Minkowski spacetime

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Abstract: In this talk we review the Minkowski spacetime which is the spacetime of Special Relativity.

Contents

1 Introduction

> This work contains many things which are new and interesting. Unfortunately, everything that is new is not interesting, and everything which is interesting, is not new.
>
> L.D. Landau

Before we deal with the concepts of Minowski spacetime, let us warm up with some basic definitions of classical mechanics.

If one wants to describe processes happening in nature, then the concept of reference systems must be taken into account, by which we understand a system of coordinates serving to indicate the space and time of a particle. Reference systems are called inertial, when in which a freely moving particle continues to move with constant velocity vector when there is no external forces act on it. In future reference, we also use quite often the concept of events, an event is described by the place where is occurred and the time when is occurred.

In Special Relativity, one studies the structure of spacetime, which plays a role as the non-dynamical background on which in the sense of high energy physics particles and fields evolve.

Due to the Michelson-Morley experiment, light propagates with the maximal speed c and the same in all inertial systems of frames. Correspondingly, we can consider the following postulates: For any two inertial systems, they shall agree on

1. whether a given particle is moving with the speed of light.
2. the distance between events which are simultaneous in both systems.

We choose our conventions such that $c = 1$ and the Minkowski metric has the signature $(-1, 1, 1, 1)$.

The plan of this talk is first we introduce some basic mathematical definitions and properties of the Minkowski spacetime, after that we will discuss some applications to Special Relativity.

2 Minkowski spacetime

Definition 2.1. A metric tensor g on a smooth manifold M is a symmetric nondegenerate $(0, 2)$ tensor field on M. A smooth manifold M equipped with a metric tensor g is called a Riemannian manifold.

We use $\langle , \rangle$ as an alternative notation for g, so that

$$g(v, w) = \langle v, w \rangle , \tag{2.1}$$

in which v, w denoted tangent vectors.

Definition 2.2. Minkowski spacetime denoted by $(\mathcal{M}, \eta)$ is the manifold $\mathbb{R}^4$ endowed with the Minkowski (pseudo) inner product $\langle , \rangle$. For a coordinate system (x^0, x^1, x^2, x^3), one gets the components of the metric tensor g

$$g_{\mu\nu} = \left\langle \frac{\partial}{\partial x^\mu}, \frac{\partial}{\partial x^\nu} \right\rangle := \eta_{\mu\nu}, \tag{2.2}$$

where $\mu, \nu = 0, 1, 2, 3$.

Theorem 2.3. If $(x^1, ..., x^n)$ is a coordinate system in M at p, then its coordinate vectors $\partial_1|_p, \cdots \partial_n|_p$ form a basis for the tangent spaces $T_p(M)$ and

$$v = \sum_{i=1}^{n} v(x^i) \partial_i|_p. \tag{2.3}$$

In four-dimensional space, the coordinates of an event (ct, x, y, z) are the conponents of a four dimensional radius vector denoted by (x^0, x^1, x^2, x^3). And a tangent vector in a tangent space at an fixed event called origin can be expressed as

$$v = \sum_{\mu} v^\mu \frac{\partial}{\partial x^\mu}, \tag{2.4}$$

For v, w two tangent vectors in $\mathcal{M}$, their Minkowski inner product then takes the form

$$\langle v, w \rangle := -v^0 w^0 + v^1 w^1 + v^2 w^2 + v^3 w^3 = \sum_{\mu,\nu} \eta_{\mu\nu} v^\mu w^\nu. \tag{2.5}$$

The coordinate basis

$$\left\{ \frac{\partial}{\partial x^0}, \frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}, \frac{\partial}{\partial x^3} \right\} \tag{2.6}$$

is an orthonormal basis for this inner product. We have defined the Minkowski inner product, so now we have to check if it is well defined, i.e. it has no dependence on the choice of the inertial reference system.

On the inertial frame called S which was defined particularly as above, if one observes a particle moving with the velocity of light, then its path shall be a straight line whose tangent vector components must yield

$$(v^0)^2 = (v^1)^2 + (v^2)^2 + (v^3)^2, \tag{2.7}$$

so that the distance traveled equals the elapsed time. With the particular choice of coordinates, v must satisfies

$$\langle v, v\rangle = 0. \tag{2.8}$$

We consider an other inertial frame S' associated with the coordinates (x'^0, x'^1, x'^2, x'^3). Similar to (2.7), on this frame the vector

$$\frac{\partial}{\partial x'^0} \pm \frac{\partial}{\partial x'^i}, \tag{2.9}$$

is tangent to the motion of a particle moving with the velocity of light. Due to the first postulate, given a particle moving with the velocity of light then any inertial frame must agree with this velocity. Thus,

