Néron-Tate heights on algebraic curves and subgroups of the modular group

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Abstract. Combining Arakelov theory with Belyi's theorem we derive that the values of the Néron-Tate height pairing for divisors on algebraic curves defined over number fields are essentially given by linear combinations of scattering constants associated to finite index subgroups of the modular group $\mathrm{PSL}_2(\mathbb{Z})$.

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0.1. Introduction. The main motivation to study the image of the Néron-Tate height pairing of degree zero divisors on algebraic curves is that these real numbers are conjecturally related to special values of L-functions coming from arithmetic. The most important result of this general picture is the theorem of Gross and Zagier which relates the Néron Tate height of Heegner divisors to the derivative of the Hasse-Weil L-function of certain elliptic curves [GZ]. In this article we relate Néron-Tate heights to scattering constants associated to finite index subgroups of the full modular group. For simplicity, we describe all results in the introduction for curves defined over $\mathbb{Q}$.

Let Γ be a finite index subgroup of $\Gamma(1) = \mathrm{SL}_2(\mathbb{Z})/\{\pm 1\}$ and let $\mathfrak{h}$ be the upper complex half plane. We denote by $X(\Gamma) = \overline{\Gamma \setminus \mathfrak{h}}$ the general modular curve associated to Γ. For each cusp S_j let b_j denotes its width and let $\gamma_j \in \Gamma(1)$ be such that $\gamma_j(S_j) = \infty$. Then the *scattering constants* C_{jk} for Γ are the following real numbers

$$C_{jk} := \lim_{s \to 1} \left(\frac{\pi^{1/2}}{(b_j b_k)^s} \frac{\Gamma(s-1/2)}{\Gamma(s)} \sum_{c>0} \left(\sum_{d \,(\mathrm{mod}\, b_k c)}^{*} 1 \right) \frac{1}{c^{2s}} - \frac{3/(\pi \cdot [\Gamma(1):\Gamma])}{s-1} \right);$$

here the inner sum is taken over c, d such that there exist $\left(\begin{smallmatrix} * & * \\ c & d \end{smallmatrix}\right) \in \gamma_j^{-1} \Gamma \gamma_k$. The union of the set of scattering constants for Γ as Γ runs over all subgroups of finite index in $\Gamma(1)$ is a countable subset of $\mathbb{R}$.

Let $X_{\mathbb{Q}}$ be a geometrically irreducible curve defined over $\mathbb{Q}$. Let $\boldsymbol{\beta} : X_{\mathbb{Q}} \to \mathbb{P}^1_{\mathbb{Q}}$ be a Belyi morphism and let $X_{\mathbb{Q}}(\mathbb{C}) \cong X(\Gamma)$ be the induced Belyi uniformization. A divisor D on $X_{\mathbb{Q}}$ is called cuspidal if its degree is zero and if the support of the induced divisor $D(\mathbb{C})$ on $X(\Gamma)$ is contained in the cusps. The group generated by cuspidal divisors will be denoted by $\mathrm{Cusp}(X_{\mathbb{Q}}, \boldsymbol{\beta})$. We note that given a finite set of points on $X_{\mathbb{Q}}$ one can find a Belyi morphism $\boldsymbol{\beta}$ such that the induced points on $X(\Gamma)$ are cusps.

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Combining the Faltings-Hriljac theorem with the extension of Arakelov theory given in [Kü], we obtain as special case of our main results:

Theorem. *Let $\boldsymbol{\beta} : X_{\mathbb{Q}} \to \mathbb{P}^1_{\mathbb{Q}}$ be a Belyi morphism with induced Belyi uniformization $X(\mathbb{C}) \cong X(\Gamma)$. Let $D_1 = \sum_j n_j S_j$, $D_2 = \sum_k m_k S_k$ be cuspidal divisors.*

(i) The Néron-Tate height pairing of D_1 and D_2 equals

$$\langle D_1, D_2 \rangle_{NT} = - \sum_{p\,\mathrm{prime}} \delta_p \log(p) - 2\pi \sum_{j,\,k} n_j m_k C_{jk},$$

here the coefficients δ_p are rational numbers.

(ii) If $\mathrm{Cusp}(X_{\mathbb{Q}}, \boldsymbol{\beta})$ generates a torsion subgroup in the Jacobian of $X_{\mathbb{Q}}$, then

$$C_{jk} = \frac{12}{\deg(\boldsymbol{\beta})} \big(12\zeta'_{\mathbb{Q}}(-1) - 1 + \log(4\pi) \big) + \sum_{p\,\mathrm{prime}} a_{p,jk} \log(p),$$

here the coefficients $a_{p,jk}$ are rational numbers.

We remark that if a regular, semi-stable model of $X(\Gamma)$ is known explicitly, then the rational numbers in question can be calculated explicitly. Moreover if Γ is a normal subgroup of $\Gamma(1)$, then for all primes p with $p \not| [\Gamma(1) : \Gamma]$ the coefficients $a_{p,jk}$ vanish.

Recall, if $X(\Gamma)$ is the modular curve for a congruence subgroup Γ, then by the Manin-Drinfeld Theorem some multiple of any cuspidal divisor is a principal divisor. We will show below that the scattering constants for congruence subgroups are always of type as in (ii). We also illustrate our results in the case of the modular curves $X_0(p)$, see Example 4.13.

If one of the cuspidal divisors D_1 or D_2 is torsion in $\mathrm{Jac}(X_{\mathbb{Q}})$, then their Néron-Tate height pairing vanishes. Thus (i) implies

$$\exp\left(-2\pi \sum_{j,\,k} n_j m_k C_{jk} \right) = \prod_{p\,\mathrm{prime}} p^{\delta_p},$$

here the exponents δ_p are rational numbers. Related to this observation is a theorem of A. J. Scholl who has observed that these assumptions on D_1 resp. D_2 are equivalent to the algebraicity of the positive Fourier coefficients of certain linear combinations of non-holomorphic Eisenstein series [Sc]. We note that our result gives a necessary condition for D_1 or D_2 to be torsion which involves the constant term of these Fourier expansion.

0.2. Leitfaden. In section one we recall the extension of Arakelov theory used in the sequel. In section two we consider non-holomorphic Eisenstein series. We use them to construct Green's functions associated to the cusps. In section three we start recalling facts on Belyi's theorem and introduce models for modular curves. In section four we use the Green's functions of section two to obtain hermitian, logarithmically singular line bundles associated to the cusps. Then we calculate the analytic contribution of the generalized arithmetic intersection numbers for these hermitian line bundles, which is the main new ingredient needed to prove our results. We finish with an example and suggestions for further research.

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1 Review of Arakelov theory for arithmetic surfaces

1.1. Notation. In order to fix notation we recall some aspects of Arakelov Theory (see e.g. [Kü], [So])

An arithmetic surface $\mathcal{X}$ is a regular scheme of dimension 2 together with a projective flat morphism $f : \mathcal{X} \to \operatorname{Spec} \mathcal{O}_E$, where $\mathcal{O}_E$ is the ring of integers of E. Moreover we assume that the generic fiber X_E of f is a geometrically irreducible, i.e., $\mathcal{X}$ is a regular model for X_E over $\operatorname{Spec} \mathcal{O}_E$.

We denote by $\mathcal{X}_\infty(\mathbb{C})$ the smooth projective manifold $\coprod_{\sigma:E\to\mathbb{C}} \mathcal{X}_\sigma(\mathbb{C})$.

1.2. Definitions. Let $\mathcal{X}$ be an arithmetic surface and $\mathcal{X}_\infty$ as above; for a finite set $\mathcal{S}$ of points $S_1, ..., S_r$ of $\mathcal{X}_\infty$ denote by $\mathcal{Y}_\infty$ the open complex manifold $\mathcal{X}_\infty \setminus \mathcal{S}$. For $S_j \in \mathcal{S}$ and $\varepsilon > 0$, denote by $B_\varepsilon(S_j) \subseteq \mathcal{X}_\infty$ the open disk of radius ε centered at S_j and $\mathcal{X}_\varepsilon = \mathcal{X}_\infty \setminus \cup_{S_j\in\mathcal{S}} B_\varepsilon(S_j)$; let t be a local parameter at S_j $(j = 1, ..., r)$. For a line bundle $\mathcal{L}$ on $\mathcal{X}$, a singular metric h on the induced complex line bundle $\mathcal{L}_\infty$ on $\mathcal{X}_\infty$ is called *hermitian, logarithmically singular (with respect to $\mathcal{S}$)*, if the following two conditions hold:

(a) h is a smooth, hermitian metric on $\mathcal{L}_\infty$ restricted to $\mathcal{Y}_\infty$;

(b) for each $S_j \in \mathcal{S}$ and any section l of $\mathcal{L}$, there exist a real number α and a positive, continuous function φ defined on $B_\varepsilon(S_j)$ and smooth away from the origin such that the equality

$$\|l(t)\| = -\log(|t|^2)^\alpha \cdot |t|^{\operatorname{ord}_{S_j}(l)} \cdot \varphi(t)$$

holds for all $t \in B_\varepsilon(S_j) \setminus \{0\}$; furthermore, there exist positive constants β and ρ such that the inequalities

$$\left|\frac{\partial\varphi(t)}{\partial t}\right| \leq \frac{\beta}{|t|^{1-\rho}}, \quad \left|\frac{\partial\varphi(t)}{\partial \bar{t}}\right| \leq \frac{\beta}{|t|^{1-\rho}}, \quad \left|\frac{\partial^2\varphi(t)}{\partial t\,\partial \bar{t}}\right| \leq \frac{\beta}{|t|^{2-\rho}}$$

hold for all $t \in B_\varepsilon(S_j) \setminus \{0\}$.

