# FLOWING THE LEAVES OF A FOLIATION

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Abstract. We study the flows of the leaves $(M_\Theta)_{\Theta>0}$ of a foliation of $\mathbb{R}^{n+1}\setminus\{0\}$ consisting of uniformly convex hypersurfaces in the direction of their outer normals with speeds $-\log(F/f)$. In the case that $F$ is a function with inverse of class $(K^*)$ and $f$ a smooth and positive function on $S^n$ we show that there is a distinct leaf $M_{\Theta_*}$ in this foliation with the property that the flow starting from $M_{\Theta_*}$ converges to a translating solution of the flow equation. Furthermore, when starting the flow from a leave inside $M_{\Theta_*}$ it shrinks to a point and when starting the flow from a leave outside $M_{\Theta_*}$ the diameters of the flow hypersurfaces tend to infinity. We show that such a behavior remains true if we assume $F=H$, that the $M_\Theta$ are rotational symmetric with respect to a fixed axis and in addition a certain property for $f$. Furthermore, under appropriate symmetry assumptions for $M_\Theta$ and $f$ we obtain in both of the above situations in the case $\theta=\theta_*$ even convergence to a hypersurface with $F$-curvature and correspondingly mean curvature equal to $f$ (when considered as a function of the normal).

## 1. Introduction and main result

Chou and Wang [6] study a logarithmic Gauss curvature flow of the leaves of a foliation of $\mathbb{R}^{n+1}\setminus\{0\}$ consisting of a homothetic family of uniformly convex hypersurfaces. While their main purpose in doing so is to provide a variational reproof of the Minkowski problem we focus in our paper on the tool of the flow of the leaves of a foliation itself in a more general context, i.e. for other flow speeds (and depending on the flow speeds under less or more restrictive assumptions on the foliation), and use for it additional geometric arguments. The flow speeds in our paper are of type $-\log(F/f)$ where $F$ is a curvature function of the principal curvatures and $f$ is a smooth positive function on $S^n$ which we consider via the Gauss map also being defined on uniformly convex hypersurfaces.

Throughout the paper we make the following assumption.

**Assumption 1.1.** Either

(i) the inverse $\tilde{F}$ of $F$ satisfies Assumption 1.3 and $f$ is a positive function on $S^n=\{x\in\mathbb{R}^{n+1}:|x|=1\}$ or

(ii) $F=H$, the $M_\Theta$ are rotational symmetric with respect to a fixed axis, let us say the latter is the $x$-axis in an appropriately chosen Euclidean coordinate system $(x,x_2,...,x_{n+1})$ in $\mathbb{R}^{n+1}$, and $f$ is a positive function on $S^n$ which depends only on the $x$-coordinate and which satisfies

$$f<\frac{n}{n-1}f(0,...,0,1). \tag{1.1}$$

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Let us recall how the mechanism in [6] works. Let $(M_\Theta)_{\Theta>0}$, $M_\Theta = \Theta M_0$, be a family of homothetic transformations of an embedded, closed, uniformly convex hypersurface $M_0$ in $\mathbb{R}^{n+1}$. There is exactly one $\Theta_* > 0$ for which the flow with (outer) normal speed $-\log(K/f)$, $K$ the Gauss curvature, converges to a translating solution of the flow equation. Here, $f$ is a positive function of the normal and $K$ the Gauss curvature of the flow hypersurfaces. The flows starting from $M_\Theta$ shrink to a point in the case $\Theta < \Theta_*$ and converge to expanding spheres in the case $\Theta > \Theta_*$. The limit speed $\xi \in \mathbb{R}^{n+1}$ for the translating limit hypersurface of the flow starting from $M_{\Theta_*}$ is obtained from the necessary condition for the Minkowski problem

$$\int_{S^n} \frac{x_i}{e^{\xi \cdot x} f(x)} = 0, \quad i = 1, ..., n+1, \tag{1.2}$$

and hence convergence to a translating hypersurface with Gauss curvature $e^{\xi \cdot x} f(x)$ is deduced. Since a necessary condition like (1.2) is not available for the problem of finding hypersurfaces with general prescribed curvatures, see [9] e.g. for the mean curvature, the natural goal without any further symmetry assumptions in the case $\Theta = \Theta_*$ for our generalization of the above mechanism is to obtain convergence to a translating solution, cf. Theorem 1.2 for our precise main result.

We mention some literature dealing with translating solutions. Translating solutions appear e.g. as limiting behavior of rescaled mean curvature flow of surfaces in the presence of type II singularities, see [11]. Furthermore, translating solutions appear in the works [10, 1] as limiting behavior of solutions of non-parametric mean curvature evolution with Neumann boundary conditions. Translating solutions appear also in the limiting behavior of the second boundary value problem for certain non-parametric curvature flows [14] of strictly convex hypersurfaces. For further aspects of and literature about translating solutions see e.g. also [15] where complete translating solutions of the mean curvature are studied and the references therein.

We introduce the setting of our paper more precisely and state our main results in Theorem 1.2. Let $(M_\Theta)_{\Theta>0}$ be a foliation of $\mathbb{R}^{n+1} \setminus \{0\}$ by embedded, closed, uniformly convex (i.e. the Gauss curvature is positive) hypersurfaces $M_\Theta$ where we assume that $\Theta$ can be viewed as a smooth function with non-vanishing gradient. W.l.o.g. we assume that the monotone ordering of the associated open convex bodies $C_\Theta$ of the $M_\Theta$ with respect to inclusion is increasing. Let $(X_\Theta)_{\Theta>0}$ be a family of embeddings $X_\Theta : S^n \to \mathbb{R}^{n+1}$ of $M_\Theta$. We consider the evolution of convex hypersurfaces $M(t)$, parametrized by $X(\cdot, t)$, so that

$$\frac{\partial X}{\partial t} = -\log(F/f)\nu \tag{1.3}$$

with

$$X(p, 0) = X_\Theta(p). \tag{1.4}$$

Here, $\nu(p,t)$ denotes the unit outer normal of $M(t)$ at $X(p,t)$, $F$, $f$ satisfy Assumption 1.1 and $F$ is evaluated at the principal curvatures $\kappa_i$ of $M(t)$. Our main result is as follows, compare with [6].

**Theorem 1.2.** *(i) Let $(M_\Theta)_{\Theta>0}$ be as above and let $(M_\Theta)_{\Theta>0}$, $F$, $f$ satisfy Assumption 1.1. Then there exists $\Theta^* > 0$ and $\xi \in \mathbb{R}^n$ so that the flow (1.3), (1.4) with initial hypersurface $X_{\Theta^*}$ converges to a translating solution of the flow equation which translates with speed $\xi$, i.e.*

$$X(\cdot, t) - \xi t \to X^* \tag{1.5}$$

*in $C^m(S^n)$, $m \in \mathbb{N}$, for $t \to \infty$ where $X^*$ is the embedding of a smooth, uniformly convex hypersurface. If $\Theta \in (0, \Theta^*)$ then the solution of (1.3), (1.4) shrinks to a point in finite time. If $\Theta \in (\Theta^*, \infty)$ then the diameters of the solutions expand to infinity as $t$ goes to infinity.*

*(ii) Moreover, in the case (i) of Assumption 1.1 the solutions converge to expanding spheres for $\Theta > \Theta_*$.*

*(iii) If under the assumptions in (i) $f$ is in addition even, the foliation $(M_\Theta)_\Theta$ is symmetric to the $\{x = 0\}$-plane and each $M_\Theta$ is rotational symmetric with respect to the $x$-axis then the translating limit hypersurface obtained for $\Theta = \Theta_*$ in (i) has $F$-curvature and accordingly mean curvature equal (via the Gauss map) to $f$ and it translates with speed zero.*

Let us make some general remarks about the differences between our setting and [6, 14] on the technical level.

First note that we generalize the assumptions in [6] and at the same time obtain also (partially) a weaker conclusion than in [6], which is as expected, as explained above.

Concerning the foliation we essentially remove in the case (i) of Assumption 1.1 compared to [6] the requirement that the hypersurfaces in the foliation emerge by homothety from each other.

Case (i) of Assumption 1.1 allows several more speed functions than only the Gauss curvature but we benefit from the fact that in this case the inverse $\tilde{F}$ of $F$ behaves similarly in the $C^2$-estimates as the Gauss curvature so that we can follow [6] with some adaptions. These include a new identification of principle curvatures as eigenvalues of certain lower dimensional matrices which appear in a well-known and special parametrization of the flow hypersurfaces. This special parametrization uses an explicit expression (Monge-Ampére equation) for the Gauss curvature of a hypersurface in terms of the second derivatives of the restriction $u$ of the homogeneous degree one extension of the support function of the hypersurface to a tangent plane, cf. [6, Equ. (1.2)] and the end of page 738 therein for such a representation of the Gauss curvature. To handle the fact that an explicit expression in terms of the second derivatives of $u$ does not seem to be available for the curvature $F$ we use instead the well-known representation (2.8) of the principal radii of a hypersurface as zeros of a determinant of a certain matrix in $\mathrm{Sym}(n+1)$ and that we can write these zeros in special cases as eigenvalues of appropriate matrices in $\mathrm{Sym}(n)$, see the proof of Lemma 2.4.

Compared with [6] we implement a different strategy which is based on a method from [14] to deduce convergence to a translating solution when the a priori estimates are available, cf. proof part (ii) on page 16. In [6] the analogous conclusion is obtained from certain estimates which depend crucially on the Gauss curvature.

Since there is overlap with the flow speeds considered in [14] and the ones in the case (i) of Assumption 1.1 we look at these more closely. The flow speed in [14] in the direction of the outer normal is given by $\log(F/f)$ where $F$ is a smooth function of the class $(\tilde{K}^*)$ of the principal curvatures and $f$ is a smooth function defined in the domain over which the flowing hypersurfaces are written as graphs. Hereby, the class $(\tilde{K}^*)$ is a slight generalization of the class $(K^*)$, see [14] for a definition of the class $(\tilde{K}^*)$ and see also the remarks following Assumption 1.3. Most prominent member of both classes is the Gauss curvature and the mean curvature belongs to neither of both. In this non-parametric case [14] a necessity as described above of our new identification of principal curvatures as eigenvalues of certain lower dimensional matrices is not at hand and not used. Compared with [14] the consideration of further symmetry assumptions which imply in our case convergence to a hypersurface with $F$-curvature and correspondingly mean curvature equal to $f$ is completely new.

In the case of Assumption 1.1 (ii) $\tilde{F}$ lacks a crucial property (compared with Assumption 1.1 (i)). Although the problem can be seen as lower dimensional in view of the symmetry of the $M_\Theta$ we follow in this case also the proof strategy from Assumption 1.1 (i) but obtain in doing so only a poor upper bound for the principal radii, cf. Lemma 2.7. Until and in Section 3 we proceed with both cases in Assumption 1.1 quite analogously. Lemma 4.1 which uses geometric arguments then compensates the poor upper bound for the principal radii so that we conclude our result also when assuming case (ii) of Assumption 1.1. To the best knowledge of the author the difficulties arising in our $F = H$ case have never been studied in the literature before.

Finally, let us remark that there has been interest in the community to study fully nonlinear versions of the mean curvature flow, especially when the flow speed is a nonlinear function of the mean curvature, cf. e.g. to [**?**, **?**, ] for the flow by powers of the mean curvature. In this sense our paper serves as a first study of the 'logarithmic mean curvature case' for the flow speed. Furthermore, we point out that our geometric arguments in the final section are not based solely on the maximum principle but require also some nonlocal arguments.

Our paper is organized as follows. The remaining part of the paper deals with the proof of Theorem 1.2. In the remaining part of this section we introduce some notations for curvature functions. Section 2 estimates the principal radii of curvature of the flow hypersurfaces from below and above as well as their inradii from below. Using these estimates we prove Theorem 1.2 in the case (i) of Assumption 1.1 in Section 3. The proof of Theorem 1.2 in the case (ii) of Assumption 1.1 can be found in Section 4.

