Universität Hamburg

Fachbereich Mathematik
Algebra und Zahlentheorie
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Algebra 2 Summer Semester 2020 Exercises

Discussion on 09.07.: Ex 10.1 - Ex 10.3
Submit your solutions for the exercises 11.1-11.- by Tuesday, 14.07. Everybody should hand in his/her own solution.

Keywords for the week 06.07.-12.07.: Tensor algebra, universal enveloping algebra.

Exercise 11.2⋆: (5* points)
Let k be a ring, M a k-module and $\mathfrak{g}$ be a Lie algebra over k. Prove that for every associative k-algebra A, we have the following isomorphisms:

$$\begin{array}{ll} (i) & \mathrm{Hom}_{k-\mathrm{mod}}(M, A) \simeq \mathrm{Hom}_{k-\mathrm{alg}}(\mathcal{T}(M), A) \\ (ii) & \mathrm{Hom}_{\mathrm{Lie}}(\mathfrak{g}, \mathrm{Lie}(A)) \simeq \mathrm{Hom}_{k-\mathrm{alg}}(\mathcal{U}(\mathfrak{g}), A) \end{array}$$

Exercise 11.1⋆: (5* points)
Let k be a ring, $\mathfrak{g}$ be a Lie algebra over k and M a $\mathfrak{g}$-module. Show that we have

$$\begin{array}{ll} (i) & H_*(\mathfrak{g}, M) \simeq \mathrm{Tor}_*^{\mathcal{U}(\mathfrak{g})}(k, M) \\ (ii) & H^*(\mathfrak{g}, M) \simeq \mathrm{Ext}^*_{\mathcal{U}(\mathfrak{g})}(k, M) \end{array}$$

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Keywords for the week 29.06.-05.07.: Lie algebra, free Lie algebra, modules over a Lie algebra, Lie algebra (co-)homology.

Exercise 10.3: (6 points)

Let f be the free Lie algebra over a ring k on a set X and M an f-module. Show that:

$$\begin{aligned} &(i) && H_0(f,k) = H^0(f,k) = k \\ &(ii) && H_1(f,k) = \bigoplus_{x\in X} k, \quad H^1(f,k) = \prod_{x\in X} k \\ &(iii) && H_n(f,M) = H^n(f,M) = 0 \ \text{ for all } n \geq 2. \end{aligned}$$

Exercise 10.2: (3+3* points)

Let $\mathfrak{g}$ be a Lie algebra.

a) Prove that the category $\mathfrak{g} - \mathbf{mod}$ of $\mathfrak{g}$-modules is an abelian category.

b*) Show that $\mathfrak{g} - \mathbf{mod}$ has enough projectives and injectives.

Exercise 10.1: (3+3* points)

Let k be a ring and A a associative k-Algebra.

a) Prove that A together with the commutator bracket

$$[a,b] = ab - ba \quad \forall a,b \in A$$

is a Lie Algebra. Deduce that we have a functor

$$\mathrm{Lie} : \mathbf{Alg}_k \to \mathbf{LieAlg}_k.$$

b*) Determine the subgroups of $\mathrm{Gl}_m(k)$ corresponding to the Lie algebras $o_m(k)$, $t_m(k)$, $n_m(k)$ and $sl_m(k)$.

Keywords for the week 22.06.-28.06.: Double complex, total complex, extensions and Baer sum, Yoneda product.

Exercise 9.2: (8 points)

Let R be a ring and M, N be R-modules. Show that

a) $\mathrm{Tor}_k^R(M,N) \simeq \widetilde{\mathrm{Tor}}_k^R(M,N)$ for all $k \geq 0$.

b) $\mathrm{Ext}_R^k(M,N) \simeq \widetilde{\mathrm{Ext}}_R^k(M,N) \simeq \mathrm{Ext'}_R^k(M,N)$ for all $k \geq 0$.