We call a line bundle $\mathcal{L}$ on $\mathcal{X}$ equipped with a logarithmically singular metric h a *hermitian, logarithmically singular line bundle* and denote it by $\overline{\mathcal{L}} = (\mathcal{L}, h)$. To indicate the dependence of the quantities α (resp. φ, β, ρ) on $l, \overline{\mathcal{L}}$ and $S_j \in \mathcal{S}$, we write instead $\alpha_{\overline{\mathcal{L}},j}$ (resp. $\varphi_{\overline{\mathcal{L}},j}, \beta_{\overline{\mathcal{L}},j}, \rho_{\overline{\mathcal{L}},j}$).

Two hermitian, logarithmically singular line bundles $\overline{\mathcal{L}}, \overline{\mathcal{M}}$ on $\mathcal{X}$ are *isomorphic*, if

$$\overline{\mathcal{L}} \otimes \overline{\mathcal{M}}^{-1} \cong (\mathcal{O}_\mathcal{X}, |\cdot|).$$

The *generalized arithmetic Picard group*, denoted by $\widehat{\mathrm{Pic}}(\mathcal{X},\mathcal{S})$, is the group of isomorphy classes of hermitian, logarithmically singular line bundles $\overline{\mathcal{L}}$ on $\mathcal{X}$ the group structure being given by the tensor product. Note, if $\mathcal{S}=\emptyset$, then $\widehat{Pic}(\mathcal{X},\emptyset)$ coincides with the classical arithmetic Picard group $\widehat{Pic}(\mathcal{X})$.

1.3. Definition. Let $\overline{\mathcal{L}},\overline{\mathcal{M}}$ be two hermitian, logarithmically singular line bundles on $\mathcal{X}$ and l,m (resp.) be non-trivial, global sections, whose induced divisors on $\mathcal{X}_\infty$ have no points in common. Then, the *generalized arithmetic intersection number* $\overline{\mathcal{L}}.\overline{\mathcal{M}}$ *of* $\overline{\mathcal{L}}$ *and* $\overline{\mathcal{M}}$ is given by

$$\overline{\mathcal{L}}.\overline{\mathcal{M}}:=(l.m)_{\mathrm{fin}}+\langle l.m\rangle_\infty; \tag{1.3.1}$$

here $(l.m)_{\mathrm{fin}}$ is defined by Serre's Tor-formula, which for l,m having proper intersection specializes to

$$(l.m)_{\mathrm{fin}}=\sum_{x\in\mathcal{X}}\log\sharp\left(\mathcal{O}_{\mathcal{X},x}/(l_x,m_x)\right),$$

where l_x, m_x are the local equations of l, m respectively at the point $x\in\mathcal{X}$ and

$$\begin{aligned}&\langle l.m\rangle_\infty=\\&-(\log\|m\|)\big[\mathrm{div}(l)-\sum_{j=1}^{r}\mathrm{ord}_{S_j}(l)\cdot S_j\big]+\sum_{j=1}^{r}\mathrm{ord}_{S_j}(l)\Big(\alpha_{\overline{\mathcal{M}},j}-\log(\varphi_{\overline{\mathcal{M}},j}(0))\Big)-\\&\lim_{\varepsilon\to 0}\left(\sum_{j=1}^{r}\mathrm{ord}_{S_j}(l)\cdot\alpha_{\overline{\mathcal{M}},j}\cdot\log(-\log\varepsilon^2)+\int_{\mathcal{X}_\varepsilon}\log\|l\|\cdot c_1(\overline{\mathcal{M}})\right).\end{aligned}\tag{1.3.2}$$

Note, in formula (1.3.2) the points $P_i\in\mathcal{S}$ with $\alpha_i=0$ behave like the metric where smooth. In [Kü] we proved:

1.4. Proposition. *The formula* (1.3.1) *induces a bilinear, symmetric pairing*

$$\widehat{\mathrm{Pic}}(\mathcal{X},\mathcal{S})\times\widehat{\mathrm{Pic}}(\mathcal{X},\mathcal{S})\longrightarrow\mathbb{R}$$

extending the pairing of Arakelov. □

1.5. Remark. We remark that the normalized generalised arithmetic intersection number $1/[E:\mathbb{Q}]\cdot\overline{\mathcal{L}}.\overline{\mathcal{M}}$ is invariant under extension of scalars from E to a finite extension of it.

1.6. Definition. We put $\widehat{Pic}(\mathcal{X},\mathcal{S})_\mathbb{Q}=\widehat{Pic}(\mathcal{X},\mathcal{S})\otimes_\mathbb{Z}\mathbb{Q}$. Let $\widehat{Pic}^0(\mathcal{X},\mathcal{S})_\mathbb{Q}\subset\widehat{Pic}(\mathcal{X},\mathcal{S})_\mathbb{Q}$ denote the subgroup generated by the hermitian line bundles $\overline{\mathcal{L}}=(\mathcal{L},\|\cdot\|)$ satisfying $\deg(\mathcal{L}|\mathcal{C}_\mathfrak{p}^{(l)})=0$ for all irreducible components $\mathcal{C}_\mathfrak{p}^{(l)}$ of the fiber $f^{-1}(\mathfrak{p})$ above $\mathfrak{p}\in\operatorname{Spec}\mathcal{O}_E$, and $\mathrm{c}_1(\overline{\mathcal{L}})=0$. Let $\widehat{Pic}^0(\mathcal{X})_\mathbb{Q}$ be the corresponding subgroup of $\widehat{Pic}(\mathcal{X})_\mathbb{Q}$ considered in classical Arakelov theory.

1.7. Proposition. *We have an equality*

$$\widehat{Pic}^0(\mathcal{X},\mathcal{S})_\mathbb{Q}=\widehat{Pic}^0(\mathcal{X})_\mathbb{Q}.\tag{1.7.1}$$

Proof. Let $(\mathcal{L}, \|\cdot\|) \in \widehat{Pic}^0(\mathcal{X}, \mathcal{S})_{\mathbb{Q}}$. Since $c_1(\overline{\mathcal{L}}) = 0$ we note that by [Kü], proposition 3.3, together with [Gr], formula (3.4), the hermitian metric $\|\cdot\|$ is in fact smooth on $\mathcal{X}_\infty$. It is unique up to multiplication by a scalar. Therefore, $\widehat{Pic}^0(\mathcal{X}, \mathcal{S})_{\mathbb{Q}}$ does not depend on $\mathcal{S}$ and hence coincides with $\widehat{Pic}^0(\mathcal{X})_{\mathbb{Q}}$. $\square$

1.8. Proposition. *Let D be a divisor on X_E with $\deg D = 0$. Then there exist a divisor $\mathcal{D}$, which may have rational coefficients, on $\mathcal{X}$ satisfying $\mathcal{D}_E = D$ and a hermitian metric $\|\cdot\|$ on $\mathcal{O}(\mathcal{D})_\infty$ such that*

$$\overline{\mathcal{O}(\mathcal{D})} = (\mathcal{O}(\mathcal{D}), \|\cdot\|) \in \widehat{Pic}^0(\mathcal{X})_{\mathbb{Q}}.$$

The divisor $\mathcal{D}$ is unique up to multiples of the fibers of f and the metric $\|\cdot\|$ is unique up to multiplication by scalars.

Proof. For the existence of $\mathcal{D}$ we refer to lemme 6.14.1 in [MB1], the existence of $\|\cdot\|$ follows from formula (3.4) in [Gr]. $\square$

1.9. Proposition. (Faltings-Hriljac) *Let $\mathcal{X} \to \operatorname{Spec} \mathcal{O}_E$ be a semi-stable arithmetic surface. Let D_1, D_2 be divisors on X_E with $\deg D_1 = \deg D_2 = 0$. Then, for all extensions $\overline{\mathcal{O}(\mathcal{D}_1)}, \overline{\mathcal{O}(\mathcal{D}_2)}$ of $\mathcal{O}(D_1)$, $\mathcal{O}(D_2)$ to $\widehat{Pic}^0(\mathcal{X})_{\mathbb{Q}}$, there is an equality*

$$-\langle D_1, D_2 \rangle_{NT} = \frac{1}{[E:\mathbb{Q}]} \cdot \overline{\mathcal{O}(\mathcal{D}_1)}.\overline{\mathcal{O}(\mathcal{D}_2)}, \tag{1.9.1}$$

where $\langle D_1, D_2 \rangle_{NT}$ is the Néron-Tate height pairing of the induced classes in the Picard group of X_E and $\overline{\mathcal{O}(\mathcal{D}_1)}.\overline{\mathcal{O}(\mathcal{D}_2)}$ denotes the generalized arithmetic intersection number (1.3.1).