In the following we recall some facts about curvature functions from [8]. Let $\Gamma \subset \mathbb{R}^n$ denote a symmetric cone, $(\Omega, \xi^i)$ a coordinate chart in $\mathbb{R}^n$, $(g_{ij})$ a fixed positive definite $T^{0,2}(\Omega)$-tensor with inverse $(g^{ij})$ and $S = \mathrm{Sym}(n)$ the subset of symmetric tensors in $T^{0,2}(\Omega)$. Let $S_\Gamma$ be the set of the tensors $(h_{ij})$ in $S$ with eigenvalues with respect to $(g_{ij})$, i.e. eigenvalues of the $T^{1,1}(\Omega)$-tensor $(g^{ik}h_{kj})$, lying in $\Gamma$. In this setting we always consider a symmetric function $F$ defined in $\Gamma$ also as a function $F(\kappa_i) \equiv F(h_{ij}, g_{ij}) \equiv F(\frac{1}{2}(h_{ij} + h_{ji}), g_{ij})$ where the last expression is defined for general $(h_{ij}) \in T^{0,2}(\Omega)$. Using these interpretations we denote partial derivatives by

$$F_i = \frac{\partial F}{\partial \kappa_i}, \quad F_{ij} = \frac{\partial^2 F}{\partial \kappa_i \partial \kappa_j} \tag{1.6}$$

and

$$F^{ij} = \frac{\partial}{\partial h_{ij}} F(\frac{1}{2}(h_{ij} + h_{ji}), g_{ij}), \quad F^{ij,kl} = \frac{\partial^2}{\partial h_{ij} \partial h_{kl}} F(\frac{1}{2}(h_{ij} + h_{ji}), g_{ij}). \tag{1.7}$$

For a symmetric function $F$ in $\Gamma_+ = \{\kappa \in \mathbb{R}^n : \kappa_i > 0\}$ we define its inverse $\tilde{F}$ by

$$\tilde{F}(\kappa_i^{-1}) = \frac{1}{F(\kappa_i)}, \quad (\kappa_i) \in \Gamma_+. \tag{1.8}$$

In the following we state an assumption which summarizes some technical properties for reference purposes.

**Assumption 1.3.** $\tilde{F}$ is a symmetric and positively homogeneous of degree $d_0$ function $\tilde{F} \in C^\infty(\Gamma_+) \cap C^0(\bar{\Gamma}_+)$ with

$$\tilde{F}_{|\partial\Gamma_+} = 0, \tag{1.9}$$

$$\tilde{F}_i = \frac{\partial \tilde{F}}{\partial \kappa_i} > 0 \quad \text{in } \Gamma_+ \tag{1.10}$$

and

$$\epsilon_0 \tilde{F} \operatorname{tr}(h_{ij}) \le \tilde{F}^{ij} h_{ik} h^k_j \quad \forall (h_{ij}) \in S_{\Gamma_+} \tag{1.11}$$

where $\epsilon_0 = \epsilon_0(\tilde{F}) > 0$ and where we raise and lower indices with respect to $(g_{ij})$.

Furthermore, (i) or (ii) hold where

(i) means that $\tilde{F}$ is concave and $d_0 = 1$ and

(ii) means that

$$\tilde{F}^{ij,kl}\eta_{ij}\eta_{kl} \le \tilde{F}^{-1}\left(\tilde{F}^{ij}\eta_{ij}\right)^2 - \tilde{F}^{ik}\tilde{h}^{jl}\eta_{ij}\eta_{kl} \quad \forall \eta \in S \tag{1.12}$$

where $(\tilde{h}^{ij})$ is the inverse of $(h_{ij})$.

Assumption 1.3 is independent from the chosen tensor $(g_{ij})$ but expressions like $\tilde{F}(h_{ij})$ depend on $(g_{ij})$ where the latter will always refer to the corresponding induced metric and will be suppressed in the notation.

Assumption 1.3 is satisfied for curvature functions $\tilde{F}$ of class $(K^*)$, cf. [8, Definition 2.2.15].

The inverse of the mean curvature would satisfy Assumption 1.3 if part (1.11) therein is removed.

## 2. A priori estimates

We recall some facts about the support function of a closed and convex hypersurface $M$ in $\mathbb{R}^{n+1}$ from [6] and follow the presentation therein closely, see also [13] and [3]. The support function $H$ of $M$ is defined on $S^n$ by

$$H(x) = \sup_{y \in M} x \cdot y \tag{2.1}$$

where the dot denotes the inner product in $\mathbb{R}^{n+1}$. It is sometimes convenient to work with the homogeneous degree one extension of $H$ in $\mathbb{R}^{n+1}$ which we also denote by $H$. $H$ is convex in $\mathbb{R}^{n+1}$ and we have

$$\sup_{S^n} |\nabla H| \le \sup_{S^n} |H| \tag{2.2}$$

since $H$ is the supremum of linear functions. If $M$ is strictly convex, i.e. for each $x$ in $S^n$ there is a unique point $p = p(x)$ on $M$ whose unit outer normal is $x$, $H$ is differentiable at $x$ and

$$p_\alpha = \frac{\partial H}{\partial x_\alpha}, \quad \alpha = 1, ..., n+1. \tag{2.3}$$

Furthermore, given an orthonormal frame fields $e_1, ..., e_n$ on $S^n$ and denoting covariant differentiation with respect to $e_i$ by $\nabla_i$ the eigenvalues of $(\nabla_i\nabla_j H + H\delta_{ij})_{i,j=1,...,n}$, are the principal radii of curvature at $p(x)$. When $H$ is viewed as a homogeneous function over $\mathbb{R}^{n+1}$, the principal radii of curvature of $M$ are also equal to the non-zero eigenvalues of the Hessian

$$\left(\frac{\partial^2 H}{\partial x_\alpha \partial x_\beta}\right)_{\alpha,\beta=1,\dots,n+1} \tag{2.4}$$

on $S^n$.

We begin with a reformulation of Equation (1.3) locally in Euclidean space, cf. Equation (2.14). Let $H(\cdot,t) : S^n \to \mathbb{R}$ be the support function of $M(t)$ where we denote its homogeneous degree one extension to $\mathbb{R}^{n+1}$ again by $H(\cdot,t)$ and let $p(\cdot) = p(\cdot,t)$ denote the inverse of the Gauss map $M(t) \to S^n$. Using

$$\frac{H}{\partial t}(x,t) = x \cdot \frac{\partial X}{\partial t}(p(x),t), \quad x \in S^n, \tag{2.5}$$

we rewrite problem (1.3) as the following initial value problem for $H$

$$\begin{aligned} \frac{\partial H}{\partial t} =& \log \frac{f}{F} = \log \tilde{F} f \\ H(x,0) =& H_\Theta(x) \end{aligned} \tag{2.6}$$

where $H_\Theta$ is the support function for $M_\Theta$ and $\tilde{F}$ a function of the principal radii $r_i = \kappa_i^{-1}$ defined by

$$F = F(\kappa_i) = \tilde{F}(\kappa_i^{-1})^{-1} = \tilde{F}(r_i)^{-1}. \tag{2.7}$$

We set $u(y,t) = H(y,-1,t)$, $y \in \mathbb{R}^n$. Then $u(\cdot,t)$ is convex and the principal radii $r_i$ of $X(\cdot,t)$ in $p(x,t)$, $x \in S^n$, are given as nonzero zeros of the equation

$$\det B = 0 \tag{2.8}$$

where $B = (B_{\alpha\beta})_{\alpha,\beta=0,\dots,n}$ with

$$(B_{\alpha\beta}) = \begin{pmatrix} -\frac{\lambda^2}{r} & y_1 & \dots & y_n \\ y_1 & \lambda u_{11} - r & \dots & \lambda u_{1n} \\ \dots & & & \\ y_n & \lambda u_{n1} & \dots & \lambda u_{nn} - r \end{pmatrix}, \tag{2.9}$$

$\lambda = (1 + y_1^2 + \dots + y_n^2)^{\frac{1}{2}}$ and $x$ and $y$ are related by

$$x = (y,-1)/\sqrt{1+|y|^2}, \tag{2.10}$$

cf. [13, page 16], and note that we have rewritten the equation therein slightly. Furthermore, we have

$$\frac{\partial u}{\partial t}(y,t) = \sqrt{1+|y|^2}\frac{\partial H}{\partial t}(x,t). \tag{2.11}$$

Extending $f$ to be a homogeneous function of degree 0 in $\mathbb{R}^{n+1}$ we obtain the local representation of (1.3) in terms of $u$

$$\frac{\partial u}{\partial t} = \sqrt{1+|y|^2}\log \tilde{F} + l(y), \quad y \in \mathbb{R}^n, \tag{2.12}$$

where

$$l(y) = \sqrt{1+|y|^2}\log f(y,-1) \tag{2.13}$$

and $\tilde{F}$ is evaluated at the zeros $r_i$ of Equation (2.8). For technical reasons we rewrite this equation slightly by using the homogeneity of $\tilde{F}$

$$\frac{\partial u}{\partial t} = \sqrt{1+|y|^2}\log \tilde{F}(\lambda^{-3} r_i) + g(y), \quad y \in \mathbb{R}^n, \tag{2.14}$$

where

$$g(y) = l(y) + 3d_0\lambda \log \lambda. \tag{2.15}$$

From the maximum principle one gets an analogous comparison principle as [6, Lemma 2.1] which implies uniqueness of a solution of (2.6).

**Lemma 2.1.** *For $i = 1, 2$ let $f_i$ be two positive $C^2$-functions on $S^n$ and $H_i$ $C^{2,1}$-solutions of*

$$\frac{\partial H_i}{\partial t} = \log \tilde{F} f_i. \tag{2.16}$$

*If $H_1(x, 0) \leq H_2(x, 0)$ and $f_1(x) \leq f_2(x)$ on $S^n$ then $H_1 \leq H_2$ for all $t > 0$ and $H_1 < H_2$ unless $H_1 \equiv H_2$.*

In the following we will always assume that $H \in C^\infty(S^n \times [0, T])$ is a solution of (2.6). We denote the outer and inner radii of the hypersurface $X(\cdot, t)$ determined by $H(\cdot, t)$ by $R(t)$ and $r(t)$, respectively, and set

$$R_0 = \sup\{R(t) : t \in [0, T]\} \tag{2.17}$$

and

$$r_0 = \inf\{r(t) : t \in [0, T]\}. \tag{2.18}$$

The goal of the present section is to estimate the principal radii of curvatures of $X(\cdot, t)$ from below and above in terms of $r_0$, $R_0$ and initial data.

Lemma 2.2, Lemma 2.3, Lemma 2.4, Corollary 2.5 and Lemma 2.6 which will follow below are concerning their formulation the same as the corresponding ones in [6] but refer here to a different flow. We state them for the convenience of the reader and present proofs when differences to [6] appear. We begin with two lemmas needed in the following.

**Lemma 2.2.** *Let $r$ and $R$ be the inner and outer radii of a uniformly convex hypersurface $X$ respectively. Then there exists a dimensional constant $C$ such that*

$$\frac{R^2}{r} \leq C \sup\{R(x, \xi) : x, \xi \in S^n\}, \tag{2.19}$$

*where $R(x, \xi)$ is the principal radius of curvature of $X$ at the point with normal $x$ and along the direction $\xi$.*

*Proof.* See [6, Lemma 2.2]. □

**Lemma 2.3.** *Let $a(t), b(t) \in C^1([0, T])$ and $a(t) < b(t)$ for all $t$. Then there exists $h(t) \in C^{0,1}([0, T])$ such that*

*i) $a(t) - 2M \leq h(t) \leq b(t) + 2M$,*

*ii) $\sup\{\frac{|h(t_1)-h(t_2)|}{|t_1-t_2|} : t_1, t_2 \in [0, T]\} \leq 2\max\{\sup_t b'(t), \sup_t(-a'(t))\}$,*

*where $M = \sup_t(b(t) - a(t))$.*

*Proof.* See [6, Lemma 2.3] □

In the following lemma we prove an upper bound for the principal radii of curvature.

