
\begin{align*}
G(1) &= \Gamma(1/2)\pi^{-1/2}v^{-1} = 1/v \\
G'(s) &= (\Gamma'(s - 1/2)/\Gamma(s - 1/2) - \log \pi - \log v)G(s) \\
G'(1) &= (\Gamma'(1/2)/\Gamma(1/2) - \log(\pi v))1/v \\
&= (1/v)(-\gamma - \log(4\pi v)),
\end{align*}
\]
and, hence,
\[
\begin{align*}
a_2(1) &= F(1)G(1) = -1/(2v) = \zeta(-1)4\pi(3/(2\pi v)) \\
a_2'(1) &= F'(1)G(1) + F(1)G'(1) = -\gamma(1/v) - (1/2)((-\gamma - \log(4\pi v))1/v \\
&= (1/2v)(-\gamma + \log(4\pi v)).
\end{align*}
\]
Finally we have the claim for the constant terms
\[
\begin{align*}
\bar{c}_0(v, 1) &= a_1(1) + a_2(1) = \zeta(-1)(-2\pi + 6/v) = \pi/6 - (1/2v) \\
\bar{c}'_0(v, 1) &= a_1'(1) + a_2'(1) \\
&= (\pi/6)(24\zeta''(-1) + \gamma - 1 + \log(4\pi v)) + (1/2v)(-\gamma + \log(4\pi v)).
\end{align*}
\]

\[\Box\]

**Some Zeta and related Tools.**

Here we assemble some well known facts which go into our computations.
B.3. Remark.

\[
\zeta^*(s) = \zeta(s)\pi^{-s/2}\Gamma(s/2) = \zeta^*(1 - s)
\]
\[
\zeta(2) = \pi^2/6, \quad \zeta(-1) = -1/12, \quad \zeta(0) = -1/2,
\]
\[
\zeta'(0) = -1/2\log 2\pi([AS]23, 2.13)
\]
\[
\zeta(s) = 1/(s - 1) + \gamma + O(s - 1)
\]
\[
\zeta^*(s) = 1/(s - 1) - (1/2)(\log(4\pi) - \gamma) + O(s - 1)
\]
\[
\zeta^*(2s - 1)/\zeta^*(2s) = (3/\pi)(1/s - 1 + 2 - 2\log(2\pi) - 24\zeta'(-1) + O(s - 1)
\]
\[
\Gamma(1/2) = \sqrt{\pi}, \quad \Gamma(-1/2) = -2\sqrt{\pi}
\]
\[
\Gamma(x)\Gamma(1 - x) = \pi/\sin(\pi x), \quad \Gamma(x)\Gamma(-x) = -\pi/(x\sin(\pi x))
\]
\[
\Gamma'(1) = -\gamma, \quad \Gamma'(1/2) = -\sqrt{\pi}(\gamma + \log 4),
\]
\[
\Gamma'(-1/2) = 2\sqrt{\pi}(\gamma + \log 4 - 2),
\]
\[
\Gamma'(3/2) = 2\sqrt{\pi}(\gamma + \log 4 - 2).
\]

The value of \(\Gamma'(1/2)\) comes from the relation

\[
\Gamma(s - 1/2)/\Gamma(s) = \sqrt{\pi}(1 - (s - 1)\log 4 + \ldots)
\]

to be found in Lang [La] p.275.

B.4. Remark. One has

\[
\zeta'(2) = 2\pi^2(\zeta'(-1) + \zeta(-1)(1 - \gamma - \log(2\pi))).
\]

(B.4.1) \[= (\pi^2/6)(12\zeta'(-1) + \log(2\pi) + \gamma - 1).\]

Proof. From

\[
\zeta(2s) = \zeta(1 - 2s)\pi^{2s-1/2}\Gamma(1/2 - s)/\Gamma(s)
\]

we get by differentiation

\[
2\zeta'(2s) = -2\zeta'(1 - 2s)\pi^{2s-1/2}\Gamma(1/2 - s)/\Gamma(s) + 2\log \pi\zeta(1 - 2s)\pi^{2s-1/2}\Gamma(1/2 - s)/\Gamma(s)
\]
\[
- \zeta(1 - 2s)\pi^{2s-1/2}\Gamma'(1/2 - s)/\Gamma(s)
\]
\[
- \zeta(1 - 2s)\pi^{2s-1/2}\Gamma'(1/2 - s)/\Gamma(s)^2
\]
and, using the facts from the Remark above,

\[ 2\zeta'(2) = -2\zeta'(-1)^2(-2) - 4\log\pi\zeta(-1)^2 \]
\[ -\zeta(-1)^2-1/2\Gamma'(-1/2) - \zeta(-1)2\pi^2\gamma \]
\[ = -2\zeta'(-1)^2(-2) - 4\log\pi\zeta(-1)^2 \]
\[ -\zeta(-1)^2-1/2\gamma + \log 4 - 2 - \zeta(-1)2\pi^2\gamma \]
\[ = 4\pi^2(\zeta'(-1) + \zeta(-1)(1 - \gamma - \log(2\pi)). \]

\[ \square \]

**B.5. Remark.** One has

\[ -\text{Ei}(-x) = -\int_{-\infty}^{-x} e^t dt/t \]
\[ = -\int_{-\infty}^{x} e^{-t} dt = \int_{x}^{\infty} e^{-t} dt = \int_{1}^{\infty} e^{-xt} dt/t \]

\[(B.5.1) \]
\[ = -\gamma - \log|x| - x + x^2/(2 \cdot 2!) - x^3/(3 \cdot 3!) \ldots. \]

and for small \( \epsilon \) and \( \kappa > 0 \) also (we shall call this *Siegel’s formula*)

\[ \int_{\epsilon}^{\infty} e^{-\kappa t} dt/t = \int_{1}^{\infty} e^{-\kappa^2 t} dt/t = -\log \kappa - \gamma - 2\log|\epsilon| + O(\epsilon). \]

\[(B.5.2) \]