Exercise 9.1: (4*+4 points)

Let $\mathcal{C}$ be an abelian category and $M, N \in \mathrm{ob}(\mathcal{C})$.

a*) Show that for all $n \geq 1$, $\mathrm{Ex}^n(M,N)$ is an abelian group with the composition $+$ defined in the lecture.

b*) Assume that $\mathcal{C}$ has enough projectives. Prove that there is a natural isomorphism of functors $\mathrm{Ex}^n \simeq \mathrm{Ext}^n$.

c) Show that the Yoneda product defined in the lecture in a well-defined, bilinear and associative multiplication.

Keywords for the week 15.06.-21.06.: Resolutions, derived functors, Tor functor, Ext functor.

Exercise 8.3: (6 points)
Let $\mathcal{A}$ be an abelian category with enough projectives and $F : \mathcal{A} \to \mathcal{B}$ a right-exact, additive functor of abelian categories. Suppose we have an exact sequence

$$0 \to M \to P_{m-1} \to P_{m-2} \to \cdots \to P_0 \to A \to 0$$

in $\mathcal{A}$, where all $P_i,\ i = 0, ..., m$, are projective. Show that for $n > m$, there are canonical isomorphisms

$$L_n F(A) \simeq L_{n-m} F(M)$$

and we have an exact sequence

$$0 \to L_m F(A) \to F(M) \to F(P_{m-1}).$$

Formulate and prove the dual statement for left-exact functors and injective objects.

Exercise 8.2: (3+3 points)
Let k be a field and $R = k[x]/(x^n)$, $n \geq 1$. Then we can consider k as a R-module, where x acts by zero on k.

a) Find a projective and injective resolution for the R-module k.

b*) Compute the Tor groups $\mathrm{Tor}_m^R(k, k)$ and the Ext groups $\mathrm{Ext}_R^m(k, k)$ for all $m \geq 0$.

Exercise 8.1: (6 points)
a) Let $m, n \geq 1$. Compute the Ext groups $\mathrm{Ext}_{\mathbb{Z}}^k(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}/m\mathbb{Z})$ for all $k \geq 0$.

b) Let G be a finitely generated, abelian group. Determine the Tor groups $\mathrm{Tor}_k^{\mathbb{Z}}(\mathbb{Q}, G)$ and the Ext groups $\mathrm{Ext}_{\mathbb{Z}}^k(G, \mathbb{Q})$ for all $k \geq 0$.

c) Let R be a principal ideal domain and M, N be R-modules. Show that $\mathrm{Ext}_R^k(N, M) = 0$ for all $k \geq 2$.

Keywords for the week 08.06.-14.06.: Category, opposite category, functors, products and coproducts in a category, additive category, kernels and cokernels in a category, abelian category, projective and injective objects in a category.

Exercise 7.3: (5 points)
Let $\mathcal{C}, \mathcal{D}$ be abelian categories. An *adjunction* $F : \mathcal{C} \leftrightarrow \mathcal{D} : G$ is a pair of functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ equipped with bijections

$$\rho_{A,B} : \mathrm{Hom}_{\mathcal{D}}(F(A), B) \to \mathrm{Hom}_{\mathcal{C}}(A, G(B))$$

for all objects $A \in \mathrm{ob}(\mathcal{C})$, $B \in \mathrm{ob}(\mathcal{D})$, such that for all morphisms $f : A' \to A$ in $\mathcal{C}$ and $g : B \to B'$ in $\mathcal{D}$ the following diagram commutes:

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{D}}(F(A), B) \arrow[r, "-\circ F(f)"'] \arrow[d, "\rho_{A,B}"'] & \mathrm{Hom}_{\mathcal{D}}(F(A'), B) \arrow[r, "g\circ -"'] \arrow[d, "\rho_{A',B}"'] & \mathrm{Hom}_{\mathcal{D}}(F(A'), B') \arrow[d, "\rho_{A',B'}"'] \\
\mathrm{Hom}_{\mathcal{C}}(A, G(B)) \arrow[r, "-\circ f"'] & \mathrm{Hom}_{\mathcal{C}}(A', G(B)) \arrow[r, "G(g)\circ -"'] & \mathrm{Hom}_{\mathcal{C}}(A', G(B'))
\end{tikzcd}$$

The functor F is called a *left adjoint* to G and G is called a *right adjoint* to F.