**Lemma 2.4.** *We assume part (i) of Assumption 1.1. For any $\gamma \in (1,2]$ there exists a constant $c_\gamma$ which may depend on initial data such that*

$$\sup\{H_{\xi\xi}(x,t) : (x,t) \in S^n \times [0,T], \xi \in T_xS^n, |\xi| = 1\} \le c_\gamma(1 + D^\gamma), \tag{2.20}$$

*where $D = \sup\{d(t) : t \in [0,T]\}$ and $d(t)$ is the diameter of $X(\cdot,t)$.*

*Proof.* We adapt the proof of [6, Lemma 2.4] by including the case of a more general speed function and implementing the novelty that we identify principal radii by zeros of certain lower dimensional matrices. Applying Lemma 2.3 to the functions $-H(-e_i,t)$ and $H(e_i,t)$ where $\pm e_i$ are the intersection points of $S^n$ with the $x_i$-axis, $i = 1,...,n+1$, we obtain $p_i(t)$ so that

$$-H(-e_i,t) - 2D \le p_i(t) \le H(e_i,t) + 2D \tag{2.21}$$

and

$$\begin{aligned} &\sup\left\{\frac{|p_i(t_1) - p_i(t_2)|}{|t_1 - t_2|} : t_1, t_2 \in [0,T]\right\} \\ &\le 2\sup\{H_t(x,t) : (x,t) \in S^n \times [0,T]\}. \end{aligned} \tag{2.22}$$

We have

$$\left|H(x,t) - \sum_{i=1}^{n+1} p_i(t)x_i\right| \le cD \quad \text{for } (x,t) \in S^n \times [0,T], \tag{2.23}$$

and by (1.1)

$$\sum_{i=1}^{n+1} |H_i(x,t) - p_i|^2 \le cD^2. \tag{2.24}$$

Let

$$\Phi(x,t) = H_{\xi\xi}(x,t) + \left[1 + \sum_{i=1}^{n+1} |H_i(x,t) - p_i(t)|^2\right]^{\frac{\gamma}{2}} \tag{2.25}$$

where $\gamma \in (1,2]$. Suppose that the supremum

$$\sup\{\Phi(x,t) : (x,t) \in S^n \times [0,T], \xi \text{ tangential to } S^n, |\xi| = 1\} \tag{2.26}$$

is attained at the south pole $x = (0,...,0,-1)$ at $t = \bar{t} > 0$ and in the direction $\xi = e_i$. For any $x$ on the south hemisphere, let

$$\xi(x) = \left(\sqrt{1 - x_1^2}, -\frac{x_1x_2}{\sqrt{1-x_1^2}}, ..., -\frac{x_1x_{n+1}}{1 - x_1^2}\right). \tag{2.27}$$

We perform the calculations in an Euclidean setting which can be achieved by considering the restriction $u$ of $H$ on $x_{n+1} = -1$. Due to the homogeneity of $H$ we obtain

$$\begin{aligned} &\sum_{i=1}^{n+1}(H_i - p_i)^2(x,t) \\ &= \sum_{i=1}^{n}(u_i(y,t) - p_i(t))^2 + \left|u(y,t) + p_{n+1} - \sum_{i=1}^{n} y_iu_i(y,t)\right|^2 \end{aligned} \tag{2.28}$$

and

$$H_{\xi\xi}(x,t) = u_{11}(y,t)\frac{(1+y_1^2+...+y_n^2)^{\frac{3}{2}}}{1+y_2^2+...+y_n^2}, \tag{2.29}$$

where $y = -(x_1, ..., x_n)/x_{n+1}$ in $\mathbb{R}^n$. The function

$$\begin{aligned}\varphi(y,t) =& u_{11}\frac{(1+y_1^2+...+y_n^2)^{\frac{3}{2}}}{1+y_2^2+...+y_n^2}\\ &+\left[1+\sum(u_i-p_i)^2+|u+p_{n+1}-\sum y_iu_i|^2\right]^{\frac{\gamma}{2}}\end{aligned} \tag{2.30}$$

attains its maximum at $(y,t) = (0,\bar{t})$ where we may w.l.o.g. assume that the Hessian of $u$ at $(0,\bar{t})$ is diagonal. Hence at $(0,\bar{t})$ we have for each $k$,

$$\begin{aligned}0 \le \varphi_t =& u_{11t} + \gamma[(u_i-p_i)(u_{it}-p_{i;t}) + (u+p_{n+1})(u_t+p_{n+1;t})]Q^{\frac{\gamma-2}{2}},\\ 0 =& \varphi_k = u_{11k} + \gamma(u_i-p_i)u_{ik}Q^{\frac{\gamma-2}{2}}\end{aligned} \tag{2.31}$$

and

$$\begin{aligned}0 \ge& \varphi_{kk} = u_{kk11} + \tau_k u_{11} + \gamma[u_{kk}^2 + (u_i-p_i)u_{ikk} - (u+p_{n+1})u_{kk}]Q^{\frac{\gamma-2}{2}}\\ &+\gamma(\gamma-2)(u_i-p_i)^2u_{ik}^2Q^{\frac{\gamma-4}{2}},\end{aligned} \tag{2.32}$$

where $Q = 1+\sum(u_i-p_i)^2 + (u+p_{n+1})^2$, $\tau_k = 1$ if $k > 1$, $\tau_1 = 3$ and $p_{i;t} = \frac{dp_i}{dt}$.

On the other hand, we are going to differentiate equation (2.14). We recall that in $(0,\bar{t})$ we have $y = 0$ and the Hessian $(u_{ij})$ is diagonal, hence $B$ is diagonal.

Let us fix $y_i = 0$, $i = 2, ..., n$, and vary $y_1$ for a moment. In this case we rewrite Equation (2.8) by using the matrices $B^1 = (B_{ij})_{i,j=1,...,n}$ and $B^2 = (B_{ij})_{i,j=2,...,n}$ as follows. We have for $r \ne 0$ that

$$\begin{aligned}&\det B = 0\\ &\Leftrightarrow \det\begin{pmatrix}-\frac{\lambda^2}{r} & y_1 & 0 & ... & 0\\ y_1 & \lambda u_{11}-r & \lambda u_{12} & ... & \lambda u_{1n}\\ 0 & \lambda u_{21} & \lambda u_{22}-r & ... & \lambda u_{2n}\\ ... & & & & \\ 0 & \lambda u_{n1} & \lambda u_{n2} & ... & \lambda u_{nn}-r\end{pmatrix} = 0\\ &\Leftrightarrow -\frac{\lambda^2}{r}\det B^1 - y_1^2\det B^2 = 0\\ &\Leftrightarrow \det B^1 + \frac{y_1^2 r}{\lambda^2}\det B^2 = 0\\ &\Leftrightarrow \det\begin{pmatrix}\lambda u_{11} - r\left(1-\frac{y_1^2}{\lambda^2}\right) & \lambda u_{12} & ... & \lambda u_{1n}\\ \lambda u_{21} & \lambda u_{22}-r & ... & \lambda u_{2n}\\ ... & & & \\ \lambda u_{n1} & \lambda u_{n2} & ... & \lambda u_{nn}-r\end{pmatrix} = 0\\ &\Leftrightarrow \det\begin{pmatrix}\lambda^3 u_{11}-r & \lambda^2 u_{12} & ... & \lambda^2 u_{1n}\\ \lambda^2 u_{21} & \lambda u_{22}-r & ... & \lambda u_{2n}\\ ... & & & \\ \lambda^2 u_{n1} & \lambda u_{n2} & ... & \lambda u_{nn}-r\end{pmatrix} = 0.\end{aligned} \tag{2.33}$$

Setting

$$(a_{ij}) = \begin{pmatrix} \lambda u_{11} & ... & \lambda u_{1n} \\ ... & & , \\ \lambda u_{n1} & ... & \lambda u_{nn} \end{pmatrix} \quad (a^1_{ij}) = \begin{pmatrix} \lambda^3 u_{11} & \lambda^2 u_{12} & ... & \lambda^2 u_{1n} \\ \lambda^2 u_{21} & \lambda u_{22} & ... & \lambda u_{2n} \\ ... & & & \\ \lambda^2 u_{n1} & \lambda u_{n2} & ... & \lambda u_{nn} \end{pmatrix} \tag{2.34}$$

the zeros of Equation (2.8) can be written as eigenvalues of the matrix $(a^1_{ij})$. Analogously, defining for $r = 1, ..., n$ the matrix $(a^r_{ij})$ as the matrix which is obtained by multiplying row $r$ and column $r$ in $(a_{ij})$ with $\lambda$ we can write the zeros of Equation (2.8) as eigenvalues of the matrix $(a^r_{ij})$ in the case where we vary $y_r$, $r$ fixed, and fix $y_i = 0$ for $i \neq r$.

Hence we may write $\tilde{F}$ in (2.14) as

$$\tilde{F} = \tilde{F}(\lambda^{-3} r_i) = \tilde{F}(\tilde{a}^r_{ij}) \tag{2.35}$$

where

$$(\tilde{a}^r_{ij}) = \lambda^{-3}(a^r_{ij}) \tag{2.36}$$

if $(y, t) = (0, ..., 0, y_r, 0, ..., 0, t)$. And we have in $(0, \bar{t})$ that

$$\frac{\partial \tilde{F}}{\partial y_k} = \tilde{F}^{ii} a^k_{ii;k} \quad \wedge \quad \frac{\partial^2 \tilde{F}}{\partial {y_k}^2} = \tilde{F}^{ii} a^k_{ii;kk} + \tilde{F}^{ij,rs} a^k_{ij;k} a^k_{rs;k} \tag{2.37}$$

where we do not sum over $k$ and where we used [8, Lemma 2.1.9] to deduce that $\tilde{F}^{ij}$ is diagonal. Here and in the following we sometimes denote partial derivatives by indices separated by a semicolon for greater clarity of the presentation

Differentiating (2.14) gives in $(0, \bar{t})$ that

$$\begin{aligned} u_{kt} =& \left(1 + |y|^2\right)^{-\frac{1}{2}} y_k \log \tilde{F} + \sqrt{1 + |y|^2} \frac{1}{\tilde{F}} \tilde{F}^{ij} \tilde{a}^k_{ij;k} + g_k \\ u_{kkt} =& \log \tilde{F} - \frac{1}{\tilde{F}^2} \tilde{F}^{ij} \tilde{a}^k_{ij;k} \tilde{F}^{rs} \tilde{a}^k_{rs;k} + \frac{1}{\tilde{F}} \tilde{F}^{ij,rs} \tilde{a}^k_{ij;k} \tilde{a}^k_{rs;k} \\ &+ \frac{1}{\tilde{F}} \tilde{F}^{ij} \tilde{a}^k_{ij;kk} + g_{kk}, \end{aligned} \tag{2.38}$$

here, we do not sum over $k$. Hence at $(0, \bar{t})$ we have

$$\begin{aligned} 0 \geq& \sum_k \frac{1}{\tilde{F}} \tilde{F}^{kl} \varphi_{kl} - \varphi_t \\ =& \sum_k \frac{1}{\tilde{F}} \tilde{F}^{kk} \varphi_{kk} - \varphi_t \\ =& \frac{1}{\tilde{F}} \sum_k \tilde{F}^{kk} u_{kk11} + \frac{1}{\tilde{F}} \tilde{F}^{kk} u_{11} \tau_k \\ &+ \gamma \{ \frac{1}{\tilde{F}} \tilde{F}^{kk} u^2_{kk} [1 + \frac{(\gamma - 2)(u_k - p_k)^2}{1 + \sum (u_i - p_i)^2 + (u + p_{n+1})^2}] \\ &+ (u_i - p_i)(\frac{1}{\tilde{F}} \tilde{F}^{rs} u_{irs} - u_{it}) - \frac{1}{\tilde{F}} \tilde{F}^{kk} u_{kk}(u + p_{n+1}) \\ &- (u + p_{n+1})(u_t + p_{n+1;t}) + (u_i - p_i) p_{i;t} \} Q^{\frac{\gamma - 2}{2}} - u_{11t} \end{aligned} \tag{2.39}$$