**C Towards the Green function integral for \( m = 0 \).**

The equation \[(B.6.0.5)\] tells us that for \( m = 0 \) we have to relate the Green function integral to

\[ A'(v,1,0) = \zeta(-1)3(\zeta'(-1)/\zeta(-1)(1 - 2/(\pi v)) \]
\[ - \log(4\pi v)(1/6 + 1/(2\pi v)) + \gamma(1/(2\pi v) - 1/6)) \]
\[ = 3\zeta'(-1) - (1/8) + (\gamma/24) + (1/24)\log(4\pi v)) \]
\[ + (1/8\pi v)(-48\zeta'(-1) - \gamma + 2 + \log(4\pi v)). \]

\[(C.0.3) \]
\[ = (-1/24 + 1/8\pi v)(6(\zeta'(-1)/\zeta(-1) + 1/2) - \gamma) \]
\[ + (1/4\pi v)(\zeta'(-1)/\zeta(-1) + 1/2) \]
\[ - (1/24 + 1/8\pi v)\log(4\pi v). \]
As this seems rather confusing, let’s look how it came up. We started with the constant term of the normalized Eisenstein series $E^*(\tau, s)$

$$c_0(v, s) = \zeta^*(2s)v^s + \zeta^*(2s - 1)v^{1-s}$$

$$= c_0^{-1}(v, 1)(s - 1)^{-1} + c_0(v, 1) + c_0'(v, 1)(s - 1) + c_0''(v, 1)(s - 1)^2/2 + \ldots$$

where the prime ’ indicates derivation with respect to $s$. Then we come to the constant term of $E_2(\tau, s)$

$$a(v, s, 0) = (-1/4\pi)\partial_v c_0(v, s)$$

$$= (-1/4\pi)\partial_v (c_0(v, 1)(s - 1)^{-1} + c_0(v, 1) + c_0'(v, 1)(s - 1) + \ldots)$$

On the other hand we have look at the Green function integral for $m = 0$ and determine

$$\int_X (\Xi^*(v, z, 0) - (1/(8\pi v))g_0(z))d\mu.$$
and
\[ g_1' = \begin{pmatrix} 1 & \beta_1' \\ 1 & \end{pmatrix}, g_2' = \begin{pmatrix} 1 & \beta_2' \\ 1 & \end{pmatrix}, \]
one has
\[ g_1 g_1' \left( a \right) t(g_2 g_2') = g_1 \left( a \right) t g_2 = a \begin{pmatrix} \alpha_1 \alpha_2 & \alpha_1 \gamma_2 \\ \gamma_1 \alpha_2 & \gamma_1 \gamma_2 \end{pmatrix}, \]
and each matrix with determinant 0 has such a form with coprime \( \alpha_1, \gamma_1 \) and \( \alpha_2, \gamma_2 \).
\[ \square \]

**Definition.** We take our standard \( \text{SL}(2, \mathbb{Z}) \)-fundamental domain \( \mathcal{F} \), the truncated domain
\[ \mathcal{F}_T = \{ z \in \mathbb{H}; |z| \geq 1, |x| \leq 1/2, y \leq T \}, \]
and
\[ \mathcal{G}_T := \{ z \in \mathbb{H}; |x| \leq 1/2, y \leq T \}, \mathcal{F}'_T := \mathcal{F} \setminus \mathcal{F}_T. \]

**C.2. Remark.** One has
\[ \int_{\mathcal{F}_T} dx \wedge dy/y^2 = \pi/3 - 1/T. \]

**Proof.**
\[ \int_{\mathcal{F}_T} dx \wedge dy/y^2 = \int_{-1/2}^{1/2} \int_{\sqrt{1-x^2}}^{T} (dy/y^2) dx \]
\[ = \int_{-1/2}^{1/2} (-1/T + 1/\sqrt{1-x^2}) dx \]
\[ = \pi/3 - 1/T. \]
\[ \square \]

**C.3. Remark.** One has
\[ \int_{\mathcal{F}_T} y \cdot dx \wedge dy/y^2 = \log T - (1/2)(3 \log 3 - 2 \log 2 - 2). \]
Proof.

\[
\int_{\mathcal{F}_T} dx \wedge dy/y^2 = \int_{-1/2}^{1/2} \int_{\sqrt{1-x^2}}^{T} (dy/y) dx \\
= \int_{-1/2}^{1/2} (\log T - \log \sqrt{1 - x^2}) dx \\
= \log T - (1/2)\left( \int_{-1/2}^{1/2} \log(1 + x) dx + \int_{-1/2}^{1/2} \log(1 - x) dx \right) \\
= \log T - \int_{1/2}^{3/2} \log u du \\
= \log T - [u \log u - u]_{1/2}^{3/2} \\
= \log T - (1/2)(3 \log 3 - 2 \log 2 - 2).
\]

\[\square\]

C.4. Corollary. Both Remarks together give

\[
\int_{\mathcal{F}_T} \int_{\mathcal{F}_T} t(s + 1/s) dx_1 \wedge dy_1/y_1^2 \wedge dx_2 \wedge dy_2/y_2^2 \\
= \int_{\mathcal{F}_T} \int_{\mathcal{F}_T} (y_1 + y_2) dx_1 \wedge dy_1/y_1^2 \wedge dx_2 \wedge dy_2/y_2^2 \\
= 2(\pi/3 - 1/T)(\log T - (1/2)(3 \log 3 - 2 \log 2 - 2)).
\]

Moreover, here we can take over some material from Zagier\[^\text{Za}\] and Kühn’s thesis\[^K\]. With

\[\varphi(s) := \zeta^*(2s - 1)/\zeta^*(2s)\]

for large \(T\) one has from\[^\text{Za}\] formula (33), p.426

\[(\text{C.4.1}) \quad \int_{\mathcal{F}_T} E(\tau, s) dx \wedge dy/y^2 = T^{s-1}/(s - 1) - \varphi(s)T^{-s}/s\]

and from\[^K\] Corollary 5.4

\[(\text{C.4.2}) \quad \int_{\mathcal{F}_T} \log \| \Delta(\tau) \| dx dy/(4\pi y^2) = (1/2) \lim_{s \to 1} \int_{\mathcal{F}_T} E(\tau, s) dx dy/y^2 - \int_{\mathcal{F}_T} \varphi(s) dx dy/y^2 = 1 - 12\zeta'(-1) - (1/2) \log(4\pi T) + O(\log T/T).\]
Hence, we can use
\[
\int_{F_T} \int_{F_T} \log \| \Delta(z_1) \Delta(z_2) \| \, dx_1 dy_1 / y_1^2 \, dx_2 dy_2 / y_2^2 \\
= \int_{F_T} \int_{F_T} (\log \| \Delta(z_1) \| + \log \| \Delta(z_2) \|) \, dx_1 dy_1 / y_1^2 \, dx_2 dy_2 / y_2^2
\]
(C.4.3)
\[
= (4\pi)(1 - 12\zeta'(-1) - (1/2) \log(4\pi T) + O(\log T/T)) (\pi/3 - 1/T).
\]