If F, G are additive functors and all bijections $\rho_{A,B}$ are group isomorphisms, we call the adjunction $F : \mathcal{C} \leftrightarrow \mathcal{D} : G$ *additive.*

(a) Let $F : \mathcal{C} \leftrightarrow \mathcal{D} : G$ be an additive adjunction. Prove that the functor F is right exact and G is left exact.

(b) Let R be a ring and M an R-module. Show that there is an additive adjunction

$$- \otimes_R M : \mathrm{Mod}_R \longleftrightarrow \mathrm{Mod}_R : \mathrm{Hom}(M, -).$$

Exercise 7.2: (5 points)
Let $\mathcal{C}$ be an abelian category. Prove that the projective elements in $\mathcal{C}$ are exactly the injective elements in the opposite category $\mathcal{C}^{\mathrm{op}}$.

Exercise 7.1: (5 points)
Let k be a field and V, W be k-vector spaces.

(a) Show that the definition of the kernel and cokernel of a linear map $f : V \to W$ is compatible with the definition of kernels and cokernels from the lecture.

(b) Prove that the category of k-vector spaces is an abelian category, i.e., show that
(i) every monomorphism is the kernel of a morphism.
(ii) every epimorphism is the cokernel of a morphism.

Keywords for the week 25.05.-31.05.: G-modules, group cohomology.

Exercise 6.3: (6 points)
Let (A^*, d_A), (B^*, d_B) be complexes. We define the complex $(A[1]^*, d_{A[1]})$ by setting $A[1]^n = A^{n+1}$ and $d^n_{A[1]} = -d^{n+1}_A$ for all $n \in \mathbb{Z}$. For a morphism $f^* : A^* \to B^*$ of complexes, the *cone* of f is given by the complex

$$\mathrm{cone}(f) = A[1] \oplus B : \quad \cdots \xrightarrow{d^{n-1}} A^{n+1} \oplus B^n \xrightarrow{d^n} A^{n+2} \oplus B^{n+1} \xrightarrow{d^{n+1}} \cdots,$$

where the differential d is defined by

$$d^n(a, b) = (-d_A(a), f^{n+1}(a) + d_B(b)), \quad (a, b) \in A^{n+1} \oplus B^n.$$

Prove that $f^* : A^* \to B^*$ is a quasi-isomorphism (i.e., the induced morphisms of f^* on the cohomology groups are isomorphisms), iff $\mathrm{cone}(f)$ is an exact complex.

Exercise 6.2: (6 points)
Let $L|K$ be a finite Galois extension and $G = \mathrm{Gal}(L|K)$. Show that $H^1(G, L^\times) = 0$.

Exercise 6.1: (6 points)
Consider the space of homogeneous polynomials $\mathbb{Q}[X, Y]_k$ of weight $k \geq 0$ equipped with the $\mathrm{Gl}_2(\mathbb{Z})$-action given by

$$\mathrm{Gl}_2(\mathbb{Z}) \times \mathbb{Q}[X, Y]_k \to \mathbb{Q}[X, Y]_k, \quad \left(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right), f(X, Y)\right) \mapsto f(aX + bY, cX + dY)$$

and let $G = \left\langle \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) \right\rangle$. Write down the complex $C^*(G, \mathbb{Q}[X, Y]_k)$ and compute the cohomology groups $H^0(G, \mathbb{Q}[X, Y]_k)$ and $H^1(G, \mathbb{Q}[X, Y]_k)$.