and we can estimate this further from below by

$$
\begin{aligned}
&\frac{1}{\tilde F}\tilde F^{kk}u_{11}-\log\tilde F+\frac{1}{\tilde F^2}\tilde F^{ij}\tilde a^1_{ij;1}\tilde F^{rs}\tilde a^1_{rs;1}-\frac{1}{\tilde F}\tilde F^{ij,rs}\tilde a^1_{ij;1}\tilde a^1_{rs;1}\\
&-\frac{1}{\tilde F}\tilde F^{kk}\tilde a^1_{kk;11}-g_{11}+\frac{1}{\tilde F}\tilde F^{kk}u_{kk11}\\
&+\gamma\{(\gamma-1)\frac{1}{\tilde F}\tilde F^{kk}u^2_{kk}-(u_i-p_i)g_i\\
&+\frac{1}{\tilde F}\tilde F^{rs}(u_{rsi}-\tilde a^i_{rs;i})(u_i-p_i)-\frac{1}{\tilde F}\tilde F^{kk}u_{kk}(u+p_{n+1})\\
&-(u+p_{n+1})(u_t+p_{n+1,t})+(u_i-p_i)p_{i;t}\}Q^{\frac{\gamma-2}{2}}\\
\geq&\frac{1}{\tilde F}\tilde F^{kk}u_{11}-\log\tilde F\\
&+\frac{1}{\tilde F}\tilde F^{kk}(u_{kk11}-\tilde a^1_{kk;11})-g_{11}\\
&+\gamma\{(\gamma-1)\epsilon_0\tilde H-(u_i-p_i)g_i+\frac{1}{\tilde F}\tilde F^{rs}(u_{rsi}-\tilde a^i_{rs;i})(u_i-p_i)\\
&-d_0(u+p_{n+1})-(u+p_{n+1})(u_t+p_{n+1,t})+(u_i-p_i)p_{i,t}\}Q^{\frac{\gamma-2}{2}}
\end{aligned}
\tag{2.40}
$$

where we used (1.11) and (1.12) or the concavity of $\tilde F$, cf. Assumption 1.3, and denoted the trace of $(u_{ij})$ by $\tilde H$. From

$$u_{ij}=\tilde a^r_{ij}\quad\wedge\quad u_{ij;k}=\tilde a^r_{ij;k}\quad\wedge\quad\tilde a^r_{rr;11}=u^r_{rr;11}\tag{2.41}$$

and

$$\tilde a^1_{ii;11}=u_{ii;11}-2u_{ii}\tag{2.42}$$

for $i\neq 1$ in $(0,\bar t)$ we conclude that

$$
\begin{aligned}
0\geq&\frac{1}{\tilde F}\tilde F^{kk}u_{11}-\log\tilde F-g_{11}\\
&+\gamma\{(\gamma-1)\epsilon_0u_{11}-(u_i-p_i)g_i-d_0(u+p_{n+1})\\
&-(u+p_{n+1})(u_t+p_{n+1,t})+(u_i-p_i)p_{i,t}\}Q^{\frac{\gamma-2}{2}}.
\end{aligned}
\tag{2.43}
$$

From (2.23) and (2.24) we deduce that $|u+p_{n+1}|\leq cD$ and $|u_i-p_i|\leq cD$ so that

$$\gamma(\gamma-1)\epsilon_0cD^{\gamma-2}u_{11}\leq\log\tilde F+c+cQ^{\frac{\gamma-2}{2}}D(1+|u_t|+|H_t|)\tag{2.44}$$

and hence

$$u_{11}\leq cD^{2-\gamma}\log u_{11}+cD^{2-\gamma}+cD(1+\log u_{11})\tag{2.45}$$

which implies the claim. □

**Corollary 2.5.** *We assume part (i) of Assumption 1.1. For any $\gamma\in(1,2]$ there exists $\delta=\delta(\gamma)>0$ such that*

$$r(t)\geq\frac{\delta R(t)^2}{1+\sup_{\tau\leq t}R^\gamma(\tau)}.\tag{2.46}$$

*Proof.* Use Lemma 2.2 and Lemma 2.4. □

In the following lemma we estimate $H_t$ from below. In view of Lemma 2.4 and Equation (2.6) this immediately implies a lower bound for the principal radii of curvature.

**Lemma 2.6.** *We assume part (i) of Assumption 1.1. There exists a constant $c$ depending only on $n$, $r_0$, $R_0$, $f$ and initial data such that*

$$\inf\{H_t(x,t) : (x,t) \in S^n \times [0,T]\} \geq -c. \tag{2.47}$$

*Proof.* We adapt the proof of [6, Lemma 2.6]. Let

$$q(t) = \frac{1}{|S^n|}\int_{S^n} xH(x,t)d\sigma(x) \tag{2.48}$$

be the Steiner point of $X(\cdot,t)$. Then there exists a positive $\delta$ which depends only on $n$, $r_0$ and $R_0$ so that

$$H(x,t) - q(t)\cdot x \geq 2\delta. \tag{2.49}$$

We assume that the function

$$\psi(x,t) = \frac{H_t(x,t)}{H(x,t) - x\cdot q(t) - \delta} \tag{2.50}$$

attains its negative infimum on $S^n \times [0,T]$ at $x = (0,...,0,-1)$ and $\bar{t} \in (0,T]$ and that $(u_{ij})$ is diagonal. Let $u$ be the restriction of $H$ to $x_{n+1} = -1$ as before. Then

$$\psi(y,t) = \frac{u_t(y,t)}{u(y,t) - q(t)\cdot(y,-1) - \delta\sqrt{1+|y|^2}} \tag{2.51}$$

attains its negative minimum at $(0,\bar{t})$. Hence in this point we have

$$0 \geq \psi_t = \frac{u_{tt}}{u + q_{n+1}(t) - \delta} - \frac{u_t(u_t + \frac{dq_{n+1}}{dt})}{(u + q_{n+1}(t) - \delta)^2}, \tag{2.52}$$

$$0 = \psi_k = \frac{u_{tk}}{u + q_{n+1}(t) - \delta} - \frac{u_t(u_k - q_k(t))}{(u + q_{n+1}(t) - \delta)^2} \tag{2.53}$$

and

$$0 \leq \psi_{kk} = \frac{u_{tkk}}{u + q_{n+1}(t) - \delta} - \frac{u_t u_{kk}}{(u + q_{n+1}(t) - \delta)^2} + \frac{\delta u_t}{(u + q_{n+1}(t) - \delta)^2}. \tag{2.54}$$

Using the notation from the proof of Lemma 2.4 we get on the other hand by differentiating (2.14) that in $(0,\bar{t})$

$$u_{tt} = \frac{1}{\tilde{F}}\tilde{F}^{ij}u_{ijt}. \tag{2.55}$$

We have in $(0,\bar{t})$ using that $(\tilde{F}^{ij})$ is diagonal

$$\begin{aligned} 0 \leq& \sum \frac{1}{\tilde{F}}\tilde{F}^{kk}\psi_{kk} - \psi_t \\ \leq& \frac{\delta u_t \frac{1}{\tilde{F}}\sum \tilde{F}^{kk} - \frac{1}{\tilde{F}}\tilde{F}^{kk}u_{kk}u_t + u_t(u_t + \frac{dq_{n+1}}{dt})}{(u + q_{n+1} - \delta)^2}. \end{aligned} \tag{2.56}$$

Since $u_t$ is negative at $(0,\bar{t})$, it follows that

$$\begin{aligned} \frac{1}{\tilde{F}}\sum_k \tilde{F}^{kk} \leq& \frac{c}{\delta}(1 + |u_t|) \\ \leq& \frac{c}{\delta}(1 + \log \tilde{F}^{-1}) \end{aligned} \tag{2.57}$$

where we used the homogeneity of $\tilde{F}$ and where $c = c(f, R_0)$.

Now we distinguish cases. In case (i) of Assumption 1.3 we have

$$\sum_k \tilde{F}^{kk} \geq F(1,1...,1) \tag{2.58}$$

in view of [8, Lemma 2.2.19]. It follows that $\tilde{F} \geq c > 0$ and

$$\begin{aligned} u_t &\geq -c + c\log\tilde{F} \\ &\geq -c \end{aligned} \tag{2.59}$$

where $c$ depends on $n$, $r_0$, $R_0$, $f$ and initial data as claimed.

In case (ii) of Assumption 1.3 we choose $i_0 \in \{1,...,n\}$ such that

$$u_{i_0 i_0} = \min_{1\leq i\leq n} u_{ii} \tag{2.60}$$

and hence

$$\tilde{F} = \frac{1}{d_0}\tilde{F}^{ii}u_{ii} \leq c\tilde{F}^{i_0 i_0}u_{i_0 i_0} \tag{2.61}$$

in view of the homogeneity of $\tilde{F}$ and [8, Lemma 2.2.4]. Hence we estimate

$$\begin{aligned} \sum_k \tilde{F}^{kk} &\geq \tilde{F}^{i_0 i_0} \\ &\geq \frac{\tilde{F}}{cu_{i_0 i_0}} \end{aligned} \tag{2.62}$$

and deduce from (2.57) that

$$(u_{i_0 i_0})^{-1} \leq c(1 + \log((u_{i_0 i_0})^{-1}) \tag{2.63}$$

so that $\tilde{F} \geq c > 0$ and the claim follows as in case (i). □

We need versions of Lemma 2.4, Corollary 2.5 and Lemma 2.6 which hold in the case (ii) of Assumption 1.1. (These analogous versions even hold for $F = H_k$, $k = 1, ..., n$, where the $H_k$ denote the elementary symmetric polynomials, general positive $f$ and $M_\Theta$ without special symmetry assumptions while being stated only in the more restrictive case for which we show Theorem 1.2.) For Lemma 2.4 we obtain the following analogon.

**Lemma 2.7.** *We assume case (ii) of Assumption 1.1. There exist constants $c_1, c_2 > 0$ such that*

$$\sup\{H_{\xi\xi}(x,t) : (x,t) \in S^n \times [0,T], \xi \in T_xS^n, |\xi| = 1\} \leq c_1(1 + D^{c_2}), \tag{2.64}$$

*where $D = \sup\{d(t) : t \in [0,T]\}$ and $d(t)$ is the diameter of $X(\cdot,t)$.*

*Proof.* Following the proof of Lemma 2.4 and adopting notation, we obtain analogously to (2.43) that we have in $(0,\bar{t})$

$$\begin{aligned} 0 \geq& \frac{1}{\tilde{F}}\tilde{F}^{kk}u_{11} - \log\tilde{F} - g_{11} \\ &+ \gamma\{(\gamma-1)\frac{1}{\tilde{F}}\tilde{F}^{kk}u_{kk}^2 - (u_i - p_i)g_i - d_0(u + p_{n+1}) \\ &- (u + p_{n+1})(u_t + p_{n+1,t}) + (u_i - p_i)p_{i,t}\}Q^{\frac{\gamma-2}{2}}. \end{aligned} \tag{2.65}$$

W.l.o.g. let us assume $u_{11} \geq ... \geq u_{nn}$ and we define $\mu = \frac{u_{11}}{u_{nn}}$. In view of [8, Lemma 2.2.4] and the homogeneity of $\tilde{F}$ we have

$$\frac{1}{\tilde{F}}\tilde{F}^{kk}u_{11} = \sum_k \frac{1}{\tilde{F}}\tilde{F}^{kk}u_{11} \geq \mu\frac{1}{\tilde{F}}\tilde{F}^{nn}u_{nn} \geq d_0\frac{\mu}{n} \tag{2.66}$$

and

$$\frac{1}{\tilde{F}}\tilde{F}^{kk}u_{kk}^2 \geq \frac{1}{\tilde{F}}\tilde{F}^{kk}u_{kk}u_{nn} = d_0\frac{u_{11}}{\mu}. \tag{2.67}$$

From (2.23) and (2.24) we deduce that $|u+p_{n+1}| \leq cD$ and $|u_i - p_i| \leq cD$. Putting (2.65), (2.66) and (2.67) together we obtain

$$cD^{\gamma-2}(\mu + \frac{u_{11}}{\mu}) \leq \log\tilde{F} + c + cQ^{\frac{\gamma-2}{2}}D(1+|u_t|+|H_t|) \tag{2.68}$$

and hence

$$\mu + \frac{u_{11}}{\mu} \leq cD^{2-\gamma}\log u_{11} + cD^{2-\gamma} + cD(1+\log u_{11}) \tag{2.69}$$

which implies the claim. □

Furthermore, we have the following analoga with proofs as before.