Proof. By our definition of $\widehat{Pic}^0(\mathcal{X})_{\mathbb{Q}}$ the statement follows immediately from the Faltings-Hriljac formula (see [Fa], [MB1]). $\square$

2 On non-holomorphic Eisenstein series

2.1. Notation. Let $\mathfrak{h} = \{\tau = x + iy \,|\, y > 0\}$ be the upper half plane. Let $\Gamma(1) = \mathrm{SL}_2(\mathbb{Z})/\{\pm 1\}$ be the modular group and let $\Gamma \subseteq \Gamma(1)$ be a subgroup of finite index. The open Riemann surface $Y(\Gamma) = \Gamma \setminus \mathfrak{h}$ can be compactified by adding the finite set of cusps. The resulting compact Riemann surface will be denoted by $X(\Gamma)$ and called a *general modular curve*. If Γ is a congruence subgroup, i.e., $\Gamma(N) \subseteq \Gamma$ for some level N, then $X(\Gamma)$ will be called simply a modular curve.

Let $\mathcal{S} = \{S_1 = \infty, \ldots, S_h\}$ be a complete set of cusps for $X(\Gamma)$. For each $S_j \in \mathcal{S}$ we let Γ_j be its stabilizer in Γ. We fix also an element $\sigma_j \in \mathrm{PSL}_2(\mathbb{R})$ such that $\sigma_j(\infty) = S_j$ and

$$\sigma_j^{-1} \Gamma_j \sigma_j = \left\{ \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix} \,|\, m \in \mathbb{Z} \right\}.$$

Let $\gamma_j \in \Gamma(1)$ be such that $\gamma_j(S_j) = \infty$, then we may take $\sigma_j = \gamma_j \cdot \begin{pmatrix} \sqrt{b_j} & 0 \\ 0 & 1/\sqrt{b_j} \end{pmatrix}$, where b_j is the width of the cusp S_j. Let $\tau \in \mathfrak{h}$, then a local parameter for the cusp S_j, considered as a point on the compact Riemann surface $X(\Gamma)$, is given by $t_j = \exp(2\pi i \sigma_j^{-1}(\tau))$; for a more detailed description of the complex structure of $X(\Gamma)$ we refer to [Kü].

2.2. Definition. For each cusp S_j there is a *non-holomorphic Eisenstein series* $E_j(\tau; s)$, which, for $s \in \mathbb{C}$, $\mathrm{Re} s > 1$, is defined by the convergent series

$$E_j(\tau; s) = \sum_{\sigma \in \Gamma_j \backslash \Gamma} \mathrm{Im}\left(\sigma_j^{-1}\sigma(\tau)\right)^s.$$

2.3. Properties. Let us recall some facts on the theory of Eisenstein series the standard reference is [Ku]. For all $j = 1, \ldots, h$ the function $E_j(\tau; s)$ has a meromorphic continuation to the s-plane, with a simple pole in $s = 1$ with residue $3/(\pi \cdot [\Gamma(1) : \Gamma])$. For all $\gamma \in \Gamma$ we have $E_j(\gamma(\tau); s) = E_j(\tau; s)$. It is an eigenfunction of the hyperbolic Laplacian with eigenvalue $s(s-1)$, i.e.,

$$\triangle E_j(\tau; s) = s(s-1)E_j(\tau; s),$$

where $\triangle = y^2\left(\partial^2/\partial x^2 + \partial^2/\partial y^2\right)$ is the hyperbolic Laplacian. The Fourier expansion of $E_j(\tau; s)$ at the cusp S_k is given by

$$E_j(\sigma_k(\tau; s)) = \sum_{n \in \mathbb{Z}} a_{jk,n}(y; s) \exp(2\pi i n x),$$

where

$$\begin{aligned} a_{jk,0}(y; s) &= \delta_{j,k} \cdot y^s + \phi_{jk,0}(s)\pi^{1/2}\frac{\Gamma(s-1/2)}{\Gamma(s)} \cdot y^{1-s}, \\ a_{jk,n}(y; s) &= \phi_{jk,n}(s) K_s(2\pi|n|y) \qquad (n \neq 0); \end{aligned}$$

here $\Gamma(s)$ is the gamma function, $K_s(t)$ the K-Bessel function of rapid decay and

$$\phi_{jk,n}(s) = \frac{1}{(b_j b_k)^s} \sum_{c>0} \left(\sideset{}{'}\sum_{d \bmod b_k c} \exp(2\pi i n d / b_k c) \right) \frac{1}{c^{2s}},$$

here the sum is taken over c, d such that there exists $\left(\begin{smallmatrix} * & * \\ c & d \end{smallmatrix}\right) \in \gamma_j^{-1}\Gamma\gamma_k$. Then the *scattering matrix*

$$\Phi_\Gamma(s) = \left(\pi^s \frac{\Gamma(s-1/2)}{\Gamma(s)} \cdot \phi_{jk,0}(s) \right)_{j,k}$$

is symmetric. Note all the coefficients of the scattering matrix are Dirichlet series in a general sense. They have a meromorphic continuation with a simple pole in $s = 1$ of residue $3/(\pi \cdot [\Gamma(1) : \Gamma])$.

2.4. Definition. For all pairs j,k we define the *scattering constant* C_{jk} to be the constant term at 1 of the Dirichlet series $(\Phi_\Gamma)_{jk}(s)$, i.e.,

$$C_{jk} := \lim_{s\to 1}\left(\Phi_\Gamma(s)_{j,k} - \frac{3/(\pi\cdot[\Gamma(1):\Gamma])}{s-1}\right). \tag{2.4.1}$$

2.5. Remark. (i) Observe that not any real number can be a scattering constant associated with a finite index subgroup Γ of $\Gamma(1)$. Indeed, the set of all such scattering constants is countable. Because the number of cusps on $X(\Gamma)$ is bounded by the index $[\Gamma(1):\Gamma]$ and there is for large N the asymptotic $\#\{\Gamma\subseteq\Gamma(1)\,|\,[\Gamma(1):\Gamma]=N\}\sim\tau_2(N)\tau_3(N)/(N-1)!$, where τ_m, $m=2,3$, is given by the generating series $\sum_{N=0}^{\infty}\frac{\tau_m(N)}{N!}x^N = \exp\left(\sum_{d|m}\frac{x^d}{d}\right)$ (see [Ve], Appendix 2).

(ii) If Γ is a congruence subgroup of certain type, then the functions $(\Phi_\Gamma)_{jk}(s)$ are determined by explicit formulas involving the the Riemann zeta function $\zeta_{\mathbb{Q}}(s)$ and Dirichlet L-series $L(s,\chi)$ for even characters (see e.g. [He], [Hu], or [Mo]). For these congruence subgroups Γ (cf. Example 4.13) one can in principle calculate the scattering constants explicitly. Our Theorem 4.8 below will in particular give a structural description for the scattering constants for all congruence subgroups.

(iii) For arbitrary finite index subgroups Γ, at least to the knowledge of the author, no such explicit formulas for the functions $(\Phi_\Gamma)_{jk}(s)$ or for the scattering constants are known. In particular for subgroups, which are given by generators and relations, new methods are needed to calculate the scattering constants.

2.6. Definition. Let Γ be a finite index subgroup of $\Gamma(1)$. Let S_j be a cusp of $X(\Gamma)$, then we define the function $g_j(\tau)$ by

$$g_j(\tau) := 4\pi\lim_{s\to 1}\Big(E_j(\tau;s) - (\Phi_\Gamma)_{j,j}(s)\Big) - \frac{12}{[\Gamma(1):\Gamma]}\log(4\pi),$$

and we denote by g_j the function on $X(\Gamma)$ induced by $g_j(\tau)$.

We denote the normalized curvature form associated to the hyperbolic metric on $\mathfrak{h}$ by ω, i.e.,

$$\int_{X(\Gamma)}\omega = \int_{\Gamma\backslash\mathfrak{h}}\frac{dxdy}{4\pi y^2} = \frac{[\Gamma(1):\Gamma]}{12}.$$

The following Proposition shows that g_j is a Green's function for the cusp S_j, which is admissible to the hyperbolic metric.