Keywords for the week 18.05.-24.05.:Homology, cohomology, de Rham cohomology.

Exercise 5.3: (6 points)
Let Γ be a finite, connected graph with vertices $(e_1, \ldots, e_E)$ and edges $(k_1, \ldots, k_K)$. We endow the edges with an orientation and obtain the incidence matrix $I_\Gamma \in \mathrm{Mat}_{E\times K}(\mathbb{Z})$ of Γ in the following way:

$$(I_\Gamma)_{ij} = \begin{cases} +1 & \text{if } k_j \text{ starts in } e_i \\ -1 & \text{if } k_j \text{ ends in } e_i \\ 0 & \text{else.} \end{cases}$$

Consider the complex C_* given by $C_0 = e_1\mathbb{Z} \oplus \ldots \oplus e_E\mathbb{Z}$, $C_1 = k_1\mathbb{Z} \oplus \ldots \oplus k_K\mathbb{Z}$ and $C_n = 0$ for all $n \geq 2$ endowed with the differential $d_1 : C_1 \to C_0$ given by multiplication with I_Γ and $d_n = 0$ for $n \neq 1$. Prove the following statements:

a) (C_*, d) is a complex.

b) The only nonzero homology modules are $H_0(C_*)$ and $H_1(C_*)$. We have $\mathrm{rk}(H_0(C_*)) = 1$ and $\mathrm{rk}(H_1(C_*)) = K - E + 1$.

c) The number of (simple) closed paths in Γ is $K - E + 1$.

Exercise 5.2: (6 points)
Let $N \in \mathbb{N}$. An *(abstract) simplicial complex* K on $\{0, 1, ..., N\}$ is a collection of subsets of $\{0, 1, ..., N\}$, such that for every $\sigma \in K$ and $\tau \subseteq \sigma$, we have $\tau \in K$. For every $n \geq 0$, define the set of *n-simplices* as

$$K_n := \{\sigma \in K \mid |\sigma| = n + 1\}.$$

Then any n-simplex $\sigma \in K_n$ can be uniquely expressed as $\sigma = \{x_0, x_1, ..., x_n\}$, where $0 \leq x_0 < x_1 < \cdots < x_n \leq N$. For every $i = 0, ..., n$, the *face map* is given by

$$\partial_i : K_n \to K_{n-1},\ \{x_0, x_1, ..., x_n\} \mapsto \{x_0, ..., x_{i-1}, x_{i+1}, ..., x_n\}.$$

For a ring R set

$$C_n(K, R) := \bigoplus_{\sigma \in K_n} Re_\sigma, \quad n \geq 0,$$

i.e., $C_n(K, R)$ is the free R-module on K_n with basis elements labelled by e_σ, and define

$$d_n : C_n(K, R) \to C_{n-1}(K, R),\ e_\sigma \mapsto \sum_{i=0}^{n} (-1)^i e_{\partial_i \sigma}, \quad n \geq 1.$$

a) Verify that

$$\cdots \xrightarrow{d_{n+1}} C_n(K, R) \xrightarrow{d_n} C_{n-1}(K, R) \xrightarrow{d_{n-1}} \cdots \xrightarrow{d_2} C_1(K, R) \xrightarrow{d_1} C_0(K, R) \to 0$$

is a complex. It is called the *simplicial chain complex.*

b) Determine the simplicial chain complex with coefficients in $\mathbb{R}$ for the simplicial complex K on $\{0, 1, 2, 3\}$ given by all subsets of cardinality ≤ 2.
Compute the homology groups $H^n(K, \mathbb{R}) = \ker(d_n)/\operatorname{im}(d_{n+1})$.

c*) For any (geometric) simplicial complex one can construct an abstract simplicial complex by only retaining the sets of vertices. Prove that, conversely, for any finite abstract simplicial complex K one can construct a (geometric) simplicial complex K'.