Concerning the equation (6.0.5) we observe:

**Lemma.** With
\[
g_0(z_1, z_2) = \log \| \Delta(z_1) \Delta(z_2) \|^2
\]
one has
\[
I_T^g = (1/(8\pi v)) \int_{F_T} \int_{F_T} g_0(z_1, z_2) \, d\mu(z_1, z_2)
\]
\[
= (1/(8\pi v))(4\pi)(1 - 12\zeta'(-1) - (1/2) \log(4\pi T) + O(\log T/T)) (\pi/3 - 1/T)
\]
(C.5.1)
\[
= (\pi/(3v))(2 - 24\zeta'(-1) - \log(4\pi T) + O(\log T/T)).
\]

Now, we are left with the determination of
\[
I^* = \int_{F \times F} (1/2) \sum_{M \in L_0} \xi(v, z, M) \, d\mu(z)
\]
resp.
\[
I_T^* = \int_{F_T \times F_T} (1/2) \sum_{M \in L_0} \xi(v, z, M) \, d\mu(z).
\]
(C.5.2)

Here we try several different approaches.

**Remark.** One has a formal calculation
\[
I^* = \int_{\Gamma \backslash \Gamma \backslash \Gamma} (1/2) \sum_{M \in L_0} \xi(v, z, M) \, d\mu(z)
\]
(C.6.1)
\[
= (1/2\pi v) \zeta(2) \int_{\Gamma \backslash \Gamma} \sum_{\beta \in \Gamma^\infty \backslash \Gamma} \im \beta y_2 (dx_2 dy_2 / y_2^2).
\]
Proof. From the Lemma above we infer

\[ I^* = \int_{\Gamma \setminus H \times \Gamma \setminus H} (1/2) \sum_{M \in L_0^*} \xi(v, z, M) d\mu(z) \]

\[ = (1/4) \int_{\Gamma \setminus H \times \Gamma \setminus H} \sum_{\gamma_1 \in (\Gamma_\infty \setminus \Gamma)} \sum_{\alpha \in \mathbb{Z} \setminus \{0\}} \sum_{\beta \in (\Gamma_\infty \setminus \Gamma) / \mathbb{Z}} \xi(v, z_1, z_2, \gamma_1 \left(\begin{array}{c} a \\ \beta \end{array}\right) \right) d\mu(z). \]

From Corollary 6.7 in Part I we have

\[ R(z, M) = R \left( g(z), M g \right), \]

i.e.,

\[ R(z, \gamma \left(\begin{array}{c} a \\ \beta \end{array}\right) \right) = R \left( (\gamma^{-1} z_1, \beta^{-1} z_2), \left(\begin{array}{c} a \\ \beta \end{array}\right) \right) \]

and with \( R(z, \left(\begin{array}{c} a \\ \beta \end{array}\right) \right) = a^2/(2y_1y_2) \) we get by unfolding with respect to \( \gamma \in \Gamma_\infty \setminus \Gamma \)

\[ I^* = (1/4) \int_{\Gamma_\infty \setminus H \times \Gamma \setminus H} \sum_{\beta \in \Gamma_\infty \setminus \Gamma} \sum_{\alpha \in \mathbb{Z} \setminus \{0\}} \sum_{\gamma \in (\Gamma_\infty \setminus \Gamma) / \mathbb{Z}} \xi(v, z_1, \beta^{-1} z_2, \left(\begin{array}{c} a \\ \beta \end{array}\right) \right) d\mu(z) \]

\[ = (1/4) \int_{\Gamma_\infty \setminus H} \int_{\Gamma \setminus H} \sum_{\beta \in \Gamma_\infty \setminus \Gamma} \sum_{\alpha \in \mathbb{Z} \setminus \{0\}} \int_1^\infty e^{-\pi v a^2 r/(y_1 \text{Im} \beta y_2)} dr / r d\mu(z_1) d\mu(z_2) \]

The \( z_1 \)-integration goes easily: With \( \alpha = \pi v a^2 r / \text{Im} \beta y_2 \) we have

\[ \int_{\Gamma_\infty \setminus H} e^{-\alpha / y} dxdy / y^2 = \int_0^1 \int_0^\infty e^{-\alpha / y} dxdy / y^2 = \int_0^\infty e^{-\alpha u} du = 1 / \alpha \]

and via \( \int_1^\infty dr / r^2 = 1 \) finally

\[ I^* = (1/2) \int_{\Gamma \setminus H} \sum_{\beta \in \Gamma_\infty \setminus \Gamma} \sum_{\alpha \in \mathbb{N}} \left( \text{Im} \beta y_2 \right) / (\pi v a^2) (dx_2 dy_2 / y_2^2) \]

\[ = \int_{\Gamma \setminus H} \sum_{\beta \in \Gamma_\infty \setminus \Gamma} (1/2\pi v) \zeta(2) \text{Im} \beta y_2 (dx_2 dy_2 / y_2^2) \]

\[ = (\pi / (12v)) \int_{\Gamma \setminus H} \sum_{\beta \in \Gamma_\infty \setminus \Gamma} \text{Im} \beta y_2 (dx_2 dy_2 / y_2^2). \]

\[ \square \]
We observe that we get something divergent near to the Eisenstein series and we moreover try as a regularization:

**Proposition C.7.** Remark. One has a formal calculation leading to

$$I^T = \int_{F \times F} \left(1/2\right) \sum_{M \in L_0^*} \xi(v, z, M) d\mu(z)$$

$$= \lim_{s \to 1, t \to 0} I_{s,t}^T$$

$$I_{s,t}^T : = \int_{F \times F} \left(1/2\right) \sum_{M \in L_0^*} \int_1^\infty e^{-2\pi v r R(z,M)^s} R(z, M)^t dr / r d\mu(z).$$