$$\left\langle \frac{\partial}{\partial x'^0} \pm \frac{\partial}{\partial x'^i}, \frac{\partial}{\partial x'^0} \pm \frac{\partial}{\partial x'^i} \right\rangle = 0. \tag{2.10}$$

which implies the following relations

$$\left\langle \frac{\partial}{\partial x'^0}, \frac{\partial}{\partial x'^0} \right\rangle = -\left\langle \frac{\partial}{\partial x'^i}, \frac{\partial}{\partial x'^i} \right\rangle, \tag{2.11}$$

$$\left\langle \frac{\partial}{\partial x'^0}, \frac{\partial}{\partial x'^i} \right\rangle = 0. \tag{2.12}$$

Analogously, consider an another tangent vector of a straight line

$$\sqrt{2}\frac{\partial}{\partial x'^0} + \frac{\partial}{\partial x'^i} + \frac{\partial}{\partial x'^j}, \qquad i \neq j. \tag{2.13}$$

We have the inner product

$$\left\langle \sqrt{2}\frac{\partial}{\partial x'^0} + \frac{\partial}{\partial x'^i} + \frac{\partial}{\partial x'^j}, \sqrt{2}\frac{\partial}{\partial x'^0} + \frac{\partial}{\partial x'^i} + \frac{\partial}{\partial x'^j} \right\rangle = 0. \tag{2.14}$$

Substituting (2.11), we find

$$\left\langle \frac{\partial}{\partial x'^i}, \frac{\partial}{\partial x'^j} \right\rangle = 0. \tag{2.15}$$

The inner product is defined to be non-degenerate, thus we deduce that there exists $\kappa \neq 0$ such that

$$\left\langle \frac{\partial}{\partial x'^\mu}, \frac{\partial}{\partial x'^\nu} \right\rangle = \kappa \eta_{\mu\nu}. \tag{2.16}$$

Now we use the second postulate to show that $\kappa = 1$. In the inertial systems S and S', for $x^0 = \text{const.}$ and $x'^0 = \text{const.}$ the simultaneity hypersurfaces are 3-planes in $\mathbb{R}^4$. So there are two cases to consider, on one hand, if they are parallel, then they will coincide for the appropriate values of x^0 and x'^0, otherwise they intersect and result a 2-plane which contains simultaneity events of both frames. Let $v \neq 0$ be a vector relating two events in the simultaneity planes. From $\mathrm{d}x^0(v) = \mathrm{d}x'^0(v) = 0$, we get

$$v = \sum_{i=1}^{3} v^i \frac{\partial}{\partial x^i} = \sum_{i=1}^{3} v'^i \frac{\partial}{\partial x'^i}. \tag{2.17}$$

The second postulate implies that

$$\sum_{i=1}^{3} (v^i)^2 = \sum_{i=1}^{3} (v'^i)^2. \tag{2.18}$$

In additions, we have

$$\langle v, v \rangle = \sum_{i=1}^{3} (v^i)^2 = \sum_{i,j} \left\langle v'^i \frac{\partial}{\partial x'^i}, v'^j \frac{\partial}{\partial x'^j} \right\rangle = \sum_{i,j} \left\langle \frac{\partial}{\partial x'^i}, \frac{\partial}{\partial x'^j} \right\rangle v'^i v'^j = \kappa \sum_{i=1}^{3} (v'^i)^2. \tag{2.19}$$

Ultimately, we find $\kappa = 1$, thus the new coordinate basis

$$\left\{ \frac{\partial}{\partial x'^0}, \frac{\partial}{\partial x'^1}, \frac{\partial}{\partial x'^2}, \frac{\partial}{\partial x'^3} \right\} \tag{2.20}$$

is indeed orthonormal, i.e. the Minkowski inner product is well defined.

Now let us study some basic properties of the Minkowski spacetime.

The **length** of a vector $v \in \mathbb{R}^4$ has the form

$$|v| = \sqrt{|\langle v, v \rangle\,|}. \tag{2.21}$$

A vector $v \in \mathbb{R}^4$ is called:

1. **timelike**, if the inner product of v with itself is negative

$$\langle v, v \rangle < 0$$

2. **spacellike**, if the inner product of v with itself is positive

$$\langle v, v \rangle > 0$$

3. **lightlike**, if the inner product of v with itself vanishes

$$\langle v, v \rangle = 0,$$

We call the set of all vectors which yield the lightlike property, the light cone. Consider a curve $s : I \in \mathbb{R} \to \mathbb{R}^4$, it is called timelike if $\langle \dot{s}, \dot{s} \rangle < 0$. A timelike curve characterised by the parameter t or a worldline inside the light cone relative to an event is nothing but the path of a nonzero mass particle moving in Minkowski spacetime. Its tangent vector takes the form

$$\frac{\mathrm{d}s(t)}{\mathrm{d}t} = \left(1, \frac{\mathrm{d}\vec{s}}{\mathrm{d}t}\right) = (1, \vec{v}), \tag{2.22}$$

in which we use $\vec{v} = v^i \partial/\partial x^i$ to denote the velocity of the particle.