**Corollary 2.8.** *We assume case (ii) of Assumption 1.1. There exist $c_1, c_2 > 0$ such that*

$$r(t) \geq \frac{c_1 R(t)^2}{1+\sup_{\tau\leq t}(R(\tau))^{c_2}}. \tag{2.70}$$

**Lemma 2.9.** *We assume case (ii) of Assumption 1.1. There exists a constant $c$ depending only on $n$, $r_0$, $R_0$, $f$ and initial data such that*

$$\inf\{H_t(x,t) : (x,t) \in S^n \times [0,T]\} \geq -c. \tag{2.71}$$

Using a comparison principle and comparing the flow (2.6) with the ODE

$$\frac{\partial\rho}{\partial t} = \log\left(\frac{\rho^{d_0}}{F(1,...,1)}\right)M, \quad \rho(0) = \rho_0, \tag{2.72}$$

where $M = \max\{f(x) : x \in S^n\}$ and $\rho_0$ sufficiently large, we obtain that $H(x,t)$ is bounded in any finite time interval. Furthermore, its gradient is also bounded by (2.2). From Krylov-Safonov estimates and parabolic regularity theory, cf. [12], one gets that problem (2.6) has for $H_\Theta \in C^{4+\alpha}(S^n)$ a unique $C^{4+\alpha,2+\frac{\alpha}{2}}$ solution in a maximal interval $[0,T^*)$, $T^* \leq \infty$ and since $H_\Theta$ is even of class $C^\infty$ in our case that this solution is also of class $C^\infty$. For the outer radius $R(t)$ of $X(\cdot,t)$ we have

$$\lim_{t\uparrow T^*} R(t) = 0 \tag{2.73}$$

if $T^*$ is finite.

## 3. Proof of Theorem 1.2 (i) and (ii) in case (i) of Assumption 1.1

We begin with some elementary properties of the foliation $(M_\Theta)_{\Theta>0}$ in the following two remarks.

**Remark 3.1.** For each $M_\Theta$ we denote the to $M_\Theta$ associated open convex body by $C_\Theta$ and have w.l.o.g. (otherwise consider $1/\Theta$)

$$\Theta_1 < \Theta_2 \Rightarrow \overline{C_{\Theta_1}} \subset C_{\Theta_2}. \tag{3.1}$$

Furthermore, all $C_\Theta$ contain 0, otherwise

$$0 < d := \inf\{\Theta > 0 : \forall_{\tilde{\Theta}\geq\Theta}\ 0 \in C_{\tilde{\Theta}}\} < \infty \tag{3.2}$$

where the last inequality is due to the fact that for $p \in \mathbb{R}^{n+1} \setminus \{0\}$ there is $\Theta(p) > 0$ so that $p, -p \in C_{\Theta(p)}$ and hence also $0 \in C_{\Theta(p)}$. We conclude $0 \in M_d$, a contradiction.

**Remark 3.2.** For all $r > 0$ exist $\Theta_1, \Theta_2 > 0$ so that

$$M_{\Theta_1} \subset B_r(0) \subset C_{\Theta_2}. \tag{3.3}$$

*Proof.* Let $r > 0$. Existence of $\Theta_2$ as claimed is clear in view of

$$\overline{B_r(0)} \subset \bigcup_{\Theta>0} C_\Theta. \tag{3.4}$$

Assume there are sequences $0 < \Theta_k \to 0$, $x_k \in C_{\Theta_k}$, $x_k \notin B_r(0)$. W.l.o.g. assume $x_k \to x \in B_r(0)^c$. Let $p = \frac{x}{2}$. There is $\Theta = \Theta(p) > 0$ so that $p \in M_{\Theta(p)}$. If $[0, x]$ meets $M_{\Theta(p)}$ tangentially in $p$ then $0 \notin C_{\Theta(p)}$ in view of the uniform convexity of $M_{\Theta(p)}$ which is a contradiction. Hence there is a neighborhood $U$ of $x$ so that for every $q \in U$ the segment $[0, q]$ meets $M_{\Theta(p)}$ non-tangentially. This implies

$$U \subset (C_{\Theta(p)})^c \subset (C_{\Theta_k})^c \tag{3.5}$$

for large $k$. On the other hand

$$x_k \in U \cap C_{\Theta_k} \tag{3.6}$$

for large $k$, a contradiction. □

*Proof of Theorem 1.2 (i) and (ii) in the case (i) of Assumption 1.1.* (i) We follow the proof of [6, Theorem A] but use different arguments to deduce convergence to a translating solution. Let $m = \inf_{S^n} f$ and $M = \sup_{S^n} f$. If the initial hypersurface $X_\Theta$ is a sphere of radius $\rho_0 > \left(\frac{F(1,\ldots,1)}{m}\right)^{\frac{1}{d_0}}$, the solution $X(\cdot, t)$ to the equation

$$\frac{\partial X}{\partial t} = -\log\frac{F}{m}\nu, \quad X(\cdot, 0) = X_\Theta, \tag{3.7}$$

remains to be spheres and the flow expands to infinity as $t \to \infty$. On the other hand, if $X_\Theta$ is a sphere of radius less than $\left(\frac{F(1,\ldots,1)}{M}\right)^{\frac{1}{d_0}}$, the solution to

$$\frac{\partial X}{\partial t} = -\log\frac{F}{M}\nu, \quad X(\cdot, 0) = X_\Theta, \tag{3.8}$$

is a family of spheres which shrinks to a point in finite time. By the comparison principle and Remark 3.2 the solution $X(x, t)$ of (1.3) will shrink to a point if $\Theta$ is small enough, and will expand to infinity if $\Theta > 0$ is large.

Hence using Corollary 2.5 we obtain that the sets

$$\begin{aligned} A =& \{\Theta > 0 : X(\cdot, t) \text{ shrinks to a point in finite time}\} \\ B =& \{\Theta > 0 : X(\cdot, t) \text{ expands to infinity as } t \to \infty\} \end{aligned} \tag{3.9}$$

are non-empty and open since the solution $X(x,t)$ of (1.3) on a fixed finite time interval $[0,T)$ depends continuously on $\Theta$. We define

$$\Theta_* = \sup A \tag{3.10}$$

and

$$\Theta^* = \inf B. \tag{3.11}$$

and deduce $\Theta_* \leq \Theta^*$ from the comparison principle.

Using Corollary 2.5 we deduce that for any $\Theta \in [\Theta_*, \Theta^*]$ the inner radii of $X(\cdot,t)$ have a uniform positive lower bound and the outer radii are uniformly bounded from above, furthermore, $T^* = \infty$ in view of (2.73). Hence (2.6) is uniformly parabolic and we have uniform bounds for $D_t^k D_x^l X(\cdot,\cdot)$ if $k+l \geq 1$, $k \geq 0$ and $l \geq 0$ on $S^n \times [0,\infty)$.

(ii) Let $\Theta \in [\Theta_*, \Theta^*]$. We shall use a method from [14] to show that our solution that exists for all positive times converges to a translating solution. The main difference from our case to [14] is that we argue on the level of a derivative of the support function while [14] uses a graphical representation of the flow hypersurfaces.

One easily checks that a family of smoothly evolving uniformly convex hypersurfaces represented by its family of support functions $\tilde{H}(\cdot,t)$ is translating iff there is $\xi \in \mathbb{R}^{n+1}$ so that

$$\tilde{H}(x,t) = \tilde{H}(x,0) + t\xi x, \quad x \in \mathbb{R}^{n+1}. \tag{3.12}$$

Let us fix $1 \leq \gamma \leq n+1$ and let $e_\gamma$ denote the corresponding standard basis vector. Differentiating the homogeneous degree one extension (not relabeled) of (3.12) with respect to $x$ in direction $e_\gamma$ we get

$$\frac{\partial}{\partial x^\gamma}\tilde{H}(x,t) = \frac{\partial}{\partial x^\gamma}\tilde{H}(x,0) + t\xi_\gamma. \tag{3.13}$$

Hence $\frac{\partial}{\partial x^\gamma}\tilde{H}(\cdot,t)$ is a scalar translating function. Conversely, if (3.13) holds then $\tilde{H}$ satisfies (3.12). Note, that $\tilde{H}(0,t) = 0$ and that $\frac{\partial}{\partial x^\gamma}\tilde{H}(\cdot,t)$ is homogeneous of degree zero.

Let $H$ be a solution of (2.6). We denote the homogeneous degree one extension of $H$ to $\mathbb{R}^{n+1}$ again by $H$ and the homogeneous degree 0 extension of $f$ to $\mathbb{R}^{n+1}\setminus\{0\}$ also by $f$. We recall the flow equation for $H$

$$\frac{\partial H}{\partial t} = \log \tilde{F} f \quad \text{in} \quad S^n \times [0,\infty), \tag{3.14}$$

where $\tilde{F} = \tilde{F}(r_i)$ and $r_i$, $i = 1, ..., n$, are the principal radii of $M(t)$ given as non-zero eigenvalues of the Hessian matrix $\left(\frac{\partial^2 H}{\partial x_\alpha \partial x_\beta}\right)_{\alpha,\beta=1,\ldots,n+1}$. Using the homogeneity of $H$ this can be rewritten as a flow equation for $H$ on $\left(\mathbb{R}^{n+1}\setminus\{0\}\right)\times[0,\infty)$

$$\begin{aligned} \frac{\partial H}{\partial t}(x,t) =& |x|\frac{\partial H}{\partial t}\left(\frac{x}{|x|},t\right) \\ =& |x| \log \tilde{F} f \end{aligned} \tag{3.15}$$

where $\tilde{F} = \tilde{F}(r_i)$ and $r_i$, $i = 1, ..., n$, are the principal radii of $M(t)$ given as non-zero eigenvalues of the matrix $\left(|x|\frac{\partial^2 H}{\partial x_\alpha \partial x_\beta}\right)_{\alpha,\beta=1,\ldots,n+1}$ at $(x,t)$ and $f = f(x)$. We will replace (formally) the curvature function $\tilde{F}$ in Equation (3.15) by a curvature function

$\hat{F}$ which depends on all eigenvalues $r_\alpha$, $\alpha = 1, ..., n+1$, of $\left(|x| \frac{\partial^2 H}{\partial x_\alpha \partial x_\beta}\right)_{\alpha,\beta=1,...,n+1}$ at $(x,t)$ and satisfies $\tilde{F}(r_i) = \hat{F}(r_\alpha)$ in order to be notational in the framework of the introduction.

a) In the case that $\tilde{F} \in C^\infty(\bar{\Gamma}_+)$ and $\tilde{F}_{|\partial\Gamma_+} = 0$ we define

$$\hat{F}(r_1, ..., r_{n+1}) = \sum_{\alpha_0=1}^{n+1} \tilde{F}(\hat{r}^{\alpha_0}) \tag{3.16}$$

where $\hat{r}^{\alpha_0} = (r_1, ..., r_{\alpha_0-1}, r_{\alpha_0+1}, ..., r_{n+1})$.