From the Lemma above we infer

$$2I_{s,t}^T = \int_{F \times F} \sum_{\gamma \in (\Gamma_{\infty} \setminus \Gamma)} \sum_{\beta \in (\Gamma_{\infty} \setminus \Gamma)} \sum_{a \in \mathbb{Z} \setminus \{0\}} \int_1^\infty e^{-2\pi v r R(z, \gamma(a)^t \beta)^s} \frac{R(z, \gamma(a)^t \beta)^t}{dr / r d\mu(z)}.$$

From Corollary 6.7 in Part I we have

$$R(z, M) = R(g(z), M),$$

i.e.,

$$R(z, \gamma(a)^t \beta) = R((\gamma^{-1} z_1, \beta^{-1} z_2), (a^{(2)}))$$

and with $$R(z, (a^{(2)})) = a^2/(2y_1 y_2)$$ we get by unfolding with respect to $$\gamma \in \Gamma_{\infty} \setminus \Gamma$$

$$4I_{s,t}^T = \int_{\Gamma_{\infty} \setminus \mathbb{H}} \int_{F \times F} \sum_{\beta \in \Gamma_{\infty} \setminus \Gamma} \sum_{a \in \mathbb{Z} \setminus \{0\}} \int_1^\infty e^{-2\pi v r (a^2/(2y_1 \Im \beta z_2))s} \frac{(a^2/(2y_1 \Im \beta z_2)) \frac{dr / r d\mu(z_1)}{d\mu(z_2)}}{dr / r d\mu(z_2)}.$$

The $$z_1$$-integration goes easily: With $$\alpha = 2\pi v r (a^2/2 \Im \beta z_2)^s$$ and $$\alpha / y^s = u$$ we have

$$\int_{\Gamma_{\infty} \setminus \mathbb{H}} e^{-\alpha / y^s} (a^2/2 \Im \beta z_2)^t dx dy / y^2 = \int_0^1 \int_0^\infty e^{-\alpha / y^s} (a^2/2 \Im \beta z_2)^t dx dy / y^2$$

$$= (1/s)\alpha^{-(1+s)} (a^2/2 \Im \beta z_2)^t \int_0^\infty e^{-u^{-1+(1+s)/s}} du$$

$$= (1/s)\alpha^{-(1+s)} (a^2/2 \Im \beta z_2)^t \Gamma((1+t)/s)$$

and via $$\int_1^\infty dr / r^\beta = 1/\beta - 1$$ finally

$$I_{s,t}^T = (1/2) \Gamma((1+t)/s) \int_{F \times F} \sum_{\beta \in \Gamma_{\infty} \setminus \Gamma} \frac{(1/s)(1/\pi v)^{(1+t)/s}(1/2)^t \zeta(2(1+t + ts)/s)(\Im \beta z_2)^{(1+t-ts)/s}(dx_2 dy_2 / y_2^2)}{dx_2 dy_2 / y_2^2}$$
For $s = 1 + t$ we stay with
\[ I^T_{s,s-1} = (1/(2sv\pi))^{2^{1-s}}\zeta(2s) \int_{F_T} \sum_{\beta \in \Gamma_\infty \backslash \Gamma} (\text{Im } \beta z_2)^{2-s} \left( dx_2 dy_2/y_2^2 \right) \]

3. Zagier’s approach

C.8. Remark. A slight variant of Zagier’s Rankin Selberg method ([Za] p.420f) goes like this. As above, we take our standard $\text{SL}(2, \mathbb{Z})$—fundamental domain $F$, the truncated domain
\[ F_T = \{ z \in \mathbb{H}; |z| \geq 1, |x| \leq 1/2, y \leq T \}, \]
and
\[ G_T := \{ z \in \mathbb{H}; |x| \leq 1/2, y \leq T \}, \quad F'_T := F \setminus F_T. \]
Then $F_T$ is a fundamental domain for the action of $\text{SL}(2, \mathbb{Z})$ on
\[ \mathbb{H}_T := \bigcup_{\gamma \in \text{SL}(2, \mathbb{Z})} \gamma F_T = \{ z \in \mathbb{H}; \max_{\gamma \in \text{SL}(2, \mathbb{Z})} \text{Im } (\gamma z) \leq T \}
= \{ z \in \mathbb{H}; y \leq T \} - \bigcup_{c \geq 1} \bigcup_{a \in \mathbb{Z}, (a,c) = 1} S_{a/c}, \]
where $S_{a/c}$ is the disc of radius $1/2c^2T$ tangent to the real axis at $a/c$. One has
\[ \text{SL}(2, \mathbb{Z}) \setminus \mathbb{H}_T = G_T - \bigcup_{c \geq 1} \bigcup_{a \mod c, (a,c) = 1} S_{a/c}. \]

Now, we look at a (sufficiently nice) function $h$ on $\mathbb{H}$ depending only on $y = \text{Im } z$. Then one has with $\Gamma := \text{SL}(2, \mathbb{Z})$
\[ \int_{F_T} \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} h(\text{Im } \gamma z) d\mu(z) = \int_{\Gamma_\infty \backslash \mathbb{H}_T} h(y) d\mu(y) \]
\[ = \int_{G_T} h(y) d\mu - \sum_{c \geq 1} \sum_{a \mod c, (a,c) = 1} \int_{S_{a/c}} h(y) d\mu. \]
Choosing an element $\gamma_0 = \left( \begin{array}{cc} a \\ c \\ \end{array} \right) \in \Gamma$ with first column $\left( \begin{array}{c} a \\ \end{array} \right)$ one has $\gamma_0 S_{a/c} = \{ z \in \mathbb{H}; y \geq T \}$ and as these elements $\gamma$ are all of the form $\gamma = \gamma_0 \left( \begin{array}{c} 1 \\ 0 \\ \end{array} \right)$ one has
\[ \int_{S_{a/c}} h(y) d\mu = \int_{\{ z \in \mathbb{H}; y \geq T \}} h(\text{Im } \gamma_0 z) d\mu(z) \]
\[ = \int_{F'_T} \sum_{n \in \mathbb{Z}} h(\gamma_0(z + n)) d\mu \]
\[ = \int_{F'_T} \sum_{\gamma = \left( \begin{array}{cc} a \\ c \end{array} \right) \gamma_0} h(\text{Im } \gamma z) d\mu. \]
Finally, we get
\[
\int_{\mathcal{F}_T} \sum_{\gamma \in \Gamma_\infty \setminus \Gamma} h(\text{Im } \gamma z) d\mu(z) = \int_{\mathcal{G}_T} h(y) d\mu(z)
\]
\[\text{eqZA} \tag{C.8.1}\]

