Penrose Singularity Theorem

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ABSTRACT: The purpose of this talk is to to study globally hyperbolic manifolds and give a rigorous proof of the Penrose singularity theorem which is one of the most important singularity theorems in differential geometry and general relativity.

Contents

1 Globally hyperbolic spacetime

In this section we are going to study globally hyperbolic spacetime, derive a certain expression of the metric for this kind of manifold and calculate the expansion of a 2-dimensional surface along null geodesics which is going to play a crucial role in the proof of the Penrose singularity theorem.

Definition 1.1. *A spacetime (M, g) is said to be stably causal if there exists a smooth function, $t : M \to \mathbb{R}$ such that gradt is timelike.*

Definition 1.2. *A stably causal spacetime possessing a time function whose level sets are Cauchy hypersurfaces is said to be globally hyperbolic.*

Let (M, g) be a globally hyperbolic spacetime, S a Cauchy hypersurface with future-pointing unit normal vector field n, and $\Sigma \subset S$ a compact 2-dimensional submanifold with unit normal vector field ν in S. Let c_p be the null geodesic with initial condition $n_p + \nu_p$ for each point $p \in \Sigma$. We can define the smooth map $\exp : (-\epsilon, \epsilon) \times \Sigma \to M$ for some $\epsilon > 0$, as $\exp(r, p) := c_p(r)$.

Definition 1.3. *The critical values of* exp *are said to be conjugate points of Σ. Conjugate points are points where geodesics starting orthognally at nearby points of Σ almost intersect.*

Let $q = \exp(r_0, p)$ be a point not conjugate to Σ. If ϕ is a local parametrization of Σ around p, then we can construct a system of local coordinates (u, r, x^2, x^3) on some open set $U \ni q$ by using the map.

$$(u, r, x^2, x^3) \longmapsto \exp(r, \psi_u(\phi(x^2, x^3)) \tag{1.1}$$

where ψ_u is the flow along the timelike geodesics orthogonal to S and the map $\exp : (-\epsilon, \epsilon) \times \psi_u(\Sigma) \to M$ is defined as the one before. Since $\frac{\partial}{\partial r}$ is tangent to null geodesics, $g_{rr} = \langle \frac{\partial}{\partial r}, \frac{\partial}{\partial r} \rangle = 0$. On the other hand we have

$$\frac{\partial g_{r\mu}}{\partial r} = \frac{\partial}{\partial r}\langle\frac{\partial}{\partial r}, \frac{\partial}{\partial x^\mu}\rangle = \langle\frac{\partial}{\partial r}, \nabla_{\frac{\partial}{\partial r}}\frac{\partial}{\partial x^\mu}\rangle = \langle\frac{\partial}{\partial r}, \nabla_{\frac{\partial}{\partial x^\mu}}\frac{\partial}{\partial r}\rangle = \frac{1}{2}\frac{\partial}{\partial x^\mu}\langle\frac{\partial}{\partial r}, \frac{\partial}{\partial r}\rangle = 0 \tag{1.2}$$

Moreover, since $g_{ru} = -1$ and $g_{r2} = g_{r3} = 0$ on $\psi_u(\Sigma)$, we have $g_{ru} = -1$ and $g_{r2} = g_{r3} = 0$ on U. Therefore the metric can be written in this coordinate system as follows

$$g = \alpha du \otimes du - du \otimes dr - dr \otimes du + \sum_{i=2}^{3} \beta_i (du \otimes dx^i + dx^i \otimes du) + \sum_{i,j=2}^{3} \gamma_{ij} dx^i \otimes dx^j \tag{1.3}$$

The determinant of the metric g is

$$\det(g) = \det \begin{pmatrix} \alpha & -1 & \beta_2 & \beta_3 \\ -1 & 0 & 0 & 0 \\ \beta_2 & 0 & \gamma_{22} & \gamma_{23} \\ \beta_3 & 0 & \gamma_{32} & \gamma_{33} \end{pmatrix} = -\det \begin{pmatrix} \gamma_{22} & \gamma_{23} \\ \gamma_{32} & \gamma_{33} \end{pmatrix} \tag{1.4}$$

But since $\det(g) < 0$ the functions $\gamma_{ij} := \langle\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\rangle$ form a positive definite matrix and thus g induces a Riemannian metric on the 2-dimensional surfaces $\exp(r, \psi_u(\Sigma))$, which are spacelike.