b) Let us consider the general case (which includes case a)). In view of our a priori estimates there are constants $b_1, b_2 > 0$ so that the non-zero eigenvalues of $\left(\frac{\partial^2 H}{\partial x_\alpha \partial x_\beta}\right)_{\alpha,\beta=1,...,n+1}$ on $S^n \times [0,\infty)$ are in the interval $[b_1, b_2]$. Having the later application of the argumentation in [14, Subsection 6.2] in mind we remark that this property carries over to the Hessians of convex combinations of $H(\cdot, t_1)$ and $H(\cdot, t_2)$ with arbitrary $t_1, t_2 > 0$. Note that the vector $x$ is a zero eigenvector of the Hessian of $H$ at every $(x,t) \in S^n \times [0,\infty)$. We define

$$\hat{F}(r_1, ..., r_{n+1}) = \tilde{F}(\hat{r}) + \check{r} \tag{3.17}$$

on the set

$$\Omega = \bigcup_{1\le\alpha\le n+1} I_\alpha \tag{3.18}$$

where

$$I_\alpha = \left(\frac{b_1}{2}, \infty\right) \times ... \times \left(\frac{b_1}{2}, \infty\right) \times \left(-\frac{b_1}{2}, \frac{b_1}{2}\right) \times \left(\frac{b_1}{2}, \infty\right) \times ... \times \left(\frac{b_1}{2}, \infty\right) \tag{3.19}$$

with factor $(-\frac{b_1}{2}, \frac{b_1}{2})$ at position $\alpha$ and where $\check{r} = r_{\alpha_0} = \min_{\alpha=1,...,n+1} r_\alpha$, $\alpha_0 \in \{1, ..., n+1\}$ suitable, and $\hat{r} = (r_1, ...r_{\alpha_0-1}, r_{\alpha_0+1}, ..., \alpha_{n+1})$. We have $\tilde{F}(r_i) = \hat{F}(r_\alpha)$. From standard arguments we deduce that $\hat{F}$ defines in the way explained in the introduction a differentiable function on the set of symmetric matrices with eigenvalues in $\Omega$.

Differentiating (3.15) we get the following equation for $H_\gamma$

$$\frac{\partial}{\partial t} H_\gamma(x,t) = |x|^2 \frac{1}{\hat{F}} \hat{F}^{\alpha\beta} \left(H_\gamma\right)_{\alpha\beta} + |x|_\gamma \log \hat{F} f + |x| \frac{f_\gamma}{f} + d_0 |x|_\gamma \tag{3.20}$$

where $\hat{F}^{\alpha\beta}$ is uniformly elliptic and the coefficients of the elliptic operator on the right-hand side depend on the derivative of $H_\gamma$ and $x$ and not explicitly on $t$ or $H_\gamma$.

Applying the argumentation from [14, Subsection 6.2] more or less word by word to the function $H_\gamma$ on $(B_{\rho_2}(0) \setminus B_{\rho_1}(0)) \times [0,\infty)$, $0 < \rho_1 < 1 < \rho_2$ both close to 1, where we use that $H_\gamma$ is homogeneous of degree zero (instead of the compactness of the spatial domain and the boundary condition when we apply maximum principles) we obtain that $H_\gamma$ converges smoothly to a translating solution of (3.20) with a translating speed $\xi = \xi(\Theta, \gamma) \in \mathbb{R}$.

(iii) We show $\Theta_* = \Theta^*$. From (ii) we know that for every $\Theta \in [\Theta_*, \Theta^*]$ the solution $X(x,t)$ of (1.3) with initial value $X_\Theta$ converges to a translating solution with a certain translating speed $\xi_\Theta \in \mathbb{R}^{n+1}$.

a) We show that there is $\xi \in \mathbb{R}^{n+1}$ so that $\xi_\Theta = \xi$ for all $\Theta \in [\Theta_*, \Theta^*]$. For this let $\Theta_* \le \Theta_1 < \Theta_2 \le \Theta^*$, differentiating (2.6) in $\Theta$ gives

$$\begin{aligned}\frac{\partial H'}{\partial t} =& A^{ij}(\nabla_i\nabla_j H' + H'\delta_{ij})\\ H'(0) =& \frac{d}{d\Theta}H_\Theta\end{aligned} \tag{3.21}$$

where $(A^{ij})$ is the inverse of $(\nabla_i\nabla_j H + \delta_{ij}H)$. By the maximum principle

$$H'(x,t) \ge \min_{S^n} \frac{d}{d\Theta}H_\Theta(x). \tag{3.22}$$

Thus

$$\begin{aligned}c(x,t) + t(\xi_{\Theta_2} - \xi_{\Theta_1})x =& H_{\Theta_2}(x,t) - H_{\Theta_1}(x,t)\\ \ge& \int_{\Theta_1}^{\Theta_2} \min_{S^n} \frac{d}{d\Theta}H_\Theta > 0\end{aligned} \tag{3.23}$$

where $c(x,t)$ is a uniformly bounded function and where we used Lemma 3.3. This implies $\xi_{\Theta_1} = \xi_{\Theta_2}$.

b) Using a) we deduce from the comparison principle that $H_* = H^*$ where $H_*$ and $H^*$ is the solution of $F = e^{\xi x}f$ starting from $H_{\Theta_*}$ and $H_{\Theta^*}$, respectively. We deduce from (3.23) with $\Theta_1 = \Theta_*$ and $\Theta_2 = \Theta^*$ by using that $H_{\Theta_2}(\cdot,t) - H_{\Theta_1}(\cdot,t)$ converges uniformly to zero as $t \to \infty$ that $\Theta_* < \Theta^*$ leads to a contradiction, hence $\Theta_* = \Theta^*$.

(iv) We show that the normalized hypersurface $X(\cdot,t)/r(t)$ converges to a unit sphere in case $\Theta > \Theta^*$ and follow for it the lines of [6, Theorem B]. Since $X$ is expanding, we may w.l.o.g. assume at $t = 0$ that it contains the ball $B_{R_1}(0)$ where $R_1 > 1 + \left(\frac{F(1,\dots,1)}{m}\right)^{\frac{1}{d_0}}$, $m = \inf_{S^n} f$, and that it is contained in the ball $B_{R_2}(0)$ where $R_2 > 0$ is sufficiently large. For $i = 1,2$ let $X_i(\cdot,t)$ be the solution of (1.3) where $f$ is replaced by $m$ and $M = \sup_{S^n} f$ respectively and $X_i(\cdot,0) = \partial B_{R_i}$. The $X_i(\cdot,t)$ are spheres and their radii $R_i(t)$ satisfy

$$c^{-1}(1+t)\log(1+t) \le R_1(t) \le R_2(t) \le c(1 + (1+t)\log^2(1+t)) \tag{3.24}$$

for some $c > 0$. We deduce from the ODEs for the $R_i$ , $i = 1,2$, that

$$\begin{aligned}\frac{d}{dt}(R_2(t) - R_1(t)) \le& d_0 \log\frac{R_2(t)}{R_1(t)} + c\\ \le& c\log\log(1+t) + c\end{aligned} \tag{3.25}$$

where the last inequality uses (3.24) and hence

$$R_2(t) - R_1(t) \le c(1 + t\log\log(1+t)) \tag{3.26}$$

so that

$$\lim_{t\to\infty} \frac{R_2(t) - R_1(t)}{R_1(t)} = 0. \tag{3.27}$$

By the comparison principle $X(\cdot,t)$ is pinched between $X_2(\cdot,t)$ and $X_1(\cdot,t)$ and, furthermore, we deduce that $X(\cdot,t)/r(t)$ converges to the unit sphere uniformly.

The proof of Theorem 1.2 (i) and (ii) is finished in the case (i) of Assumption 1.1. $\square$

**Lemma 3.3.**

$$\frac{d}{d\Theta} H_\Theta > 0. \tag{3.28}$$

*Proof.* Let $0 < \Theta_1 < \Theta_2 < \infty$, $x \in S^n$. In view of $D\Theta \neq 0$ there is $c_0 = c_0(\Theta_1) > 0$ so that

$$\operatorname{dist}(M_{\Theta_1}, M_{\Theta_2}) \geq c_0(\Theta_2 - \Theta_1). \tag{3.29}$$

For $x \in S^n$ let $y_x \in M_{\Theta_1}$ be so that

$$H_{\Theta_1}(x) = x y_x > 0 \tag{3.30}$$

and hence also

$$c_1 = \inf_{x \in S^n} x \frac{y_x}{|y_x|} > 0. \tag{3.31}$$

Let $y$ be the intersection of the ray starting in 0 through $y_x$ with $M_{\Theta_2}$ then

$$x \cdot y \geq x \cdot y_x + c_0 c_1 (\Theta_2 - \Theta_1) \tag{3.32}$$

hence

$$\begin{aligned} H_{\Theta_2}(x) \geq & x \cdot y_x + c_0 c_1 (\Theta_2 - \Theta_1) \\ = & H_{\Theta_1}(x) + c_0 c_1 (\Theta_2 - \Theta_1) \end{aligned} \tag{3.33}$$

6 which implies

$$\left( \frac{d}{d\Theta} H_\Theta(x) \right)_{|\Theta = \Theta_1} > 0. \tag{3.34}$$

□

Combining the proofs of [6, Theorem A] and Theorem 1.2 in the case (i) of Assumption 1.1 we get the following corollary which generalizes [6, Theorem A] to our more general foliation.

**Corollary 3.4.** *In the situation of Theorem 1.2 (i) with $F = K$ the translating speed $\xi$ is uniquely determined by*

$$\int_{S^n} \frac{x_i}{e^{\xi \cdot x} f(x)} d\sigma(x) = 0, \quad i = 1, ..., n+1. \tag{3.35}$$

*And the Gauss curvature of $X^*$, when regarded as a function of the normal, is equal to $e^{\xi \cdot x} f(x)$.*

## 4. Proof of Theorem 1.2 (i) in case (ii) of Assumption 1.1 & proof of Theorem 1.2 (iii)

Throughout this section (with an exception in a short passage below which we indicate explicitly) we assume that case (ii) of Assumption 1.1 holds and prove Theorem 1.2 in this case by presenting (only) the arising differences to the proof of Theorem 1.2 (i) and (ii) under Assumption 1.1 (i) in the previous section. The crucial difference is that the set $B$, cf. (3.9), has now to be redefined in view of Corollary 2.8 (which states only a poor lower bound for the inradii) and that its redefined version cannot be identified as open immediately, so further work is necessary. Deviating from (3.9) as far as $B$ is concerned we define the intervals $A$ and $B$ (intervals due to a comparison principle) now as

$$\begin{aligned} A =& \{\Theta > 0 : X(\cdot, t) \text{ shrinks to a point in finite time}\} \\ B =& \{\Theta > 0 : \operatorname{diam} X(\cdot, t) \text{ converges to infinity as } t \to \infty\} \end{aligned} \tag{4.1}$$

and $\Theta_* = \sup A$ and $\Theta^* = \inf B$. Similarly as in the proof of part (i) of Theorem 1.2 under Assumption 1.1 (i) one obtains that $[\Theta_*, a_1]$ is empty whenever $\Theta_* < a_1 < \Theta^*$ and hence $\Theta_* = \Theta^*$. Clearly, by comparing with spheres, $A$ is open. The openness of $B$ which is a priori not clear under our present assumptions given by case (ii) of Assumption 1.1 follows from the following Lemma 4.1 which will be proven by using geometric observations. Once this openness is established following the lines of the proof in the previous section we obtain convergence to a translating solution which translates with a certain speed $\xi \in \mathbb{R}^{n+1}$ and with mean curvature when considered as a function of the normal given by

$$e^{\xi \cdot x} f, x \in S^n, \tag{4.2}$$

where $\xi = (\xi_x, 0, 0)$, $\xi_x \in \mathbb{R}$, in view of the symmetry of $f$ and $M_\Theta$. Hence, inclusively Lemma 4.1, the proofs of Theorem 1.2 (i) and (ii) are complete.