or, going back to Zagier’s deduction, with \( \gamma = \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in \Gamma \)
\[
\int_{\mathcal{F}_T} \sum_{\gamma \in \Gamma_\infty \setminus \Gamma} h(\text{Im } \gamma z) d\mu(z) = \int_{\mathcal{G}_T} h(y) d\mu(z)
\]
\[\text{eqZa1} \tag{C.8.2}\]

\[-\int_{\{z \in \mathbb{H}; y \geq T\}} \sum_{c \geq 1 \text{ mod } c, (a,c) = 1} h(\text{Im } \gamma z) d\mu.\]

We will abbreviate this by
\[
\int_{\mathcal{F}_T} \sum_{\gamma \in \Gamma_\infty \setminus \Gamma} h(\text{Im } \gamma z) d\mu(z) = \int_{\mathcal{G}_T} h(y) d\mu(z)
\]
\[\text{eqZa2} \tag{C.8.3}\]

and call the first integral the \textit{main part} and the second one the \textit{side part}.

4. \textbf{An Approach using two times Zagier’s formula}

For \[\text{Greenint0} \tag{C.8.4}\]

\[
I_1^T = \frac{1}{2} \int_{\mathcal{F}_T^2} \sum_{M \in L_0^2} \xi(v, z, M) d\mu(z)
\]

\[
= \frac{1}{2} \int_{\mathcal{F}_T^2} \sum_{\gamma \in \Gamma_\infty \setminus \Gamma} \sum_{\beta \in \Gamma_\infty \setminus \Gamma} \sum_{a \in \mathbb{Z}, a \neq 0} \int_1^\infty e^{-\pi a^2 t/(\text{Im } \gamma_1 \text{Im } \beta z_2)} dr d\mu(z).
\]

we use Zagier’s formula \[\text{eqZA} \tag{C.8.3}\] in both variables \( z_1, z_2 \) and get as double
main part

\[ I_{11} = \frac{1}{2} \int_{G_T^2} \sum_{a \in \mathbb{Z} \setminus \{0\}} \int_1^\infty e^{-\pi v a^2 r / (y_1 y_2)} dr / r \mu(z) \]

\[ = \frac{1}{2} \sum_{a \in \mathbb{Z} \setminus \{0\}} \int_0^T \int_0^T \int_1^\infty e^{-\pi v a^2 r / (y_1 y_2)} dr / r y_1 dy_1 \]

\[ = \frac{1}{2} \sum_{a \in \mathbb{Z} \setminus \{0\}} \int_0^T \int_1^\infty e^{-\pi v a^2 r / (y_2)} du / u \]

\[ = \frac{1}{2} \sum_{a \in \mathbb{Z} \setminus \{0\}} \int_0^T \int_1^\infty e^{-\pi v a^2 r / (y_2)} dr / r (\pi v a^2) \]

For the interior integral one may use Siegel’s formula (eqSie1 B.5.2)

\[ \int_\epsilon^\infty e^{-\kappa \epsilon t} dt / t = \log \kappa - \gamma - 2 \log |\epsilon| + O(\epsilon) \]

where \( \kappa \) is a positive constant. With \( \epsilon = 1 / T \) for large \( T \) we get

\[ I_{11} \approx (1 / 2) \sum_{a \in \mathbb{Z} \setminus \{0\}} \int_1^\infty (-\log(\pi v a^2 r) - \gamma - 2 \log(1 / T)) dr / r^2 / (\pi v a^2). \]

\[ \approx (1 / (2 \pi v)) \sum_{a \in \mathbb{Z} \setminus \{0\}} \int_1^\infty (-\log(\pi v a^2) - \gamma - 2 \log(1 / T) - \log r) dr / r^2 (1 / a^2). \]

\[ \approx (1 / (2 \pi v)) \sum_{a \in \mathbb{Z} \setminus \{0\}} (-\log(\pi v) - \gamma - 2 \log(1 / T) - 1 - 2 \log a) (1 / a^2). \]

\[ \approx (1 / (\pi v)) (-\zeta(2) (\log(\pi v) + \gamma - 2 \log T + 1) + 2 \zeta'(2)). \]

\[ \approx (\pi / (6 v)) (-\log(\pi v) - \gamma + 2 \log T - 1 + (12 / \pi^2) \zeta'(2)). \]

For \( \zeta'(2) \) one may use the formula (eqdivzeta B.4.1)

\[ \zeta'(2) = 2 \pi^2 (\zeta'(-1) + \zeta(-1) (1 - \gamma - \log(2 \pi))) \]

\[ = \pi^2 / 6 (12 \zeta'(-1) + \gamma + \log(2 \pi) - 1) \]

and we get

\[ I_{11} \approx (\pi / (6 v)) (-\log(\pi v) - \gamma + \log T - 1) + 24 \zeta'(-1) + 2 \gamma - 2 + \log(2 \pi)). \]

\[ \approx (\pi / (6 v)) (-\log(\pi v) + 2 \log(2 \pi) + \gamma + \log T - 3 + 24(\zeta'(-1)). \]
We compare this with (C.5.1), where we get
\[
I = \frac{1}{(8\pi v)} \int_{\mathcal{F}^+} \int_{\mathcal{F}^+} g_0(z_1, z_2) d\mu(z_1, z_2)
\]
\[
= \frac{1}{(3v)} (2 - 24\zeta'(-1) - \log(4\pi) - \log T)
\]
we get
\[
I_{11} + I^0 \simeq \frac{\pi}{6v} (1 + \gamma - 24\zeta'(-1) - \log(4\pi v)).
\]
which has some resemblance to the constant term of the Eisenstein derivative (5.7)
\[
A'(v, 1, 0) = a(v, 1, 0)\psi'(1) - a'(v, 1, 0)
\]
\[
= \zeta(-1)3(\zeta'(-1)/\zeta(-1)(1 - 2/(\pi v))
\]
\[
- \log(4\pi v)(1/6 + 1/(2\pi v)) + \gamma(1/(2\pi v) - 1/6))
\]
\[
= 3\zeta'(-1) - 1/8 + \gamma/24 + (1/24) \log(4\pi v)
\]
\[
+ (1/8\pi v)(-48\zeta'(-1) + 2 - \gamma + \log(4\pi v)).
\]
We have to look how the other terms from Zagier’s approach participate but here still didn’t come to a happy end.