2.7. Proposition. *The function g_j is invariant under complex conjugation and satisfies the equality*

$$dd^c g_j + \delta_{S_j} = \frac{12}{[\Gamma(1):\Gamma]}\cdot\omega.$$

Proof. The Fourier expansion of $E_j(\tau; s)$ implies that for $\tau \in \mathfrak{h}$ the Fourier expansion of $g_j(\tau)$ at S_j is given by

$$g_j(\sigma_j(\tau)) = 4\pi y - \frac{12}{[\Gamma(1):\Gamma]} \log(4\pi y) + \sum_{m\neq 0} a_{jj,m}(y;1) \exp(2\pi i m x).$$

At the cusp S_k it is given by

$$g_j(\sigma_k(\tau)) = \frac{-12}{[\Gamma(1):\Gamma]} \log(4\pi y) + 4\pi(C_{jk} - C_{jj}) + \sum_{m\neq 0} a_{jj,m}(y;1) \exp(2\pi i m x).$$

Now, using the identity $K_{1/2}(x) = \sqrt{\pi/(2x)} \exp(-x)$, we get that $g_j(\tau)$ is a smooth function on $\mathfrak{h}$ which is invariant under the complex conjugation F_∞. Furthermore, since $E_j(\tau)$ is an eigenfunction of the hyperbolic Laplacian, we derive that for all $\tau \in \mathfrak{h}$ the equality

$$dd^c g_j(\tau) = \frac{1}{4\pi} \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) g_j(\tau) = \frac{12}{[\Gamma(1):\Gamma]} \cdot \frac{1}{4\pi y^2}$$

holds. At the cusps S_j and S_k we have in the corresponding local parameter

$$g_j(t_j) = -\log|t_j|^2 - \frac{12}{[\Gamma(1):\Gamma]} \log(-\log|t_j|^2) + f_{jj}(t_j), \tag{2.7.1}$$

$$g_j(t_k) = -\frac{12}{[\Gamma(1):\Gamma]} \log(-\log|t_k|^2) + f_{jk}(t_k), \tag{2.7.2}$$

where f_{jj} and f_{jk} are continuous functions, smooth outside outside the elliptic fixed points. They have the special values

$$f_{jj}(0) = 0 \tag{2.7.3}$$

$$f_{jk}(0) = 4\pi\left(C_{jk} - C_{jj}\right). \tag{2.7.4}$$

The claim follows from the above description of g_j in local coordinates. □

For later use we state

2.8. Lemma. *Let $B_\varepsilon(S_j) = \{P \in X(\Gamma) \,|\, |t_j(P)| < \varepsilon\}$ be a small ε-neighborhood of S_j. Then there is an equality*

$$\lim_{\varepsilon\to 0} \left(\log(-\log|\varepsilon|^2) - \int_{X(\Gamma)\setminus B_\varepsilon(S_j)} g_j\, \omega \right) = \frac{[\Gamma(1):\Gamma]}{12} \cdot 4\pi C_{jj} + 2\log(4\pi). \tag{2.8.1}$$

Proof. Let us fix $\varepsilon = \exp(-2\pi T)$, $T >> 0$ and set $X_\varepsilon = X(\Gamma) \setminus B_\varepsilon(S_j)$. Let us choose a fundamental domain $\mathcal{F}_\Gamma$ for the action of Γ on $\mathfrak{h}$, then after possible conjugation with σ_j a pre-image of $B_\varepsilon(S_j)$ is given by the set

$$\mathcal{F}_{\sigma_j,\varepsilon} = \{x + iy \in \mathfrak{h} \,|\, y > T,\ 0 \le x < 1\}.$$

As $E_j(s)$ is an eigenfunction of the hyperbolic Laplacian we get by means of Green's formula

$$\begin{aligned}\int_{X_\varepsilon} 4\pi E_j(s)\,\omega &= \frac{1}{s(s-1)}\int_{X_\varepsilon} \triangle E_j(s)\,\frac{dxdy}{y^2} = \frac{1}{s(s-1)}\int_{\partial B_\varepsilon(S_j)} \frac{\partial E_j(s)}{\partial\eta}dl \\ &= \frac{1}{s(s-1)}\int_0^1 \frac{\partial E_j(\sigma_j^{-1}(\tau);s)}{\partial y}dx\Bigg|_{y=T} = \frac{T^{s-1}}{s-1} - (\Phi_\Gamma)_{j,j}(s)\cdot\frac{T^{-s}}{s}\end{aligned}$$

Then, using the Laurent expansion of $(\Phi_\Gamma)_{j,j}(s)$ we get

$$\begin{aligned}\int_{X_\varepsilon} 4\pi E_j(s)\,\omega = &\left(1 - \frac{3}{\pi[\Gamma(1):\Gamma]}\cdot\frac{1}{T}\right)(s-1)^{-1} \\ &+\log(T) + O\left(\frac{\log T}{T}\right) + O(s-1).\end{aligned}$$

Furthermore, since we have

$$\int_{X_\varepsilon}\omega = \left(\frac{[\Gamma(1):\Gamma]}{12} - \frac{1}{4\pi T}\right),$$

we easily determine

$$\begin{aligned}&\int_{X_\varepsilon}\left(4\pi\left(E_j - (\Phi_\Gamma)_{j,j}(s)\right) - \frac{12\log(4\pi)}{[\Gamma(1):\Gamma]}\right)\omega \\ &= \left(\left(1-\frac{3}{\pi[\Gamma(1):\Gamma]}\cdot\frac{1}{T}\right) - \left(\frac{[\Gamma(1):\Gamma]}{12} - \frac{1}{4\pi T}\right)\cdot\frac{12}{[\Gamma(1):\Gamma]}\right)(s-1)^{-1} \\ &\quad - \frac{[\Gamma(1):\Gamma]}{12}\cdot 4\pi\cdot C_{jj} + \log(T) - \log(4\pi) + O\left(\frac{\log T}{T}\right) + O(s-1).\end{aligned}$$

Resorting the above terms and taking the limit $s\to 1$ leads to

$$\log(4\pi T) - \int_{X_\varepsilon} g_j\,\omega = \frac{[\Gamma(1):\Gamma]}{12}\cdot 4\pi\cdot C_{jj} + 2\log(4\pi) + O\left(\frac{\log T}{T}\right).$$

Now the claimed formula follows as $4\pi T = -\log|\varepsilon|^2$. □

2.9. Remark. Similar calculations maybe found in [Ku], page 19; compare also the special case $\Gamma = \Gamma(1)$ in [Za].

3 Review of Belyi's theorem

Recall the following formulation of Belyi's theorem [Se].

3.1. Theorem. *Let X be a non-singular projective curve over $\mathbb{C}$. The following are equivalent.*

(i) X is definable over $\overline{\mathbb{Q}}$, i.e., X arises by extensions of scalars from an algebraic curve over an number field E with an fixed embedding $E \hookrightarrow \mathbb{C}$.

(ii) There is a finite covering $\boldsymbol{\beta} : X \to \mathbb{P}^1$ unramified outside $\{0, 1728, \infty\}$.

(iii) There is a subgroup Γ of $\Gamma(1)$ of finite index such that $X \cong X(\Gamma)$. □

We remind the reader of the main ideas in the proof of this famous theorem. The passage from (i) to (ii) is given by a constructive algorithm due to Belyi. It associates to a function $f \in k^*(X_E)$ a polynomial $\boldsymbol{\beta}(f) \in E[f]$, so that the morphism $\boldsymbol{\beta}(f) : X_E \to \mathbb{P}^1_E$ induced by the evaluation map has branch points $0, 1, \infty$.

The step from (ii) to (i) is a general fact about coverings (see e.g. [Kö]). Indeed, since the branch points of $\boldsymbol{\beta}$ are rational points, $\boldsymbol{\beta}$ has to be the base change from a morphism of algebraic curves $\boldsymbol{\beta} : X_E \to \mathbb{P}^1_E$ defined over a number field E to a morphism of the associated compact Riemann surfaces.

The passage from (ii) to (iii) follows from the fact that the fundamental group of $\mathbb{P}^1 \setminus \{0, 1, \infty\}$ is a subgroup of $\Gamma(1)$. The passage from (iii) to (ii) is given by the j-function since $X(1) \cong \mathbb{P}^1(\mathbb{C})$.

3.2. Definition. We call a morphism $\boldsymbol{\beta} : X_E \to \mathbb{P}^1_E$ with branch points $0, 1, \infty$ a *Belyi morphism* and an isomorphism as in (iii) a *Belyi uniformization.* A point $P \in \boldsymbol{\beta}^{-1}(\infty)$ is called a *cusp.* A divisor on X_E of degree zero with support in the cusps is called a *cuspidal divisor*. The group generated by cuspidal divisors will be denoted by $\mathrm{Cusp}(X_E, \boldsymbol{\beta})$.

We note that a Belyi uniformization is by no means unique. Indeed, for a given finite set $\mathscr{D}$ of algebraic points of X_E there always exists a function f on X_E defined over E whose polar divisor contains $\mathscr{D}$, but then by construction $\mathscr{D} \subseteq \mathrm{Cusp}(X_E, \boldsymbol{\beta}(f))$. So in particular every divisor D on X_E of degree zero is a cuspidal divisor for some Belyi morphism (depending on D of course).

If a Belyi morphism $\boldsymbol{\beta} : X_E \to \mathbb{P}^1_E$ is given, then for each embedding $\sigma : E \to \mathbb{C}$ we obtain Belyi uniformizations $X_\sigma(\mathbb{C}) \cong X(\Gamma_\sigma)$. Observe that $[\Gamma(1) : \Gamma_\sigma] = \deg(\boldsymbol{\beta})$ for all embeddings σ. If X_E is a modular curve, then all Γ_σ with $X_\sigma(\mathbb{C}) \cong X(\Gamma_\sigma)$ are congruence subgroups. This is because $X(N)_\sigma(\mathbb{C}) \cong X(\Gamma(N))$ for all $\sigma : \mathbb{Q}(\zeta_N) \to \mathbb{C}$.