Before we prove the crucial Lemma 4.1 below let us complete in this short passage the remaining proof of Theorem 1.2 (iii) and assume for the duration of this passage the assumptions therein. First note that the representation of the $F$- and accordingly mean curvature of the limit hypersurface by (4.2) holds. Since $f$ is even and the foliation $(M_\Theta)_\Theta$ is symmetric to the $\{x = 0\}$-plane we have the identity

$$e^{\xi_x} f(1, 0, 0) = e^{-\xi_x} f(-1, 0, 0), \tag{4.3}$$

which implies

$$\xi_x = 0. \tag{4.4}$$

This proves Theorem 1.2 (iii).

**Lemma 4.1.** *If* $\Theta = \Theta_*$ *then* $\operatorname{diam} X(\cdot, t)$ *is uniformly bounded for all times.*

*Sketch of the proof:* We assume the contrary. Obviously, the flow preserves the symmetry of the initial hypersurfaces. We firstly show that the flow hypersurfaces are enclosed by an infinite long cylinder with fixed cross-section diameter. Then we show that this diameter can be chosen so small that the barrier consisting of the cylinder (more precisely, a finitely long part of it) equipped with spherical caps serves for a certain time as a barrier with decreasing diameter. Adapting this barrier when necessary (without increasing its diameter) ensures that we have a barrier which decreases its diameter 'permanently' and obtain thereby a contradiction.

*Proof of Lemma 4.1.* (i) For simplicity of the presentation we assume that $n = 2$ and note that the general case works similarly. We prove the lemma indirectly, set $d(t) = \operatorname{diam} X(\cdot, t)$ and assume that there is a sequence of times $0 < t_k \to \infty$ such that $d(t_k) \to \infty$. We first note that the flow preserves the rotational symmetry and choose the corresponding axis of rotational symmetry as the $x$-axis of an Euclidean coordinate system $(x, y, z)$ which we fix from now on. For $t \geq 0$ we let $p(t), q(t) \in \mathbb{R}$ so that $p(t) < q(t)$, $(p(t), 0, 0), (q(t), 0, 0) \in M(t)$ and so that $q(t) - p(t)$ maximal under these conditions. Furthermore, we denote the inradius of $M(t)$ by $r(t)$, its smallest principal curvature by $\kappa_{min}(t)$ and its largest principal curvature by $\kappa_{max}(t)$. From our a priori estimates we obtain positive functions $c_1(t) \equiv c_1(d(t))$,

$c_2(t) \equiv c_2(d(t))$ and $c_3(t) \equiv c_3(d(t))$ depending via the diameter continuously on $t$ such that

$$r(t) > c_1(t), \quad c_2(t) \le \kappa_{min}(t) \le \kappa_{max}(t) \le c_3(t)$$

for all $t > 0$. If $d(t)$ becomes large $c_1$, $c_2$ or $c_3$ might leave every compact interval of positive numbers.

(ii) **Claim 1:** *There is $d_0 > 0$ so that for each $t > 0$ holds*

$$M(t) \subset C_{d_0} = \{(x, y, z) \in \mathbb{R}^3 : |y|^2 + |z|^2 \le d_0^2\}. \tag{4.5}$$

*Proof of Claim 1:* If this is not the case there is a time $t_1 > 0$ so that the following holds: Firstly, due to the minimality of $\Theta_*$ and the convexity of $M(t_1)$, $M(t_1)$ lies between two parallel planes perpendicular to the $x$-axis in a distance not greater than a certain constant from each other. Secondly, there are $\bar{x} \in \mathbb{R}$, $l_1, l_2 > 0$, $l_2$ arbitrarily large and $l_1$ appropriate, and an ellipse $E(l_1, l_2)$ in the $(x, z)$-plane with middle point $(\bar{x}, 0, 0)$, vertex $(\bar{x} + l_1, 0, 0)$ and co-vertex $(\bar{x}, 0, l_2)$ so that $\tilde{E}(l_1, l_2)$, the surface of revolution obtained by rotating $E(l_1, l_2)$ around the $x$-axis, is enclosed by $M(t_1)$. We now borrow a tool from [2], namely that there is a compact, convex ancient solution (parametrized over the time interval $(-\infty, 0)$) of the mean curvature flow in $\mathbb{R}^3$ with $O(1) \times O(2)$ symmetry that lies between two parallel planes in distance $\pi$ from each other. A suitable parabolic rescaling of this flow allows to replace the distance $\pi$ in this statement by an arbitrary positive number. Restricting such a flow to the interval $[-T, 0)$ where $T > 0$ is sufficiently large we can construct a mean curvature flow which exists on the time interval $[t_1, t_1 + T]$ where $T$ is arbitrary large and with flow hypersurfaces being enclosed by $\tilde{E}(l_1, l_2)$ and enclosing $\tilde{E}(\tilde{l}_1, \tilde{l}_2)$ for $t \in [t_1, t_1 + T]$ where $0 < \tilde{l}_1 < l_1$ with $\tilde{l}_1$ appropriate and $0 < \tilde{l}_2 < l_2$ large. W.l.o.g. (rescale to handle the function $f$ in the normal speed appropriately when necessary) we may assume that this mean curvature flow acts as an inner barrier for the $M(t)$ during the interval $[t_1, t_1 + T]$. Due to the convexity of the $M(t)$ there is a square $U = [\bar{x} - a, \bar{x} + a] \times [\bar{x} - a, \bar{x} + a]$ in the $(y, z)$-plane with $a > 0$ so that a corresponding portion $B(t)$ of $M(t)$ can be written as graph over $U$ in the direction of the $x$-axis during the interval $[t_1, t_1 + T]$, i.e.

$$B(t) = \operatorname{graph} u(t, \cdot)_{|U} \tag{4.6}$$

where $u$ is a smooth function on $[t_1, t_1 + T] \times U$ with slope $|Du|$ w.l.o.g. small. Using Jensen's inequality we find

$$\begin{aligned}
\frac{1}{|U|} \int_U \dot{u} &= -\frac{|B(t)|}{|U|} \frac{1}{|B(t)|} \int_U \sqrt{1 + |Du|^2} \log\left(\frac{H}{f}\right) \\
&\ge -\frac{|B(t)|}{|U|} \log\left(\frac{1}{|B(t)|} \int_U \sqrt{1 + |Du|^2} \frac{H}{f}\right) \\
&\ge -\frac{|B(t)|}{|U|} \log\left(\frac{1}{|B(t)|} \frac{2}{\min_{S^n} f} \int_U H\right)
\end{aligned} \tag{4.7}$$

where $|B(t)|$ denotes the measure of $B(t)$ which is w.l.o.g. at least 1. We estimate by divergence theorem that

$$
\begin{aligned}
0 <& \int_U H \\
&= \int_U \operatorname{div}\left(\frac{Du}{\sqrt{1+|Du|^2}}\right) \\
&= \int_{\partial U} \left\langle \frac{Du}{\sqrt{1+|Du|^2}}, \omega \right\rangle \\
\ll& 1
\end{aligned}
\tag{4.8}
$$

where $\omega$ denotes the outer unit normal of $\partial U$. This shows that the (spatial) average vertical speed (outward directed) of the graph over $U$ is greater than a positive constant. Integrating inequality (4.7) over $[t_1, t_1+T]$ shows that there is at least one point of the graph which moves (in average) upwards from time $t_1$ to $t_1+T$, in fact w.l.o.g. an arbitrary long distance. In view of the convexity of the $M(t)$ we conclude a contradiction to the choice of $\Theta_*$ which proves Claim 1.

(iii) **Claim 2:** *For every time interval length $T > 0$ exists a distance $\delta > 0$ so that for all $t_0 > 0$ the following holds. If $p \in \mathbb{R}^3$ is a point with* $\operatorname{dist}(p, M(t_0)) > \delta$ *then $p \notin M(t)$ for all $t \in [t_0, t_0+T]$.*

*Proof of Claim 2:* We construct a barrier for the flow starting from $M(t_0)$. The barrier consists of a cylinder with middle axis equal to the $x$-axis and cross-section diameter $2d_0+1$ and two spheres around $(-1+p(t_0), 0, 0)$ and $(q(t_0)+1, 0, 0)$ with radii $2d_0$ where $d_0$ is as in Claim 1. Starting the flows from $M(t_0)$, from the cylinder and from the two spheres simultaneously at time $t_0$ according to the equation (1.3), the flows of the last three serve as a barrier for $M(t)$ in the sense that $M(t)$ cannot leave the convex hull of the union of the two spherical flows during the time interval $[t_0, t_0+\delta]$ where $\delta > 0$ depends only on $d_0$, cf. also Remark 4.2. This gives an upper bound for

$$
\sup_{t\in[t_0,t_0+\delta], p\in M(t)} \operatorname{dist}(M(t_0), p) \tag{4.9}
$$

which depends only on $d_0$. Now we adjust a new barrier around $M(t_0+\delta)$ in the same fashion as before and repeat the argument yielding an upper bound for

$$
\sup_{t\in[t_0+\delta,t_0+2\delta], p\in M(t)} \operatorname{dist}(M(t_0+\delta), p) \tag{4.10}
$$

which depends only on $d_0$. Iterating this argument together with the triangle inequality proves Claim 2.

(iv) For the rest of the proof we fix a time interval $I = [t_0, t_1]$ and assume w.l.o.g. that

(1) $\Lambda_0 = d(t_0) \le d(t)$ for $t \in I$,
(2) $|I|$ is large and $\Lambda_0$ is large compared to $|I|$,
(3) denoting $C(t, x') = \{x = x'\} \cap M(t)$, $C(t, \frac{1}{10}\Lambda_0)$ and $C(t, \frac{9}{10}\Lambda_0)$ are both nonempty for all $t \in I$ (which is possible in view of Claim 2, the connectedness of $M(t)$ and since $\Lambda_0$ is large).

(v) **Claim 3:** There is $\delta > 0$ which can be chosen small for $\Lambda_0$ sufficiently large so that

$$\operatorname{diam} C(t, x') \le \frac{2+\delta}{f(0,0,1)} \tag{4.11}$$

for all $t \in \tilde{I} = [t_0, \frac{t_0+t_1}{2}]$ and all $x' \in [\frac{2}{10}\Lambda_0, \frac{8}{10}\Lambda_0]$.

*Proof of Claim 3:* We prove this indirectly and assume that $\Lambda_0$ is large, $\delta > 0$ small and that there are $\tilde{t} \in \tilde{I}$, $x' \in [\frac{2}{10}\Lambda_0, \frac{8}{10}\Lambda_0]$ so that (4.11) does not hold for $t = \tilde{t}$ and $x'$. We fix a width $0 < w < 1$ and consider the surface parts

$$R = R(t, x', \delta) = \{x' - w \le x \le x' + w\} \cap M(t), \quad t \in I. \tag{4.12}$$

Our aim is to show that the average normal speed on $R$ during the time interval $[\tilde{t}, t_1]$ is outward directed with a positive minimum speed, which is a contradiction to the minimality of $\Theta_*$. By using similar arguments as in step (ii) it suffices for it to show that there is $\mu > 0$ so that

$$\frac{1+\tilde{\epsilon}}{f(0,0,1)|R|}\int_R H < 1 - \mu \quad \forall t \in [\tilde{t}, t_1] \tag{4.13}$$

where $\tilde{\epsilon} = \tilde{\epsilon}(Dv)$ (and $v$ constitutes the representation of $R$ as a graph with respect to appropriately chosen cylindrical coordiantes) is a positive constant which can be made small if $\Lambda_0$ is sufficiently large. Here we used, that $f_{|R}$ (clear from context) converges uniformly to the constant $f(0,0,1)$ for $\Lambda_0 \to \infty$ and that this convergence is also uniform within the time interval $I$.