5. An Approach using an old idea of Kühn

We use Lemma C.1 and get
\[
I^*_1 = \frac{1}{2} \int_{\mathcal{F}^+} \sum_{M \in L_0^+} \xi(v, z, M) d\mu(z)
\]
\[
= \frac{1}{2} \int_{\mathcal{F}^+} \sum_{\gamma \in \Gamma_\infty \setminus \Gamma} \sum_{\beta \in \Gamma_\infty \setminus \Gamma} \sum_{a \in \mathbb{Z}, a \neq 0} \xi(v, z, \gamma(a)\beta) d\mu(z)
\]
\[
= \frac{1}{2} \int_{\mathcal{F}^+} \sum_{\gamma \in \Gamma_\infty \setminus \Gamma} \sum_{\beta \in \Gamma_\infty \setminus \Gamma} \sum_{a \in \mathbb{Z}, a \neq 0} \xi(v, (\gamma z_1, \beta z_2), (a)) d\mu(z)
\]
\[
= \frac{1}{2} \int_{\mathcal{F}^+} \sum_{\gamma \in \Gamma_\infty \setminus \Gamma} \sum_{\beta \in \Gamma_\infty \setminus \Gamma} \sum_{a \in \mathbb{Z}, a \neq 0} \int_1^\infty e^{-\pi a^2 r/(\Im \gamma z_1 \Im \beta z_2)} dr d\mu(z).
\]
As in the proof of the \( m \neq 0 \)--Theorem 4.2 in the integral we change coordinates: For \( g_{z_2} \) with \( g_{z_2}(i) = z_2 \) we take
\[
z = (z_1, z_2) \mapsto g(z) = (g_{z_2}(z_1), g_{z_2}^{-1}(z_2) =: (z'_1, i)
and get
\[ R(z, (a)) = R(g(z), (a)) = R((z', i), (a)) = (1/(2y'))\]
i.e.
\[ I^* = \frac{1}{2} \int_{\mathbb{F}_T} \sum_{\gamma \in \Gamma_{\infty}} \sum_{\beta \in \Gamma_{\infty}} \sum_{a \in \mathbb{Z}, a \neq 0} \int_{1}^{\infty} e^{-\pi v a^2 r/(\text{Im} \, \gamma z_{1} \text{Im} \, \beta z_{2})} dr/rd\mu(z). \]
\[ = \frac{1}{2} \int_{\mathbb{F}_T} \sum_{\gamma \in \Gamma_{\infty}} \sum_{c,d \in \mathbb{Z}, (c,d) = 1} \sum_{a \in \mathbb{Z}, a \neq 0} \int_{1}^{\infty} e^{-\pi v(c^2+d^2)a^2 r/\text{Im} \, \gamma z_{1}} dr/rd\mu(z). \]
\[ = \frac{1}{2} \int_{\mathbb{F}_T} \sum_{\gamma \in \Gamma_{\infty}} \sum_{c,d \in \mathbb{Z}, (c,d) = 1} \sum_{a \in \mathbb{Z}, a \neq 0} \int_{1}^{\infty} e^{-\pi v(c^2+d^2)a^2 r/\text{Im} \, \gamma z_{1}} dr/rd\mu(z_{1})(\pi/3 - 1/T). \]

Now one has probably several possibilities how to proceed. We try to use Zagier’s formula \[ (C.8.1) \] for the \( z_{1} \)-integral. From
\[ \int_{\mathbb{F}_T} \sum_{\gamma \in \Gamma_{\infty}} h(\text{Im} \, \gamma z) d\mu(z) = \int_{\mathbb{F}_T} h(y) d\mu(z) \]
\[ - \int_{\mathbb{F}_T} \left( \sum_{\gamma \in \Gamma_{\infty}} h(\text{Im} \, \gamma z) - h(y) \right) d\mu \]
\[ =: I_T(h) + II_T(h) \]
with
\[ h = \sum_{c,d \in \mathbb{Z}, (c,d) = 1} \sum_{a \in \mathbb{Z}, a \neq 0} \int_{1}^{\infty} e^{-\pi v(c^2+d^2)a^2 r/y} dr/r \]
\[ = \sum_{c,d \in \mathbb{Z}, c^2+d^2 \neq 0} \int_{1}^{\infty} e^{-\pi v(c^2+d^2)r/y} dr/r \]
with \( A = \pi v(c^2 + d^2) \) get

\[
2I_T(h) = \sum_{c,d \in \mathbb{Z}, c^2 + d^2 \neq 0} \int_0^T \int_1^\infty e^{-Ar/y} dr/dx dy/y^2
\]

\[
= \sum_{c,d \in \mathbb{Z}, c^2 + d^2 \neq 0} \int_0^T \int_1^\infty e^{-Ar/y} dr/rdy/y^2
\]

\[
= \sum_{c,d \in \mathbb{Z}, c^2 + d^2 \neq 0} \int_1^T \int_1^\infty e^{-Ar} dr/rdu
\]

\[
= \sum_{c,d \in \mathbb{Z}, c^2 + d^2 \neq 0} \int_1^\infty e^{-Ar/T}(1/A)dr/r^2
\]

\[
= \sum_{c,d \in \mathbb{Z}, c^2 + d^2 \neq 0} \left( e^{-A/T}/A - \int_1^\infty e^{-Ar/T}(1/T)dr/r \right)
\]