3.3. Definition. After possibly replacing E by a finite extension of E, we assume from now on that all cusps of X_E are E-rational points. We have $X(1) \cong \mathbb{P}^1(\mathbb{C})$ and the natural model $\mathcal{X}(1)$ for $X(1)$ is $\mathbb{P}^1_{\mathbb{Z}}$ where the embedding is given by the modular forms $j \cdot \Delta$ and Δ. By means of the morphism $\boldsymbol{\beta} : X_E \to \mathbb{P}^1_E$ we define $X_{\mathcal{O}_E}$ to be the normalization of $\mathcal{X}(1) \times_{\mathbb{Z}} \mathcal{O}_E$ in function field of $X(\Gamma)_E$. Finally, if the genus of X_E is different from zero we let

$$f : \mathcal{X} \longrightarrow \operatorname{Spec} \mathcal{O}_E$$

be a semi-stable, regular model for X_E (possibly replacing E by a finite extension of E), otherwise $\mathcal{X}$ will be any regular model. We call $\mathcal{X}$ an *arithmetic surface for X_E*.

3.4. Definition. We denote by $\mathcal{P}_{\boldsymbol{\beta}}$ the set of primes $\mathfrak{p}$ of bad reduction of $X_{\mathcal{O}_E}$ and let $b_{\boldsymbol{\beta}}$ be the smallest positive integer with $(b_{\boldsymbol{\beta}}) \subseteq \mathfrak{p}$ for all primes $\mathfrak{p} \in \mathcal{P}_{\boldsymbol{\beta}}$.

Recall that $\mathcal{X}$ is obtained by blowing up singularities of $X_{\mathcal{O}_E}$ and blowing down (-1) curves. Note this process does not increase the set of primes of bad reduction, but the proper morphism induced by $\boldsymbol{\beta}$

$$\mathcal{X}_{\mathcal{O}_E[1/b_{\beta}]} \longrightarrow \mathcal{X}(1)_{\mathcal{O}_E[1/b_{\beta}]},$$

does in general not extend to a proper morphism of schemes defined over $\operatorname{Spec}\mathcal{O}_E$.

3.5. Definition. The cusps of X_E are algebraic points and we denote by s_j the Zariski closure of the cusp S_j in the scheme $\mathcal{X}$. We denote by $\mathcal{P}_{\mathscr{C}}$ the set of primes $\mathfrak{p}$ for which there exist two different cusps S_j and S_k such that

$$s_j \cap f^{-1}(\mathfrak{p}) = s_k \cap f^{-1}(\mathfrak{p}).$$

3.6. Remark. In general the sets $\mathcal{P}_{\mathscr{C}}$ and $\mathcal{P}_{\beta}$ are unrelated. But if $\beta : X_E \to \mathbb{P}^1_E$ is a Galois covering, then $\mathcal{P}_{\mathscr{C}} \subseteq \mathcal{P}_{\beta}$ and $p \in \mathcal{P}_{\beta}$ implies $p|\deg(\boldsymbol{\beta})$ (see e.g. [Co], Théorème 5). Note that $\beta : X_E \to \mathbb{P}^1_E$ is a Galois cover if and only if all the subgroups Γ_σ of $\Gamma(1)$ given by the Belyi uniformisation $X_\sigma(\mathbb{C}) \cong X(\Gamma_\sigma)$ are normal subgroups.

4 Generalized arithmetic intersection numbers associated to the cusps

4.1. Notation. Let X_E be a geometrically irreducible, smooth, projective curve defined over a number field E together with a Belyi morphism $\boldsymbol{\beta} : X_E \to \mathbb{P}^1_E$. We let $f : \mathcal{X} \to \operatorname{Spec}\mathcal{O}_E$ be its associated arithmetic surface. Let us denote by $\widehat{Pic}(\mathcal{X}, \mathcal{S})$ the group of hermitian, logarithmically singular line bundles, having singularities at most at the cusps on each $X(\Gamma_\sigma)$ and at the elliptic points on each$X(\Gamma_\sigma)$.

4.2. Definition. Let S_j be a cusp of X_E. Then we metrize the induced line bundle $\mathcal{O}(S_j^\sigma)$ on $X(\Gamma_\sigma)$ by defining the norm of the canonical section by $\|1_{S_j^\sigma}\|^2_{hyp} = \exp(-g_j^\sigma)$, where g_j^σ is as in Definition 1.2. As S_j corresponds to a E-rational point of X_E, we can define a metric on $\mathcal{O}(s_j)_\infty$ over $\mathcal{X}_\infty$ by metrizing each component with the above metric. We put

$$\overline{\mathcal{O}(s_j)} = (\mathcal{O}(s_j), \|\cdot\|_{hyp}).$$

4.3. Proposition. *The line bundle $\overline{\mathcal{O}(s_j)}$ is a hermitian, logarithmically singular line bundle, i.e.,*

$$\overline{\mathcal{O}(s_j)} \in \widehat{Pic}(\mathcal{X}, \mathcal{S}).$$

Proof. Using the local descriptions (2.7.1), (2.7.2) for each g_j^σ given in Proposition 2.7 we derive that $\|\cdot\|_{hyp}$ is a hermitian, logarithmically singular metric in the sense of definition 1.2. Because the norm of the canonical section 1_{s_j} is outside the set of cusps and the set of elliptic points on each copy of $X(\Gamma)$ a smooth function. It has at the cusp S_k^σ on $X(\Gamma_\sigma)$ the following expansion

$$\|1_{s_j}\|(t_k) = (-\log|t_k|^2)^{\frac{6}{[\Gamma(1):\Gamma]}} \cdot |t_k|^{\delta_{jk}} \cdot e^{f_{jk}^\sigma(t_k)/2}, \tag{4.3.1}$$

here δ_{jk} is the Kronecker delta. Thus for all cusps we have $\alpha = \frac{6}{[\Gamma(1):\Gamma_\sigma]}$. At elliptic points the norm is a continous function, thus we have $\alpha = 0$ at these points. □

4.4. Theorem. *With notation above*

$$\frac{1}{[E:\mathbb{Q}]} \cdot \overline{\mathcal{O}(s_j)}.\overline{\mathcal{O}(s_k)} = \sum_{\mathfrak{p}\in\mathcal{P}_{\boldsymbol{\beta}}\cup\mathcal{P}_{\mathscr{C}}} \alpha_{\mathfrak{p}} \log N(\mathfrak{p}) + \frac{2\pi}{[E:\mathbb{Q}]} \sum_{\sigma:E\to\mathbb{C}} \left(C_{jk}^\sigma - C_{kk}^\sigma - C_{jj}^\sigma\right) + \frac{6-12\log(4\pi)}{\deg(\boldsymbol{\beta})},$$

where all $\alpha_{\mathfrak{p}}$ are rational numbers and the real numbers C_{jk}^σ are the scattering constants for the groups Γ_σ.

Proof. We first assume $j \neq k$, then 1_{s_j} and 1_{s_k} are non-trivial, global sections of $\mathcal{O}(s_j)$ and $\mathcal{O}(s_k)$ whose divisors on $\mathcal{X}_\infty$ are disjoint. Therefore by Definition 1.2 we have

$$\overline{\mathcal{O}(s_j)}.\overline{\mathcal{O}(s_k)} = (s_j, s_k)_{\text{fin}} + \langle s_j, s_k\rangle_\infty;$$

here $(s_j, s_k)_{\text{fin}} = (1_{s_j}, 1_{s_k})_{\text{fin}}$ is the usual intersection number at the finite places and $\langle s_j, s_k\rangle_\infty = \langle 1_{s_j}, 1_{s_k}\rangle_\infty$ is the generalized arithmetic intersection number at the infinite places.

As two different cusps meet at most in the fibers above $\mathfrak{p} \in \mathcal{P}_{\mathscr{C}}$ we obtain that there exist some $\alpha_{\mathfrak{p}} \in \mathbb{Q}$ with

$$(s_j, s_k)_{\text{fin}} = [E:\mathbb{Q}] \cdot \sum_{\mathfrak{p}\in\mathcal{P}_{\mathscr{C}}} \alpha_{\mathfrak{p}} \log \mathrm{N}(\mathfrak{p}). \tag{4.4.1}$$

Since all components of $\mathcal{O}(s_j)_\infty$ and $\mathcal{O}(s_k)_\infty$ are metrized in a similar way, we perform the remaining calculations at the primes at the infinite places only in one of the $[E:\mathbb{Q}]$ components of $\mathcal{X}_\infty$. We calculate $\langle s_j, s_k\rangle_\sigma = \langle s_j, s_k\rangle_\infty|_{\mathcal{X}_\sigma(\mathbb{C})}$ using $-\log\|1_{s_j}\| = g_j/2$ and $-\log\|1_{s_k}\| = g_k/2$ with formula (1.3.2). In our case the first term of (1.3.2) vanishes, the data to determine the second and third term can be read of the local expansion (4.3.1) possibly after interchanging j and k with each other. The remaining integration is done in Lemma 2.8. Therefore, we derive with the use of (2.7.4)

$$\begin{aligned} 2\cdot\langle s_j, s_k\rangle_\sigma &= 1\cdot\Big(\frac{12}{[\Gamma(1):\Gamma_\sigma]} + f_{kj}^\sigma(0)\Big) - \\ &\lim_{\varepsilon\to 0}\left(1\cdot\frac{12}{[\Gamma(1):\Gamma_\sigma]}\cdot\log(-\log\varepsilon^2) - \int_{X_\varepsilon} g_j^\sigma \cdot \frac{12}{[\Gamma(1):\Gamma_\sigma]}\cdot\omega\right) \\ &= 4\pi\left(C_{jk}^\sigma - C_{kk}^\sigma - C_{jj}^\sigma\right) + \frac{12}{[\Gamma(1):\Gamma_\sigma]} - \frac{24\log(4\pi)}{[\Gamma(1):\Gamma_\sigma]}. \end{aligned} \tag{4.4.2}$$