We rewrite the mean curvature $H$ of $M(t)$ as follows, cf. [2, Section 3]. If we parametrize the strictly convex curve $\{z = 0\} \cap M(t)$ (in the $(x, y)$-plane) with respect to its turning angle by $\gamma = \gamma(\theta) = (x(\theta), y(\theta))$, $\gamma : S^1 \to \{z = 0\}$, then we have with obvious notations that

$$H(\gamma(\theta)) \equiv H(\theta) = \kappa(\theta) - \frac{\cos\theta}{y(\theta)} \tag{4.14}$$

for $\theta \notin \{\frac{\pi}{2}, -\frac{\pi}{2}\}$. Here, $S^1$ is considered to be parametrized by $(\cos\theta, \sin\theta)$, $\kappa(\theta)$ is the curvature of $\gamma$ and $-\frac{\cos\theta}{y(\theta)}$ is the contribution to the mean curvature which results from the rotation. By assumption we can estimate

$$H(\theta) \le \kappa(\theta) + \frac{f(0,0,1)}{1+\frac{\delta}{2}} \tag{4.15}$$

and this yields the desired inequality (4.13) initially, i.e. for $t = \tilde{t}$, and, clearly, then also for all $t \in [\tilde{t}, t_1]$.

(vi) We consider now at time $t_0$ a (infinite long) cylinder $C$ around the $x$-axis of radius $\frac{1+\frac{3}{4}\delta}{f(0,0,1)}$ and two spheres $S_1$ and $S_2$ with centers in $p_0 = (p(t_0), 0, 0)$ and $q_0 = (q(t_0), 0, 0)$, respectively, and radii $r = \frac{1+\delta}{f(0,0,1)}$. By assumption (1.1) we can find $\tilde{\delta} > 0$ such that

$$f(0,0,1) > \frac{f}{2}(1+\tilde{\delta}). \tag{4.16}$$

Evaluating the expression $\frac{H}{f}$ on the sphere $S_1$ (or $S_2$) gives

$$\begin{aligned} \frac{H}{f} &= \frac{2f(0,0,1)}{(1+\delta)f} \\ &> \frac{1+\tilde{\delta}}{1+\delta} \\ &> 1 \end{aligned} \tag{4.17}$$

provided $\delta > 0$ is sufficiently small which can be achieved by assuming that $\Lambda_0$ is sufficiently large.

In view of (4.17) and by continuity there exists a time $\delta t > 0$ so that the flow (1.3) starting from $S_1$ at time $t = t_0$ (with flow hypersurfaces $S_1(t)$) exists during the time interval $[t_0, t_0 + \delta t]$, remains uniformly convex, and is at time $t = t_0 + \delta t$ enclosed by a sphere around $p_0$ of radius $r - \eta$, $\eta > 0$ small. Furthermore, we may assume that the $C^1$-distance from $S_1(t)$ to $S_1$ is small during the interval $[t_0, t_0 + \delta t]$ (choose $\eta$ smaller if necessary).

Correspondingly, we define $S_2(t)$. Furthermore, the flow (1.3) starting from $C$ (with flow hypersurfaces $C(t)$) remains a cylinder and we may assume that $C^1$-distance of $C(t)$ to $C$ is small during the time interval $[t_0, t_0 + \delta t]$ (choose $\delta t > 0$ smaller if necessary).

We remark that $M(t)$ remains contained in the convex hull of $S_1(t) \cup S_2(t)$ for all $t \in [t_0, t_0 + \delta t]$, cf. Remark 4.2.

At time $t = t_0 + \delta t$ we choose a new barrier consisting of $C$ and spheres $\tilde{S}_1$ and $\tilde{S}_2$ with centers in $p_0 = (p(t_0) + \frac{\eta}{2}, 0, 0)$ and $q_0 = (q(t_0) - \frac{\eta}{2}, 0, 0)$, respectively, and radii $r = \frac{1+\delta}{f(0,0,1)}$. We now consider the flow (1.3) starting from $C$, $\tilde{S}_1$ and $\tilde{S}_2$ at time $t = t_0 + \delta t$ during the time interval $[t_0 + \delta t, t_0 + 2\delta t]$ which serves again in the same fashion as before as a barrier for $M(t)$.

Iteration of this procedure leads to an estimate of the form

$$d(t) \le C - \frac{\eta}{\delta t}(t - t_0), \quad t \in \tilde{I}, \tag{4.18}$$

where $C$ is a suitable constant, a contradiction.

The proof of the lemma is complete. $\square$

The following remark can be seen as a variant of the well-known avoidance principle.

**Remark 4.2.** Let $\tilde{x}_1$, $\tilde{x}_2$, $0 < r_1 < r_2$ be real numbers. Let $S_{r_2}(\tilde{x}_i)$, $i = 1, 2$, be spheres with radii $r_2$ around $(\tilde{x}_1, 0, 0)$ and $(\tilde{x}_2, 0, 0)$ and let $C$ be an infinite long cylinder with middle axis equal to the $x$-axis and cross-section radius $r_1$. Let $\tilde{C}$ be the (closed) enclosed volume of $C$, $B$ the convex hull of $S_{r_2}(\tilde{x}_1) \cup S_{r_2}(\tilde{x}_2)$ and

$$M_0 \subset \mathring{B} \cap \mathring{\tilde{C}} \tag{4.19}$$

a closed uniformly convex surface in $\mathbb{R}^3$. Let $0 \le t_0 < t_1$ and denote by $S_{r_2}(\tilde{x}_i, t)$, $C(t)$ and $M(t)$ the corresponding images at time $t \in [t_0, t_1]$ of $S_{r_2}(\tilde{x}_i)$, $C$ and $M_0$, respectively, under the flow (1.3) starting from the latter objects at time $t_0$. With obvious notation we analogously define the quantities $\tilde{C}(t)$ and $B(t)$. If $S_{r_2}(\tilde{x}_i, t)$ are sufficiently (depending only on $r_1$, $r_2$) $C^1$-close to $S_{r_2}(\tilde{x}_i)$ and $C(t)$ sufficiently (depending only on $r_1$, $r_2$) $C^1$-close to $C$ for all $t \in [t_0, t_1]$ then we have

$$M(t) \subset \overset{\circ}{B}(t) \cap \overset{\circ}{\tilde{C}}(t) \tag{4.20}$$

for all $t \in [t_0, t_1]$.

*Proof.* We prove this indirectly. Let $0 < t \le t_1$ be minimal such that (4.20) does not hold. Then we have

$$M(t) \cap (C(t) \cup S_{r_2}(\tilde{x}_1, t) \cup S_{r_2}(\tilde{x}_2, t)) \neq \emptyset \tag{4.21}$$

and since the flow (1.3) preserves the uniform convexity there is a $p \in \mathbb{R}^3$ and a neighborhood $\tilde{U}$ of $p$ in $\mathbb{R}^3$ such that

$$\tilde{U} \cap X \cap M(t) = \{p\} \tag{4.22}$$

with $X = X(t)$ either equal to $C(t)$, $S_{r_2}(\tilde{x}_1, t)$ or $S_{r_2}(\tilde{x}_2, t)$. In each of these three cases the outer normals of $X(t)$ and $M(t)$ agree in $p$.

We now proceed with standard arguments, i.e. we write corresponding portions of $M(\tilde{t})$ and $X(\tilde{t})$ for all $\tilde{t} \in [t - \delta, t]$, $\delta > 0$ appropriately small, as graphs (in the direction of the inner normal of $M(t)$ at $p$) of functions $u_1$ and $u_2$, respectively, over a small, open and connected neighborhood $U$ of $p$ in the common tangent plane $T_pX(t) = T_pM(t)$. W.l.o.g. we may assume that the gradients $|Du_1|$ and $|Du_2|$ are small in $U$. The difference $w = u_1 - u_2$ is then positive in

$$([t - \delta, t] \times U) \setminus \{(t, p)\} \tag{4.23}$$

and $w(t, p) = 0$, which is a contradiction as can be shown by the strong maximum principle. □

## 5. Further remarks

A simple inspection shows that the line of arguments in our paper for the case $F = H = H_1$ can be easily modified so that the analogous statement for the cases $F = H_k$, $k = 2, ..., n$, can be obtained. One only needs to change the assumption (1.1) slightly, namely instead of (1.1) we require there now that for each of the cases $F = H_k$, $k = 2, ..., n$, there holds

$$f < cf(0) \tag{5.1}$$

where $c > 1$ depends on the specific case, i.e. $c = c(k)$, $k = 2, ..., n$.

Note that the only place where the arguments in the proof require a worth mentioning modification is when we use the suitably parabolically rescaled ancient solution of the mean curvature flow as a comparison flow. Here, we use now a different argument. Assume the real line between -1000 and 1000 lies within the surface (of a certain minimum thickness) for which we construct an inner barrier. Choose small balls around -999 and 999 with radii $\varepsilon > 0$ and let everything flow until the radius of these balls is $\frac{\varepsilon}{2}$. Then choose new balls of radii $\varepsilon$ around -500 and 500 and let everything flow until their radii are of size $\frac{\varepsilon}{2}$. Then choose the balls around -250 and 250 and so on. Starting from an initial diameter of size $c_0 2^l$ gives one the time $l\delta t$ during which the corresponding surface has diameter larger than $c_0$ where $l \in \mathbb{N}$ and $\delta t$ denotes the time which is needed to shrink a ball of radius $\varepsilon$ to a ball of radius $\frac{\varepsilon}{2}$ under our flow.

Especially, the outcome of this extended version of our main result is an existence proof for convex hypersurfaces for a certain class of prescribed functions on $S^n$.

These latter also satisfy the assumptions of the general existence result [9, Theorem 1.1], which we recall in the following.

**Theorem 5.1.** *Assume $f \in C^{l,1}(S^n)$, $l \geq 1$, is a positive function. Suppose $f$ is invariant under an automorphic group $G$ of $S^n$ without fixed points; i.e.*

$$f(g(x)) = f(x) \tag{5.2}$$

*for all $g \in G$ and $x \in S^n$. Then there exists a $C^{l+2,\alpha}$ (for all $0 < \alpha < 1$) closed, strictly convex hypersurface $\Sigma$ in $\mathbb{R}^{n+1}$ with*

$$H_k(n^{-1}(x)) = f(x), \quad x \in S^n, \tag{5.3}$$

*where*

$$n : \Sigma \to S^n \tag{5.4}$$

*denotes the Gauss map and $H_k$, $k = 1, ..., n$, the $k$-th elementary symmetric polynomial.*

In the following we will discuss what we can say on the basis of our paper in the case that we omit the eveness for $f$ and the symmetry with respect to the $\{x = 0\}$ plane of the foliation while keeping all other assumptions as before. Then there is a distinct leaf in the foliation from which the flow convergences to a translating solution with speed $\xi = (\xi_x, 0, 0)$, $\xi_x \in \mathbb{R}$. Furthermore, the translating limit geometry has $F = H_k$-curvature equal to

$$g(x) = e^{x\xi_x} f(x). \tag{5.5}$$

We may draw the following interesting conclusion, namely we have the cases

(i) $g(\sigma) = g(\eta_\sigma)$ for all $\sigma \in \{-1, 1\}$ and suitable $\sigma \neq \eta_\sigma \in [-1, 1]$ or

(ii) else.

In case (i) we have

$$e^{\eta_\sigma \xi_x} f(\eta_\sigma) = e^{\sigma \xi_x} f(\sigma) \tag{5.6}$$

so that we get the representation of $\xi_x$ as

$$\xi_x = \frac{1}{-\sigma + \eta_\sigma} \log\left(\frac{f(\sigma)}{f(\eta_\sigma)}\right); \tag{5.7}$$

the latter becomes even more explicit if $f$ is a non-constant linear function since then $\eta_\sigma = -\sigma$.

In case (ii) $g$ does not satisfy the assumptions from the existence Theorem 5.1 while we obtain existence of a surface with curvature equal to $g$.

Both cases (i) and (ii) are interesting, the first is a kind of extension of our previous existence result, the second is a kind of extension of Theorem 5.1. Unfortunately, it is not clear which case occurs indeed given a concrete $f$. Hence and summarized we neither extend our previous result nor Theorem 5.1 but obtain an additional interesting conclusion.

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