We estimate the sums by comparing with integrals in the usual polar coordinates \((r, \varphi)\) and with \( r^2 = u \) get

\[
\sum_{c,d \in \mathbb{Z}, c^2 + d^2 \neq 0} e^{-A/T}/A \approx 2/v \int_1^\infty e^{-\pi vr^2/T}/r^2 rdr
\]

\[
= 1/v \int_1^\infty e^{-\pi vu/T} du/u
\]

\[
= -1/v \text{Ei}(-\pi v/T) \approx 1/v(-\gamma - \log(\pi v/T) + \ldots)
\]

and

\[
\sum_{c,d \in \mathbb{Z}, c^2 + d^2 \neq 0} \int_1^\infty e^{-A/T}/T dt/t \approx 2\pi/v \int_1^\infty \int_1^\infty e^{-\pi vr^2/T} r dr dt/t
\]

\[
= \pi/T \int_1^\infty \int_1^\infty e^{-\pi vu/T} dudt/t
\]

\[
= 1/v \int_1^\infty e^{\pi u/T} dt/t^2
\]

\[
= 1/v(e^{-\pi v/T} + (\pi v/T) \text{Ei}(-\pi v/T)).
\]

Hence, for \( T \to \infty \) one has

\[
\text{eqKuehn} \quad (C.8.7) \quad 2I_T(h) \to (1/v)(-\gamma - \log(\pi v/T) - 1).
\]
The second term is more difficult. With

\[
 h = \sum_{c,d \in \mathbb{Z}, (c,d) = 1} \sum_{a \in \mathbb{Z}, a \neq 0} \int_1^\infty e^{-\pi v(c^2 + d^2) |a|^2} \frac{dr}{r}
\]

one has

\[
 II_T(h) = \sum_{c,d \in \mathbb{Z}, c^2 + d^2 \neq 0} \sum_{\gamma' \in \Gamma \setminus \Gamma} \int_1^\infty \int_T e^{-\pi v(c^2 + d^2)t(\mu^2 + \nu^2)/y^2} dt \, dx \, dy / y^2
\]

\[
 = \sum_{c,d \in \mathbb{Z}, c^2 + d^2 \neq 0} \sum_{\mu > 0, \nu \neq 0, (\mu, \nu) = 1} \int_1^\infty \int_T e^{-\pi v(c^2 + d^2) t(\mu^2 + \nu^2)/y^2} dt \, dx \, dy / y^2
\]

\[
 + \sum_{c,d \in \mathbb{Z}, c^2 + d^2 \neq 0} \sum_{\mu \neq 0} \int_1^\infty \int_T e^{-\pi v(c^2 + d^2) \mu^2 / y^2} dt \, dx \, dy / y^2
\]

\[
 = II_1 + II_2 + II_3
\]

By inspection of [Za], the third term doesn’t exist and, with \( \gamma = (\lambda \ast \mu) \) one also can write

\[
 II_T(h) = \sum_{c,d \in \mathbb{Z}, c^2 + d^2 \neq 0} \sum_{\lambda \text{ mod } \mu, (\lambda, \mu) = 1} \int_1^\infty \int_T \int_0^\infty e^{-\pi v(c^2 + d^2) t((\mu z + \nu)^2 + \mu^2 y^2)/y^2} dt \, dx \, dy / y^2
\]

\[
 = \sum_{c,d} \int_1^\infty \int_T \int e^{-\pi v(c^2 + d^2) t \mu^2 y^2 / y^3 / 2} dt / t^{3/2} \, dy / y^{3/2} \, (1/\mu) 1/\sqrt{v(c^2 + d^2)}
\]

Here, one has to cope with the nasty integral

\[
 B_T(\alpha) = \int_1^\infty \int_T e^{-\alpha t} dy / y^{3/2} dt / t^{3/2}
\]

We have

\[
 II_T(h) = \sum_{c,d \in \mathbb{Z}, c^2 + d^2 \neq 0} \sum_{\mu \geq 1} \sum_{\lambda \text{ mod } \mu, (\lambda, \mu) = 1} B_T(\alpha)(1/\sqrt{\alpha})(1/\mu)
\]
with $\alpha = \pi v(c^2 + d^2)\mu$. To investigate $B_T$, we change variables by
\[ u := ty, \ v := t/y \]
and get
\[
B_T(\alpha) = \int_{u \geq T} \int_{v=1/u}^{v=u/T^2} e^{-\alpha u} dv/\mu (2u^{3/2}) \\
= \int_{u \geq T} e^{-\alpha u} [\log v]_{v=1/u}^{v=u/T^2} du/(2u^{3/2}) \\
= \int_{u \geq T} e^{-\alpha u} \log udu/u^{3/2} - \log T \int_{u \geq T} e^{-\alpha u} du/u^{3/2} \\
=: B_T^1(\alpha) - B_T^2(\alpha)
\]
where by partial integration
\[
B_T^2(\alpha) = \log T \int_{u \geq T} e^{-\alpha u} du/u^{3/2} \\
= \log T ((e^{-\alpha u}u^{-1/2}(-2))]_T^\infty - \int_{u \geq T} e^{-\alpha u} (du/u^{1/2})(2\alpha)) \\
= \log T (2e^{-\alpha T}/\sqrt{T} - (2\alpha) \int_{u \geq T} e^{-\alpha u} (du/u^{1/2})) \\
= \log T (2e^{-\alpha T}/\sqrt{T} - (4\sqrt{\alpha}) \int_{\sqrt{\alpha}T}^{\infty} e^{-t^2} dt)
\]
and
\[
B_T^1(\alpha) = \int_{u \geq T} e^{-\alpha u} \log udu/u^{3/2} \\
= [e^{-\alpha u}(-2/\sqrt{u})(\log u - 2)]_T^\infty - \int_T^\infty e^{-\alpha u}(-2/\sqrt{u})(\log u - 2) du(-\alpha) \\
= 2(e^{-\alpha T}/\sqrt{T}) \log T - 4e^{-\alpha T}/\sqrt{T} + 4\alpha \int_{\sqrt{T}}^{\infty} e^{-\alpha u}(1/\sqrt{u}) du \\
- 2\alpha \int_{T}^{\infty} e^{-\alpha u}(\log u/\sqrt{u}) du.
\]
We are left with

\[ C^1_T(\alpha) = 2\alpha \int_T^\infty e^{-\alpha u} \left( \log u \sqrt{u} \right) du \]