Exercise 5.1: (6 points)

Let M, N be smooth manifolds.

a*) Show that the dimension of the 0-th de Rham cohomology group $H^0_{dR}(M)$ equals the number of connected components of M.

b*) Let $f, g : M \to N$ be homotopic, smooth maps. Prove that the induced maps $\tilde{f}, \tilde{g} : \Omega^*(N) \to \Omega^*(M)$ of complexes (on $\Omega^p(N)$ they are given by the pullbacks f^*, g^*) are homotopic. In particular, we have $H^p_{dR}(f) = H^p_{dR}(g)$ for all $p \geq 0$.

c) Compute the de Rham cohomology groups for the n-dimensional sphere S^n.
Hint: Use the result for S^1 from the lecture and induction.

d) Let $v, w \in \mathbb{R}^n$. Compute the de Rham cohomology groups for $\mathbb{R}^n \backslash \{v\}$ and $\mathbb{R}^n \backslash \{v, w\}$.

Keywords for the week 11.05.-17.05.: Five Lemma, Snake Lemma, complexes.

Exercise 4.5: (2 points)
Let $0 \to V_1 \to V_2 \to \ldots \to V_n \to 0$ be an exact sequence of finite dimensional vector spaces. Show that $\sum(-1)^i \dim V_i = 0$.

Exercise 4.4: (3 points)
Let $\ldots \longrightarrow M_{r-1} \xrightarrow{\alpha_r} M_r \xrightarrow{\alpha_{r+1}} M_{r+1} \longrightarrow \ldots$ be an exact sequence of modules. Prove that the induced short sequences $0 \to L_r \to M_r \to L_{r+1} \to 0$ are exact, where $L_r = \operatorname{im} \alpha_r$.

Exercise 4.3: (5 points)
Consider the following commutative diagram of R-modules

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & M_1' \arrow[r, "\gamma_1"'] \arrow[d, "\alpha'"'] & M_1 \arrow[r, "\rho_1"'] \arrow[d, "\alpha"'] & M_1'' \arrow[r] \arrow[d, "\alpha''"'] & 0 \\
0 \arrow[r] & M_2' \arrow[r, "\gamma_2"'] \arrow[d, "\beta'"'] & M_2 \arrow[r, "\rho_2"'] \arrow[d, "\beta"'] & M_2'' \arrow[r] \arrow[d, "\beta''"'] & 0 \\
0 \arrow[r] & M_3' \arrow[r, "\gamma_3"'] \arrow[d] & M_3 \arrow[r, "\rho_3"'] \arrow[d] & M_3'' \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

where all of the columns and the middle row are exact. Show that the first row is exact, iff the third row is exact.

Exercise 4.2: (6 points)
Let R be a ring and $I, J \subseteq R$ be ideals.

(a) Prove that there is a short exact sequence of R-modules

$$0 \to I \cap J \to I \oplus J \to I + J \to 0.$$

(b) Use the Snake Lemma to deduce from part (a) an exact sequence

$$0 \to R/(I \cap J) \to R/I \oplus R/J \to R/(I + J) \to 0.$$

(c) Show that the sequences of (a) and (b) are in general not split exact.

Exercise 4.1: (4 points)
Let N be a submodule of a finitely generated R-module M.

(a) Show that N is not finitely generated in general.

(b) Prove that N is finitely generated if it is the kernel of a surjective R-module homomorphism $\Phi : M \to R^n$ for some $n \in \mathbb{N}$.

Hint: Show and use that the sequence $0 \to N \to M \xrightarrow{\Phi} R^n \to 0$ is split exact.

Keywords for the week 04.05.-10.05.: projective modules, flat modules, injective modules.

Exercise 3.4: (6 points)

a) Give an example for a short exact sequence

$$0 \to A \to B \to C \to 0$$

of R-modules, which do not split.

b) Can you give an example as in part a), such that only A resp. B resp. C is free?

c) Is there an example as in part a), such that two resp. three of the modules A, B, C are free?