We then compute the inner product of the tangent vector with itself as follows

$$\left\langle \frac{\mathrm{d}s(t)}{\mathrm{d}t}, \frac{\mathrm{d}s(t)}{\mathrm{d}t} \right\rangle = -c^2 + \vec{v}^2. \tag{2.23}$$

Since $s(t)$ is timelike, thus we have $v < c$. In other words, c is the maximum possible velocity.

Consider two inertial frames, one at rest and one moves relative to the other. Assume that we are in the inertial reference system which is at rest, and observe clocks in the other system. On one hand, in an infinitesimal time interval $\mathrm{d}t$ read in the rest frame, then the other frame go a distance $\sqrt{\mathrm{d}\vec{s}^2}$, and on the other hand, in a system of the moving clocks yields $\mathrm{d}x' = \mathrm{d}y' = \mathrm{d}z' = 0$. We know that ds^2 is invariant of the choice of inertial frames, thus we have

$$\mathrm{d}s^2 = -c^2\mathrm{d}t + \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2 = -c^2\mathrm{d}\tau^2, \tag{2.24}$$

from which

$$\mathrm{d}\tau = \frac{1}{c}\frac{\mathrm{d}s}{\mathrm{d}t}\mathrm{d}t = \mathrm{d}t\left(1 - \frac{\mathrm{d}\vec{s}^2}{c^2\mathrm{d}t^2}\right)^{1/2} = \mathrm{d}t\left(1 - \frac{\vec{v}^2}{c^2}\right)^{1/2}, \tag{2.25}$$

in which τ is the time read in the inertial frame of the moving clocks and $\vec{v}$ is its velocity. Integrating this expression, we obtain for $c = 1$

$$\tau = \int \mathrm{d}t|\dot{s}(t)| = \int \mathrm{d}t\left(1 - \vec{v}^2\right)^{1/2}. \tag{2.26}$$

This is the proper time of the moving object expressed in terms of the time for a system from which the motion of the object is observed.

3 Lorentz transformations

In the last section, we have studied some basic definitions and properties of the Minkowski spacetime. In this section, we attempt to investigate the Lorentz transformations relating two inertial reference systems.

For simplicity, we consider the 2-dimensional Minkowski spacetime described by the spacetime coordinates (t, x), and $T \in SO_0(1,1)$ is a Lorentz change of basis to a new inertial reference system with the coordinates (t', x'), by which $SO_0(1,1)$ is called the proper Lorentz group in two dimensions. Let T act on the orthonormal basis $(\partial/\partial t, \partial/\partial x)$, we obtain

$$\frac{\partial}{\partial t'} = T\frac{\partial}{\partial t} = \cosh u\frac{\partial}{\partial t} + \sinh u\frac{\partial}{\partial x}, \tag{3.1}$$

$$\frac{\partial}{\partial x'} = T\frac{\partial}{\partial x} = \sinh u\frac{\partial}{\partial t} + \cosh u\frac{\partial}{\partial x}, \tag{3.2}$$

in which $u \in \mathbb{R}$ is called the rapidity or angle of rotation. The change of the basis can be represented by the following matrix

$$S = \begin{pmatrix} \cosh u & \sinh u \\ \sinh u & \cosh u \end{pmatrix} \tag{3.3}$$

and its inverse matrix is obtained readily by changing u to $-u$

$$S^{-1} = \begin{pmatrix} \cosh u & -\sinh u \\ -\sinh u & \cosh u \end{pmatrix} \tag{3.4}$$

In more details, the relation between the old and the new coordinates is descibed by the change of tangent vector elements under Lorentz transformations given by

$$t' = t\cosh u - x\sinh u, \tag{3.5}$$

$$x' = x\cosh u - t\sinh u. \tag{3.6}$$

These transformations express the information that with respect to an inertial reference system chosen to be at rest, the moving frame then has velocity $v = \tanh u$. One can choose the inertial reference system called S' at rest and use the inverse matrix S^{-1} to express (t, x) in terms of t', x' and u. Furthermore, one shall note the following relations