\[ = 2\alpha \left( [e^{-\alpha u} 2\sqrt{u} \log u - 2]\right) |_T^\infty - \int_T^\infty e^{-\alpha u} 2\sqrt{u} \log u - 2) du(-\alpha) \]

\[ = 2\alpha \left( -e^{-\alpha T} 2\sqrt{T} \log T - 2\right) - \int_T^\infty e^{-\alpha u} 2\sqrt{u} \log u - 2) du(-\alpha) \]

\[ = 2\alpha \left( -e^{-\alpha T} 2\sqrt{T} \log T - 2\right) - 4\alpha \int_T^\infty e^{-\alpha u} \sqrt{u} du + 2\alpha \int_T^\infty e^{-\alpha u} 2\sqrt{u} \log u du \]

\[ = 2\alpha \left( -e^{-\alpha T} 2\sqrt{T} \log T - 4/\sqrt{T} \int_{\sqrt{T}}^\infty e^{-t^2} dt + 2\alpha \int_T^\infty e^{-\alpha u} 2\sqrt{u} \log u du \right), \]

i.e.

\[ B^1_T(\alpha) = 2(e^{-\alpha T} / \sqrt{T}) \log T - 4e^{-\alpha T} / \sqrt{T} + 4\alpha \sqrt{T} \log Te^{-\alpha T} \]

\[ + 4\alpha^2 \int_T^\infty e^{-\alpha u} \left( \log u \sqrt{u} \right) du. \]

where (in Maple) for the log-integral one can find with Meijer’s G-function.

\[ \int_1^\infty e^{-\alpha u} \left( \log u \sqrt{u} \right) du = (1/\sqrt{a})MeijerG(3, 3/2, \alpha). \]

We get

\[ B_T(\alpha) = 2(e^{-\alpha T} / \sqrt{T}) \log T - 4e^{-\alpha T} / \sqrt{T} + 4\alpha \sqrt{T} \log Te^{-\alpha T} \]

\[ + 4\alpha^2 \int_T^\infty e^{-\alpha u} \left( \log u \sqrt{u} \right) du. \]

\[ - \log T \left( 2e^{-\alpha T} / \sqrt{T} - (4\sqrt{\alpha}) \int_{\sqrt{T}}^\infty e^{-t^2} dt \right) \]

\[ = - 4e^{-\alpha T} / \sqrt{T} + 4\alpha \sqrt{T} \log Te^{-\alpha T} \]

\[ + 4\alpha^2 \int_T^\infty e^{-\alpha u} \left( \log u \sqrt{u} \right) du. \]

\[ - (4\sqrt{\alpha}) \int_{\sqrt{T}}^\infty e^{-t^2} dt) \]
To proceed another way, starting with (C.8.8) one may try the standard estimation

\[ \max_{\gamma \in \Gamma} \gamma z = \max_{(\mu, \nu) = 1} y |\mu z + \nu|^2 \leq \max(1/y, y) \]

which probably is too rough. Further, for \( II_1 \) we may try the estimation from Rankin [Ra] p.135. For \( z = r e^{i\phi} \in F_T \) and \( \mu > 0, \nu \neq 0 \) we have

\[ |\mu z + \nu|^2 \geq 4\mu |\nu| r^2 \sin^2(\epsilon/2) =: C\mu |\nu| r \]

with \( \tan \epsilon = 2T \), i.e. \( \epsilon \to \pi/2 \) for \( T \to \infty \). Using this estimation, one gets

\[
II_1 \leq \sum_{c, d \in \mathbb{Z}, c^2 + d^2 \neq 0} \sum_{\gamma \in \Gamma \setminus \Gamma_1} \int_{-1/2}^{1/2} \int_{-1/2}^{1/2} \int_{-1/2}^{1/2} \int_{1/2}^{1/2} e^{-\pi v(c^2 + d^2) t C|\nu| r/y} dt dy dx dy/y^2.
\]

One may evaluate this in different ways but should finish with a finite \( 1/T \)-sum.

One has

\[
\begin{align*}
II_2 &= \sum_{c, d \in \mathbb{Z}, c^2 + d^2 \neq 0} \sum_{\mu \neq 0} \int_{-1/2}^{1/2} \int_{-1/2}^{1/2} \int_{-1/2}^{1/2} \int_{1/2}^{1/2} e^{-\pi v(c^2 + d^2) t |\mu z|^2 / y} dt dy dx dy/y^2 \\
&= \sum_{c, d \in \mathbb{Z}, c^2 + d^2 \neq 0} \sum_{\mu \neq 0} \int_{-1/2}^{1/2} \int_{-1/2}^{1/2} \int_{-1/2}^{1/2} \int_{1/2}^{1/2} e^{-\pi v(c^2 + d^2) t \mu^2 (x^2 + y^2) / y} dt dy dx dy/y^2 \\
&\leq \sum_{c, d \in \mathbb{Z}, c^2 + d^2 \neq 0} \sum_{\mu \neq 0} \int_{-1/2}^{1/2} \int_{-1/2}^{1/2} \int_{1/2}^{1/2} e^{-\pi v(c^2 + d^2) t \mu^2 y^2 / y} dt dy dx dy/y^2 \\
&\simeq 2\pi \int_{1}^{\infty} \sum_{\mu \neq 0} \int_{1}^{\infty} e^{-\pi v r^2 t \mu^2 y} dr dt dy/y^2 \\
&\simeq 2\pi (1/2) \int_{1}^{\infty} \sum_{\mu \neq 0} \int_{1}^{\infty} e^{-\pi v \mu^2 y} d\mu dt dy/y^2 \\
&\simeq 2\pi (1/2) \int_{1}^{\infty} \sum_{\mu \neq 0} \int_{1}^{\infty} e^{-\pi v \mu^2 y} (\pi v \mu^2 y) dt dy/y^2 \\
&\simeq (1/v) \int_{1}^{\infty} \sum_{\mu \neq 0} \int_{1}^{\infty} e^{-\pi v \mu^2 y} / \mu^2 dt dy/y^3 \\
&\leq (1/v) \zeta(2) \int_{1}^{\infty} dy/y^3 \int_{1}^{\infty} dt/t^2 = (1/v) \zeta(2) (1/T^2)
\end{align*}
\]
which leads to a finite $1/T$–sum.

References