Exercise 3.3: (6 points)

Prove or disprove the following statements:

a) Projective modules are flat.

b) Flat modules are projective.

c) Free modules are projective.

d) Flat modules are free.

e) Free modules are injective.

Exercise 3.2: (6 points)

Show that for a R-module M the following conditions are equivalent:

i) For every exact sequence $0 \to N_1 \to N_2$ of R-modules and any module morphism $f : N_1 \to M$, there exists a morphism $g : N_2 \to M$, such that the following diagram commutes:

$$\begin{tikzcd} 0 \arrow[r] & N_1 \arrow[r] \arrow[d, "f"'] & N_2 \arrow[dl, "g"] \\ & M & \end{tikzcd}$$

ii) Every short exact sequence $0 \to M \to N_1 \to N_2 \to 0$ of R-modules splits.

iii) For every short exact sequence $0 \to T' \to T \to T'' \to 0$ of R-modules, also the sequence

$$0 \to \mathrm{Hom}(T'', M) \to \mathrm{Hom}(T, M) \to \mathrm{Hom}(T', M) \to 0$$

is exact.

A module M satisfying those properties is called an injective module.

Exercise 3.1: (6 points)

Let $R = \mathbb{Z}[\sqrt{-5}] = \{a + b\sqrt{-5} \in \mathbb{C} \mid a, b \in \mathbb{Z}\}$. Further, let A_1 be the ideal generated by $1 + \sqrt{-5}$ and 2 in R and A_2 the ideal generated by 2.

Which of the R-modules A_1, A_2, A_1A_1 and A_1A_2 are projective resp. flat?

Keywords for the week 27.04.-03.05.: Tensor product, tensor algebra, localisation.

Exercise 2.5: (2 points)
Let R be a principal ideal domain and K its field of fractions, i.e., we have $K = S^{-1}R$ where $S = R\backslash\{0\}$. Show, if M is a finitely generated R-module, then we have $S^{-1}M \cong K^n$ where n is the rank of $M/T(M)$.

Exercise 2.4: (2 + 2* points)
a) Let A be an integral domain and $S \subset A$ a multiplicative subset. Show that the assignments

$$\mathfrak{q} \to \mathfrak{q}S^{-1} \text{ and } \mathfrak{Q} \to \mathfrak{Q} \cap A$$

give a 1-1-correspondence between the prime ideals $\mathfrak{q} \subseteq A \setminus S$ of A and the prime ideals $\mathfrak{Q}$ of AS^{-1}.

b*) Let A be an integral domain and $\mathfrak{p}$ a prime ideal of A. Prove that the ring $S^{-1}A$, where $S = A \setminus \mathfrak{p}$, is a local ring.

Exercise 2.3: (2 points)
Let V be a k-vector space of dimension n. Show that the tensor algebra $T(V)$ is isomorphic to the free algebra $k\langle X\rangle$ over an alphabet $X = \{x_1, ..., x_n\}$ with n letters.
Tip: See L. Foissy - Algebres de Hopf combinatoires.

Exercise 2.2: (4 + 2 + 2* points)
Let $(*)\,:\, 0 \longrightarrow M' \xrightarrow{f} M \xrightarrow{g} M'' \longrightarrow 0$ be a short exact sequence of A-modules. Prove the following statements:

a) For every A-module F, the sequence ("tensoring with $F \otimes_A -$")

$$F \otimes_A M' \xrightarrow{f} F \otimes_A M \xrightarrow{g} F \otimes_A M'' \longrightarrow 0$$

is also exact.

b) If $(*)$ splits, then for every A-module F, the sequence

$$0 \longrightarrow F \otimes_A M' \xrightarrow{f} F \otimes_A M \xrightarrow{g} F \otimes_A M'' \longrightarrow 0$$

is also exact.

c*). What changes, if we apply $- \otimes_A F$ to $(*)$?