$$\cosh u = (1 - v^2)^{-1/2}, \qquad \sinh u = v(1 - v^2)^{-1/2}. \tag{3.7}$$

Originally, we have for the Lorentz boost

$$S[v] = \begin{pmatrix} (1 - v^2)^{-1/2} & -v(1 - v^2)^{-1/2} \\ -v(1 - v^2)^{-1/2} & (1 - v^2)^{-1/2} \end{pmatrix}, \tag{3.8}$$

this matrix is dependent of v. So if one tries to compute the composition of $S[v_1], S[v_2]$, then one get

$$S[v_1]S[v_2] = S\left[\frac{v_1 + v_2}{1 + v_1 v_2}\right] \tag{3.9}$$

which is not what we expected since due to the Galileo group we have

$$G[v_1]G[v_2] = G[v_1 + v_2] \tag{3.10}$$

Therefore we re-parameterize v appropriately in terms of the rapidity as in (3.7), in order to get $S[u_1]S[u_2] = S[u_1 + u_2]$. Checking this relation is straightforward.
By substituting (3.7), then (3.5) and (3.6) can be re-expressed as

$$t' = t(1 - v^2)^{-1/2} - x(1 - v^2)^{-1/2}v, \tag{3.11}$$

$$x' = x(1 - v^2)^{-1/2} - t(1 - v^2)^{-1/2}v. \tag{3.12}$$

Obviously, if we consider the non-relativistic regime, by taking $c \to \infty$, then we run back to the very well known case, the Galileo transformations

$$t' = t, \qquad x' = x - vt. \tag{3.13}$$

We can say that the Galileo group is a subgroup of the Lorentz group.

In the last chapter, we have used one approach to derived the proper time. With the Lorentz transformations, we now seek to deal with concepts of proper length, proper volume and again the proper time.

Consider an inertial frame S at rest, in this frame a length of a rod is measured to be $\Delta x = x_2 - x_1$ where x_1, x_2 are positions indicated by two ends of the rod. We now determine what happens if we measure the length of this rod in the S' system moving along the x-axis with velocity v. According to (3.12), we have

$$x_1 = \frac{x_1' + vt'}{\sqrt{1-v^2}}, \qquad x_2 = \frac{x_2' + vt'}{\sqrt{1-v^2}}, \tag{3.14}$$

from which

$$\Delta x = x_2 - x_1 = \frac{\Delta x'}{\sqrt{1-v^2}}. \tag{3.15}$$

Let us denote the proper time (the length of a rod measured in the rest system) by $l_0 = \Delta x$, and its length measured in any other reference system S' by l, then we can rewrite (3.11) as

$$l = l_0\sqrt{1-v^2} = \frac{l_0}{\cosh u} < l_0. \tag{3.16}$$

This result implies that a maximum possible length of a rod can be achieved if it is measured in a reference system which is at rest. This effect is called the Lorentz contraction. Very similar, the volume $\mathfrak{V}$ of a body decreases in terms of the proper volume $\mathfrak{V}_0$ as follows

$$\mathfrak{V} = \mathfrak{V}_0\sqrt{1-v^2}. \tag{3.17}$$

Lastly, from (3.12) we have

$$t_1 = \frac{t_1' + vx'}{\sqrt{1-v^2}}, \qquad t_2 = \frac{t_2' + vx'}{\sqrt{1-v^2}} \tag{3.18}$$

subtracting t_1 from t_2,

$$\tau = \Delta t' = \Delta t\sqrt{1-v^2}. \tag{3.19}$$

This result implies that the moving clocks run slower than the ones at rest. This effect is called the time dilatation.

Appendices

Lorentz group

The Lorentz transformations form a group defined by the group of linear insometries of Minkowski space $(\mathcal{M}_4, \eta_{\mu\nu})$

$$\mathbf{L} = \{\Lambda \in M(4,\mathbb{R}) | \langle \Lambda x, \Lambda y\rangle = \langle x, y\rangle\}$$

that requires the following constraint on Λ

$$\Lambda^t \eta \Lambda = \eta, \qquad \text{or} \qquad \sum_{\mu,\nu} \eta_{\mu_\nu} \Lambda^\mu_\rho \Lambda^\nu_\sigma = \eta_{\rho\sigma}. \tag{.20}$$

In fact, by taking the determinant of (3.1), we end up with the two values of $\det(\Lambda) = \pm 1$, and (3.1) also implies $\Lambda^0_0 \geq 1$ or $\Lambda^0_0 \leq -1$, therefore **L** is a 6-dimensional non-compact Lie group $O(1,3)$ which consists of four connected components. For example, one component can be restricted by taking $\det \Lambda = 1$ and $\Lambda^0_0 \geq 1$ which is denoted by $SO(1,3)^\uparrow$. The proper Lorentz group $SO_0(1,3)$ is the group of linear maps which preserve orientation.

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