Adding the two quantities (4.4.1) and (4.4.2) gives the formula

$$\frac{1}{[E:\mathbb{Q}]}\cdot\overline{\mathcal{O}(s_j)}.\overline{\mathcal{O}(s_k)}=\sum_{\mathfrak{p}\in\mathcal{P}_{\mathscr{C}}}\alpha_{\mathfrak{p}}\log\mathrm{N}(\mathfrak{p})+\frac{2\pi}{[E:\mathbb{Q}]}\sum_{\sigma:E\to\mathbb{C}}\left(C^{\sigma}_{jk}-C^{\sigma}_{kk}-C^{\sigma}_{jj}\right)$$
$$+\frac{6-12\log(4\pi)}{\deg(\boldsymbol{\beta})}. \tag{4.4.3}$$

If $j=k$ we have to move s_k by the divisor $\operatorname{div}h$ of a rational function h in order to get proper intersection on the generic fiber of $\mathcal{X}$. For this procedure we will use the morphism $\boldsymbol{\beta}:\mathcal{X}_{\mathcal{O}_E[1/b_{\boldsymbol{\beta}}]}\longrightarrow\mathcal{X}(1)_{\mathcal{O}_E[1/b_{\boldsymbol{\beta}}]}$. Note by bilinearity of the generalized arithmetic intersection number we may work with $\mathbb{Q}$ powers of line bundles and divisors with $\mathbb{Q}$ coefficients. Thus we may proceed if there were a rational function h such that

$$h|_{\mathcal{X}_{\mathcal{O}_E[1/b_{\boldsymbol{\beta}}]}}=\boldsymbol{\beta}^*(j)^{1/b_k},$$

where j is the j-function and b_k is the width of the cusp S_k. Let $\operatorname{div}h=\operatorname{div}h^++\operatorname{div}h^-$ be the decomposition of $\operatorname{div}h$ into its positive and negative part. Then by the projection formula we have on $\mathcal{X}_{\mathcal{O}_E[1/b_{\boldsymbol{\beta}}]}$ that

$$(s_k,\operatorname{div}h^+)_{\mathrm{fin},\mathcal{X}_{\mathcal{O}_E[1/b_{\boldsymbol{\beta}}]}}=(\infty,0)_{\mathrm{fin},\mathbb{P}^1_{\mathcal{O}_E[1/b_{\boldsymbol{\beta}}]}}=0,$$

therefore only the primes $\mathfrak{p}\in\mathcal{P}_{\boldsymbol{\beta}}$ may give a contribution to the intersection number at the finite places $(s_k,\operatorname{div}h^+)_{\mathrm{fin}}$. As furthermore s_k and $s_k+\operatorname{div}h^-$ intersect at most in the fibers above $\mathfrak{p}\in\mathcal{P}_{\boldsymbol{\beta}}\cup\mathcal{P}_{\mathscr{C}}$, there exist some $\alpha_{\mathfrak{p}}\in\mathbb{Q}$ with

$$(s_k,s_k+\operatorname{div}h)_{\mathrm{fin}}=[E:\mathbb{Q}]\cdot\sum_{\mathfrak{p}\in\mathcal{P}_{\boldsymbol{\beta}}\cup\mathcal{P}_{\mathscr{C}}}\alpha_{\mathfrak{p}}\log\mathrm{N}(\mathfrak{p}). \tag{4.4.4}$$

Since each component of $\mathcal{O}(s_k)_\infty$ is metrized in a similar way, we perform the remaining calculations only in one of the $[E:\mathbb{Q}]$ components of $\mathcal{X}_\infty$. Recall the well-known Fourier expansion of the j-function. Then we obtain by (4.3.1) for the local expansion of the rational section $1_{s_k}\cdot h$ of $\mathcal{O}(s_k)$ at the cusp S^σ_k the formula

$$-\log\|1_{s_k}\cdot h^\sigma\|_{hyp}=\frac{-12}{[\Gamma(1):\Gamma_\sigma]}\log(-\log|t_k|^2)+f^\sigma_{kk}(t_k)+O(t_k).$$

Using this modified data we proceed as we did for (4.4.2) and derive

$$\begin{aligned}2\langle s_k,s_k+\operatorname{div}h\rangle_\sigma&=1\cdot\Big(\frac{12}{[\Gamma(1):\Gamma_\sigma]}\Big)-\\&\lim_{\varepsilon\to0}\left(1\cdot\frac{12}{[\Gamma(1):\Gamma_\sigma]}\cdot\log(-\log\varepsilon^2)-\int_{\mathcal{X}_\varepsilon}g^\sigma_k\cdot\frac{12}{[\Gamma(1):\Gamma_\sigma]}\cdot\omega\right)\\&=-4\pi C^\sigma_{kk}+\frac{12}{[\Gamma(1):\Gamma_\sigma]}-\frac{24\log(4\pi)}{[\Gamma(1):\Gamma_\sigma]}.\end{aligned} \tag{4.4.5}$$

Adding the above quantities (4.4.4) and (4.4.5) gives

$$\frac{1}{[E:\mathbb{Q}]}\cdot\overline{\mathcal{O}(s_k)}.\overline{\mathcal{O}(s_k)}=\sum_{\mathfrak{p}\in\mathcal{P}_{\boldsymbol{\beta}}\cup\mathcal{P}_{\mathscr{C}}}\alpha_{\mathfrak{p}}\log\mathrm{N}(\mathfrak{p})-\frac{2\pi}{[E:\mathbb{Q}]}\sum_{\sigma:E\to\mathbb{C}}C^{\sigma}_{kk}+\frac{6-12\log(4\pi)}{\deg(\boldsymbol{\beta})}. \tag{4.4.6}$$

Since $j=k$, we have $C^{\sigma}_{jj}=C^{\sigma}_{jk}=C^{\sigma}_{kk}$. Therefore we may write in the above formula the quantity $-C^{\sigma}_{kk}$ as $C^{\sigma}_{jk}-C^{\sigma}_{jj}-C^{\sigma}_{kk}$. □

4.5. Remark. If $\Gamma=\Gamma(1)$, then there is only one cusp S_∞. We have by Kronecker's limit formula $g_\infty=-\log\|\Delta\|^2_{Pet}$ and

$$-2\pi C_{\infty,\infty}=\lim_{s\to1}\left(\frac{\widehat{\zeta_{\mathbb{Q}}}(2s-1)}{\widehat{\zeta_{\mathbb{Q}}}(2s)}-\frac{3/\pi}{s-1}\right)=12\left(12\zeta'_{\mathbb{Q}}(-1)-1+\log(4\pi)\right),$$

where $\widehat{\zeta_{\mathbb{Q}}}(s)=\pi^{-s/2}\Gamma(s/2)\zeta_{\mathbb{Q}}(s)$ is the completed Riemann zeta function, see e.g. [Kü], section 5. Furthermore since $\mathcal{X}(1)$ is smooth we obtain

$$\overline{\mathcal{O}(s_\infty)}.\overline{\mathcal{O}(s_\infty)}=12^2\left(\frac{1}{2}\zeta_{\mathbb{Q}}(-1)+\zeta'_{\mathbb{Q}}(-1)\right).$$

This formula was a major step in the proof of Theorem 6.1 of [Kü].

4.6. Proposition. *Let $D=\sum_j n_jS_j$ be a cuspidal divisor. Then, there exist rational numbers $n^{(l)}_{\mathfrak{p}}$, non-zero at most for $\mathfrak{p}\in\mathcal{P}_{\boldsymbol{\beta}}$ such that the class of*

$$\overline{\mathcal{O}(\mathcal{D})}=\bigotimes_{\mathfrak{p}\in\mathcal{P}_{\boldsymbol{\beta}}}\left(\bigotimes_{(l)}\mathcal{O}\left(\mathcal{C}^{(l)}_{\mathfrak{p}}\right)^{\otimes n^{(l)}_{\mathfrak{p}}}\right)\bigotimes_j\overline{\mathcal{O}(s_j)}^{\otimes n_j} \tag{4.6.1}$$

in $\widehat{Pic}(\mathcal{X},\mathcal{S})_{\mathbb{Q}}$ is an extension of $\mathcal{O}(D)$ to $\widehat{Pic}^0(\mathcal{X})_{\mathbb{Q}}$.