Exercise 2.1: (8 points)
Let $(*) : 0 \longrightarrow M' \xrightarrow{f} M \xrightarrow{g} M'' \longrightarrow 0$ be a short exact sequence of A-modules.

a) We say that $(*)$ splits if one of the following equivalent conditions i)-iii) is satisfied:

i) There is a morphism $\phi \in \mathrm{Hom}_A(M'', M)$, such that $g \circ \phi = \mathrm{id}$ ("g has a section").

ii) There is a morphism $\psi \in \mathrm{Hom}_A(M, M')$, such that $\psi \circ f = \mathrm{id}$ ("f has a retract").

iii) There is an isomorphism $h : M \to M' \oplus M''$, such that the following diagram commutes

$$\begin{tikzcd}
0 \arrow[r] & M' \arrow[r, "f"] \arrow[d, "\mathrm{id}"] & M \arrow[r, "g"] \arrow[d, "h"] & M'' \arrow[r] \arrow[d, "\mathrm{id}"] & 0\,, \\
0 \arrow[r] & M' \arrow[r, "\iota_1"] & M' \oplus M'' \arrow[r, "\mathrm{pr}_2"] & M'' \arrow[r] & 0
\end{tikzcd}$$

where ι_1 is the canonical embedding and pr_2 is the canonical projection onto the second component.
Show that i) -iii) are equivalent.

b) Prove, if $(*)$ splits, then there are isomorphisms $M \cong \mathrm{im}\, f \oplus \ker \psi$ and $M \cong \ker g \oplus \mathrm{im}\, \phi$.

Exercise 2.0 = old Exercise 1.3 (3 points)

Compute the following tensor products:

a) $\mathbb{Z}/n\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/m\mathbb{Z}$.

b) $\mathbb{Z}/n\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Q}$.

c) $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{R}$.

Keywords for the week 20.04.-26.04.: Rings, integral domain, units, modules, direct sum, direct product, free modules, rank, isomorphism theorems.

Exercise 1.6: (6 points)

a) Let $f : A \to B$ be a ring morphism. Show that B is an A-module.

b) Show that any module is a $\mathbb{Z}$-module.

Exercise 1.5: (2 points)

Prove or disprove the following statements

a) The direct product of rings $\prod_{i \in I} R_i$ is again a ring.

b) The direct sum of rings $\bigoplus_{i \in I} R_i$ is again a ring.

Exercise 1.4: (8 points)

Prove the isomorphism theorems:

a) Let $f : M \to N$ be a R-module morphism. Then the is a canonical isomorphism $M/\ker f \cong \operatorname{im} f$.

b) Let $N, N' \subset M$ be submodules. Then there exists a canonical isomorphism $(N + N')/N \cong N'/(N \cap N')$.

c) If $N' \subset N \subset M$ are submodules, then we have $(M/N')/(N/N') \cong M/N$.

d) Compute the above statements a)-c) for $N' = 6\mathbb{Z}$, $N = 3\mathbb{Z}$ and $M = \mathbb{Z}$.

Exercise 1.3: (moved to next week)

Exercise 1.2*: (3 points)

Let $R = \mathbb{Z}[\sqrt{-5}] = \{a + b\sqrt{-5} \in \mathbb{C} \mid a, b \in \mathbb{Z}\}$. Further, let A_1 be the ideal generated by $1 + \sqrt{-5}$ and 2 in R and A_2 the ideal generated by 2.

Compute the R-modules $A_1 + A_2$, A_1A_1 and A_1A_2. Which of these modules is free?

Exercise 1.1*: (2 points)

Let k be a field. Prove that a $k[X]$-module M is equivalent to a k-vector space V equipped with an endomorphism $\phi : V \to V$.
Tip: Multiplying by $\sum a_i x^i$ on M is equivalent to applying $\sum a_i \phi^i$ on V.