Proof. By Proposition 1.8 the divisor D extends to a divisor $\mathcal{D}$ with rational coefficients on $\mathcal{X}$, which has zero degree on all irreducible components of the fibers of f. In particular there exist rational numbers $n^{(l)}_{\mathfrak{p}}$, non-zero at most for $\mathfrak{p}\in\mathcal{P}_{\boldsymbol{\beta}}$ such that $\mathcal{O}(\mathcal{D})=\bigotimes_{\mathfrak{p}\in\mathcal{P}_{\boldsymbol{\beta}}}\left(\bigotimes_{(l)}\mathcal{O}\left(\mathcal{C}^{(l)}_{\mathfrak{p}}\right)^{\otimes n^{(l)}_{\mathfrak{p}}}\right)\bigotimes_j\mathcal{O}(s_j)^{\otimes n_j}$. It remains to show that for the right hand side of (4.6.1) the first Chern form vanishes, but since $\sum_j n_j=0$ this follows from Proposition 2.7. □

4.7. Remark. The fact that $g_D=\sum_j m_jg_j$ is a harmonic Green's function for the cuspidal divisor $D=\sum_j m_jS_j$, i.e. $dd^cg_D+\delta_D=0$, is also a key point in the article [Sc].

4.8. Theorem. *Let X_E be a geometrically irreducible curve defined over a number field E. Let $\boldsymbol{\beta}:X_E\to\mathbb{P}^1_E$ be a Belyi morphism and let $X_\sigma(\mathbb{C})\cong X(\Gamma_\sigma)$ be the induced Belyi* *uniformizations. Assume that* $\mathrm{Cusp}(X_E, \boldsymbol{\beta})$ *generates a torsion subgroup in the Jacobian of* X_E*, then the scattering constants* C^{σ}_{jk} *for the groups for the groups* Γ_σ *satisfy the equality*

$$\frac{1}{[E:\mathbb{Q}]}\sum_{\sigma:E\to\mathbb{C}} C^{\sigma}_{jk} = \frac{12}{\deg(\boldsymbol{\beta})}\big(12\zeta'_{\mathbb{Q}}(-1)-1+\log(4\pi)\big)+\sum_{\mathcal{P}_{\boldsymbol{\beta}}} a_{\mathfrak{p},jk}\log N\mathfrak{p}, \tag{4.8.1}$$

where $a_{\mathfrak{p},jk}$ *are rational numbers.*

Proof. Since the Neron Tate height pairing as well as the arithmetic intersection pairing are non-degenerate pairings, we deduce as in Proposition 4.6 that for all cusps S_j there exists rational numbers $n^{(l)}_{\mathfrak{p}}$ (depending on S_j) and real numbers ρ^{σ}_j such that in $\widehat{Pic}(\mathcal{X},\mathcal{S})_{\mathbb{Q}}$

$$\bigotimes_{\mathfrak{p}\in\mathcal{P}_{\boldsymbol{\beta}}}\left(\bigotimes_{(l)}\mathcal{O}\left(\mathcal{C}^{(l)}_{\mathfrak{p}}\right)^{\otimes n^{(l)}_{\mathfrak{p}}}\right)\bigotimes\left(\mathcal{O}(s_j),\|\cdot\|_{hyp,\rho^{\sigma}_j}\right)^{\otimes\deg(\boldsymbol{\beta})}=\boldsymbol{\beta}^*\overline{\mathcal{O}(s_\infty)},$$

where $\|\cdot\|_{hyp,\rho^{\sigma}_j}$ means that on $X_\sigma(\mathbb{C})$ the metric is scaled by $\exp(-\rho^{\sigma}_j)$. Thus for any $\sigma: E\to\mathbb{C}$ there is an algebraic modular form f^σ for Γ_σ of weight 12 such that $\deg(\boldsymbol{\beta})(g^{\sigma}_j+\rho^{\sigma}_j)=\log\|f^\sigma\|^2_{Pet}$. The Fourier coefficients of f^σ at each cusp S_k are algebraic numbers (not necessarily algebraic integers!), i.e., $f^\sigma(t_k)=\sum_{n\geq 0} b(n,k)^\sigma t_k^n$. Since $\mathrm{div}(f^\sigma)$ does at most contain fibers above $\mathfrak{p}\in\mathcal{P}_\beta$, we get that the coefficients $b(1,j)^\sigma$ and $b(0,k)^\sigma$ for $k\neq j$ are units in $\mathcal{O}_{E[1/b_{\boldsymbol{\beta}}]}$. We deduce from the formulas (2.7.3) and (2.7.4) that $\rho^{\sigma}_j=\log|b(1,j)^\sigma|^2$ and $C^{\sigma}_{jk}-C^{\sigma}_{jj}=\log|b(0,k)^\sigma|^2-\log|b(1,j)^\sigma|^2$.

Let h be the modular function f^σ/Δ for Γ_σ, then $-\log|h|^2=\deg(\boldsymbol{\beta})(g^{\sigma}_j+\rho^{\sigma}_j)-\boldsymbol{\beta}^*g_\infty$. On $\mathcal{X}_\infty$ the intersection of $\boldsymbol{\beta}^*(\mathrm{div}(j)+s_\infty)$ and S_k is proper for all cusps S_k and the class of the trivial bundle equipped with the trivial metric is in the kernel of the arithmetic intersection pairing. We derive $0=(\mathrm{div}(h),\boldsymbol{\beta}^*(\mathrm{div}(j)+s_\infty))_{\mathrm{fin}}+\langle\mathrm{div}(h),\boldsymbol{\beta}^*(\mathrm{div}(j)+s_\infty)\rangle_\infty$. Calculations as in the proof of Theorem 4.4 show

$$\frac{1}{[E:\mathbb{Q}]}(\mathrm{div}(h),\boldsymbol{\beta}^*(\mathrm{div}(j)+s_\infty))_{\mathrm{fin}}=\sum_{\mathfrak{p}\in\mathcal{P}_{\boldsymbol{\beta}}}\alpha_{\mathfrak{p}}\log\mathrm{N}(\mathfrak{p})$$

and

$$\frac{1}{[E:\mathbb{Q}]}\langle\mathrm{div}(h),\boldsymbol{\beta}^*(\mathrm{div}(j)+s_\infty)\rangle_\infty=2\pi\deg(\boldsymbol{\beta})\left(C_{\infty\infty}-\frac{\deg(\boldsymbol{\beta})}{[E:\mathbb{Q}]}\sum_{\sigma:E\to\mathbb{C}}C^{\sigma}_{jj}+\rho^{\sigma}_j\right).$$

Using the fact that $-2\pi C_{\infty\infty}=12(12\zeta'_{\mathbb{Q}}(-1)-1+\log(4\pi))$ we derive the claim. □

4.9. Remark. (i) Note that, given for all primes $\mathfrak{p}$ the intersection matrix of the irreducible components $\mathcal{C}^{(l)}_{\mathfrak{p}}$ of the fibers $f^{-1}(\mathfrak{p})$, then one can calculate the all the multiplicities $a_{\mathfrak{p}}$ in Formula (4.8.1) explicitly.

(ii) Observe that, if the genus of X_E is zero, then any cuspidal divisor is a principal, thus we get non-trivial relations among the scattering constants.

(iii) We note that, if X_E is a modular curve, then by the Manin-Drinfeld theorem some multiple of any cuspidal divisor is a principal divisor.

4.10. Theorem. *Let X_E be a geometrically irreducible curve defined over a number field E. Let $D_1 = \sum_j n_j S_j$, $D_2 = \sum_k m_k S_k$ be two divisors on X_E of degree 0. Choose a Belyi morphism $\boldsymbol{\beta} : X_E \to \mathbb{P}^1_E$ such that D_1 and D_2 are cuspidal divisors on X_E and let $X_\sigma(\mathbb{C}) \cong X(\Gamma_\sigma)$ be the induced Belyi uniformizations. Then we have the following formula for the Néron-Tate height pairing $\langle D_1, D_2\rangle_{NT}$ of the classes of D_1, D_2 in $Pic^0(X_E)$*

$$-\langle D_1, D_2\rangle_{NT} = \sum_{\mathfrak{p}\in\mathcal{P}_{\boldsymbol{\beta}}\cup\mathcal{P}_{\mathscr{C}}} \delta_{\mathfrak{p}} \log N(\mathfrak{p}) + \frac{2\pi}{[E:\mathbb{Q}]}\left(\sum_{\sigma:E\to\mathbb{C}}\sum_{j,k} n_j m_k C^{\sigma}_{jk}\right), \tag{4.10.1}$$

where all $\delta_{\mathfrak{p}}$ are rational numbers and the real numbers C^{σ}_{jk} are the scattering constants for the groups Γ_σ.

Proof. Let $\overline{\mathcal{O}(\mathcal{D}_1)}$ and $\overline{\mathcal{O}(\mathcal{D}_2)}$ be extensions of D_1, D_2 to $\widehat{Pic}^0(\mathcal{X})_{\mathbb{Q}}$ provided by Proposition 4.3. Applying the Faltings-Hriljac Formula 1.9 we derive

$$\begin{aligned}
-\langle D_1, D_2\rangle_{NT} &= \frac{1}{[E:\mathbb{Q}]}\cdot\overline{\mathcal{O}(\mathcal{D}_1)}.\overline{\mathcal{O}(\mathcal{D}_2)}\\
&= \frac{1}{[E:\mathbb{Q}]}\left(\bigotimes_{\mathfrak{p}\in\mathcal{P}_{\boldsymbol{\beta}}}\left(\bigotimes_{(l)}\mathcal{O}\left(\mathcal{C}^{(l)}_{\mathfrak{p}}\right)^{\otimes n^{(l)}_{\mathfrak{p}}}\right)\bigotimes_j\overline{\mathcal{O}(s_j)}^{\otimes n_j}\right).\\
&\qquad\left(\bigotimes_{\mathfrak{p}\in\mathcal{P}_{\boldsymbol{\beta}}}\left(\bigotimes_{(l)}\mathcal{O}\left(\mathcal{C}^{(l)}_{\mathfrak{p}}\right)^{\otimes m^{(l)}_{\mathfrak{p}}}\right)\bigotimes_k\overline{\mathcal{O}(s_j)}^{\otimes m_k}\right)\\
&= \sum_{\mathfrak{p}\in\mathcal{P}_{\boldsymbol{\beta}}}\beta_{\mathfrak{p}}\log \mathrm{N}(\mathfrak{p}) + \frac{1}{[E:\mathbb{Q}]}\left(\sum_{j\neq k} n_j m_k\overline{\mathcal{O}(s_j)}.\overline{\mathcal{O}(s_k)}\right).
\end{aligned}$$

For the last equality we used the bilinearity of the generalized arithmetic intersection number. The generalized arithmetic intersection numbers involved in this formula are determined in Theorem 4.4. Collecting all the terms and using the fact $\sum_j n_j = \sum_k m_k = 0$ we derive our main result. □

4.11. Remark. (i) One should remark that $\langle D_1, D_2\rangle_{NT}$ does of course not depend on the choice of the Belyi morphism $\boldsymbol{\beta}$, but the individual terms in our formula may.

(ii) Note that, given for all primes $\mathfrak{p}$ the intersection matrix of the irreducible components $\mathcal{C}^{(l)}_{\mathfrak{p}}$ of the fibers $f^{-1}(\mathfrak{p})$ and the intersection multiplicities of the divisors s_j with s_k, then one can calculate the all the multiplicities $\delta_{\mathfrak{p}}$ in Formula (4.10.1) explicitly.

4.12. Corollary. *Let $D_1 = \sum_j m_j S_j$, $D_2 = \sum_k n_k S_k$ be cuspidal divisors on X_E and set*

$$\rho_{D_1,D_2} = \exp\left(-\frac{2\pi}{[E:\mathbb{Q}]}\left(\sum_{\sigma:E\to\mathbb{C}}\sum_{j,k} m_j n_k C^{\sigma}_{jk}\right)\right).$$

If some non-zero multiple of D_1 or D_2 is principal, then there exist rational numbers $\delta_{\mathfrak{p}}$ such that

$$\rho_{D1,D_2} = \prod_{\mathfrak{p}\in\mathcal{P}_{\beta}\cup\mathcal{P}_{\mathscr{C}}} N(\mathfrak{p})^{\delta_{\mathfrak{p}}}.$$

Proof. If some non-zero multiple of D_1 or D_2 is a principal divisor then its Néron-Tate height vanishes. The claim then follows by exponentiating equation (4.10.1). □

4.13. Example. Let us illustrate in an example how one could recover the Manin-Drinfeld theorem for certain modular curves. To ease notation we assume p is a prime such that $p \equiv 1 \mod 12$ and $X_0(p)$ is the modular curve associated to the congruence subgroup $\Gamma_0(p)$ defined by

$$\Gamma_0(p) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(1) \,\middle|\, c \equiv 0 \mod p \right\}.$$

The curve $X_0(p) = X(\Gamma_0(p))$ is defined over $\mathbb{Q}$. It has two cusps denoted by S_∞ and S_0 of widths 1 and p respectively. Clearly $C(\Gamma_0(p))$ is torsion, since $\operatorname{div}(\Delta(p\tau)/\Delta(\tau)) = (p-1)(S_\infty - S_0)$. Let us denote the cuspidal divisor $S_0 - S_\infty$ by D. We now show $\langle D, D\rangle_{NT} = 0$ by means of the methods developed in this article.

The semi-stable model $f : \mathcal{X}_0(p) \longrightarrow \operatorname{Spec}\mathbb{Z}$ of $X_0(p)$ is defined over $\mathbb{Z}$. The only prime of bad reduction is the prime p, see e.g. [DeRa]. The fiber above p contains two irreducible components $\mathcal{C}_p^{(\infty)}$ and $\mathcal{C}_p^{(0)}$. Both cusps determine disjoint sections s_∞ and s_0 of f. The intersection matrix is given by

	$\mathcal{C}_p^{(0)}$	$\mathcal{C}_p^{(\infty)}$	s_0	s_∞
$\mathcal{C}_p^{(0)}$	$-\frac{p-1}{12}$	$\frac{p-1}{12}$	1	0
$\mathcal{C}_p^{(\infty)}$	$\frac{p-1}{12}$	$-\frac{p-1}{12}$	0	1

.

From this we get that the extension $\mathcal{D} = s_0 - s_\infty + 12/(p-1)\cdot\mathcal{C}_p^{(0)}$ of D is perpendicular to all the irreducible components of the fibers of f. As in the proof of theorem 4.10 the Néron-Tate height of D is therefore given by

$$\begin{aligned}
-\langle D, D\rangle_{NT} &= \overline{\mathcal{O}(\mathcal{D})}.\overline{\mathcal{O}(\mathcal{D})} \\
&= \left(\frac{12}{p-1}\right)^2 \cdot \mathcal{O}(\mathcal{C}_p^{(0)}).\mathcal{O}(\mathcal{C}_p^{(0)}) + 2\frac{12}{p-1}\cdot \mathcal{O}(\mathcal{C}_p^{(0)}).\left(\mathcal{O}(s_0)\otimes\mathcal{O}(s_\infty)^{-1}\right) \\
&\quad + \overline{\mathcal{O}(s_0)}.\overline{\mathcal{O}(s_0)} - 2\overline{\mathcal{O}(s_0)}.\overline{\mathcal{O}(s_\infty)} + \overline{\mathcal{O}(s_\infty)}.\overline{\mathcal{O}(s_\infty)} \\
&= \frac{12}{p-1}\log p + \overline{\mathcal{O}(s_0)}.\overline{\mathcal{O}(s_0)} - 2\overline{\mathcal{O}(s_0)}.\overline{\mathcal{O}(s_\infty)} + \overline{\mathcal{O}(s_\infty)}.\overline{\mathcal{O}(s_\infty)}.
\end{aligned}$$

Applying formula (4.4.3), we get since the cusp s_0 and s_∞ do never meet each other

$$\overline{\mathcal{O}(s_0)}.\overline{\mathcal{O}(s_\infty)} = -2\pi(C_{00} + C_{\infty\infty} - C_{0\infty}) + \frac{6}{p+1} - \frac{12\log(4\pi)}{p+1}.$$

The natural morphism $\boldsymbol{\beta} : X_0(p) \to X(1)$ extends to the semi-stable model and the cusps never meet $div(\boldsymbol{\beta}^* j)^+$, therefore formula (4.4.6) implies

$$\overline{\mathcal{O}(s_0)}.\overline{\mathcal{O}(s_0)} = -2\pi C_{00} + \frac{6}{p+1} - \frac{12\log(4\pi)}{p+1}$$
$$\overline{\mathcal{O}(s_\infty)}.\overline{\mathcal{O}(s_\infty)} = -2\pi C_{\infty\infty} + \frac{6}{p+1} - \frac{12\log(4\pi)}{p+1}.$$

Recall, see e.g. [He] p. 536, that the scattering matrix for $\Gamma_0(p)$ is given by the equality

$$\Phi_{\Gamma_0(p)}(s) = \frac{1}{p^{2s}-1}\begin{pmatrix} p-1 & p^s - p^{1-s} \\ p^s - p^{1-s} & p-1 \end{pmatrix} \cdot \frac{\widehat{\zeta_{\mathbb{Q}}}(2s-1)}{\widehat{\zeta_{\mathbb{Q}}}(2s)},$$

where $\widehat{\zeta_{\mathbb{Q}}}(s) = \pi^{-s/2}\Gamma(s/2)\zeta_{\mathbb{Q}}(s)$. We calculate (cf. Theorem 4.8)

$$-2\pi C_{00} = -2\pi C_{\infty\infty} = \frac{12}{p+1}\big(12\zeta_{\mathbb{Q}}'(-1) - 1 + \log(4\pi)\big) + \frac{12p^2\log(p)}{(p+1)(p^2-1)}$$
$$-2\pi C_{0\infty} = -2\pi C_{\infty 0} = \frac{12}{p+1}\big(12\zeta_{\mathbb{Q}}'(-1) - 1 + \log(4\pi)\big) + \frac{6(p^2-2p-1)\log(p)}{(p+1)(p^2-1)}$$

With this description we obtain

$$-2\pi\left(2C_{\infty 0} - C_{\infty\infty} - C_{00}\right) = -\frac{12}{p-1}\log p$$

and therefore $\langle D, D\rangle_{NT} = 0$.

4.14. Remark. (i) The Fermat curves $F_n : X^n + Y^n = Z^n$ have a Belyi uniformization associated to non-congruence groups such that the group of cuspidal divisors is a torsion subgroup in $Pic^0(F_n)$ (see e.g. [MR]). Hence by Theorem 4.8 we know the shape of the scattering constants.

(ii) To determine a Belyi morphism for a curve together with a finite set of points (say both given by equations) is a straightforward calculation, however to determine the associated scattering constants seems to be a challenge.

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