Now we want to calculate the Ricci tensor for this globally hyperbolic spacetime, so first we calculate the Christoffel symbols for this metric.

$$\begin{aligned} &\Gamma^u_{ur} = \Gamma^u_{rr} = \Gamma^u_{ri} = \Gamma^r_{rr} = \Gamma^i_{rr} = 0 \\ &\Gamma^i_{rj} = \sum_{k=2}^{3} \gamma^{ik}\beta_{kj} \end{aligned} \tag{1.5}$$

where $\beta_{ij} = \frac{1}{2}\frac{\partial \gamma_{ij}}{\partial r}$ Consequently,

$$R_{rr} = R^u_{urr} + \sum_{i=2}^{3} R^i_{irr} = \sum_{i=2}^{3}(-\frac{\partial \Gamma^i_{ir}}{\partial r} - \sum_{j=2}^{3} \Gamma^j_{ir}\Gamma^i_{rj}) = -\frac{\partial}{\partial r}(\sum_{i,j=2}^{3}\gamma^{ij}\beta_{ij}) - \sum_{i,j,k,l=2}^{3}\gamma^{jk}\gamma^{il}\beta_{ki}\beta_{lj} \tag{1.6}$$

The quantity

$$\theta := \sum_{i,j=2}^{3} \gamma^{ij}\beta_{ij} \tag{1.7}$$

is called the expansion of the null geodesics, and has an important geometric meaning

$$\theta = \frac{1}{2} tr((\gamma_{ij})^{-1} \frac{\partial}{\partial r}(\gamma_{ij})) = \frac{1}{2}\frac{\partial}{\partial r} \log \gamma = \frac{\partial}{\partial r} \log \gamma^{\frac{1}{2}} \tag{1.8}$$

where $\gamma := \det(\gamma_{ij})$. Thus the expansion yelds the variation of the area element of the 2-dimensional surfaces $\exp(r, \psi_u(\Sigma))$, and more importantly the singularity of the expansion indicates a zero of γ which means that there is a conjugate point to $\psi_u(\Sigma)$.

2 Null energy condition

In this section we are going to introduce the *null energy condition* and prove one very important consequence of this condition.

Definition 2.1. *A spacetime (M, g) is said to satisfy the null energy condition if the Ricci tensor satisfies the condition $Ric(V, V) \geq 0$ for any null vector field V.*

Proposition 2.2. *Let (M, g) be a globally hyperbolic spacetime satisfying the null energy condition, $S \subset M$ a Cauchy hypersurface, $\Sigma \subset S$ a compact 2-dimensional submanifold with a unit normal vector field ν in S and $p \in S$ a point where $\theta = \theta_0 < 0$. Then the null geodesic c_p contains at least a point conjugate to Σ, at an affine parameter distance of at most $-\frac{2}{\theta_0}$ to the future of Σ.*

Proof. Since (M, g) satisfies the null energy condition and we have already calculated one of the components of the Ricci tensor, we have

$$Ric(\frac{\partial}{\partial r}, \frac{\partial}{\partial r}) \geq 0 \implies R_{rr} \geq 0 \implies \frac{\partial \theta}{\partial r} + \sum_{i,j,k,l=2}^{3} \gamma^{jk}\gamma^{il}\beta_{ki}\beta_{lj} \leq 0 \tag{2.1}$$

we choose an orthonormal basis, such that $\gamma^{ij} = \delta^{ij}$, and using the inequality for $n \times n$ matrices

$$(trA)^2 \leq n\, tr(A^t A) \tag{2.2}$$

we get the followig result

$$\sum_{i,j,k,l=2}^{3} \gamma^{jk}\gamma^{il}\beta_{ki}\beta_{lj} = \sum_{i,j=2}^{3} \beta_{ji}\beta_{ij} = tr(\beta^t\beta) \geq \frac{1}{2}\theta^2 \tag{2.3}$$

So consequently θ has to satisfy

$$\frac{\partial \theta}{\partial r} + \frac{1}{2}\theta^2 \leq 0 \tag{2.4}$$

Integrating we get

$$\frac{1}{\theta} \geq \frac{1}{\theta_0} + \frac{r}{2} \tag{2.5}$$

but for $r = -\frac{2}{\theta_0}$

$$\frac{1}{\theta} \geq 0 \implies \theta \longrightarrow \infty \tag{2.6}$$

So since the expansion blows up at a point this means that the area element γ becomes zero and consequently the null geodesic c_p contains a least a conjugate point to Σ.

□

3 Chronological and causal future

In this section we are going to study some properties of chronological and causal future, so first we give the definition of these notions.

Definition 3.1. *A point p chronologically precedes q, also denoted as $p \ll q$, if there exists a future-directed chronological (timelike) curve from p to q. The chronological future of a point $p \in M$ is the set $I^+(p)$ of all points to which p can be connected by a future directed timelike curve, $I^+(p) := \{q \in M | p \ll q\}$.*

Definition 3.2. *A point p causally precedes q, also denoted as $p < q$, if there exists a future-directed causal (non-spacelike) curve from p to q. The causal future of a point $p \in M$ is the set $J^+(p)$ of all points to which p can be connected by a future directed causal(non-spacelike) curve, $J^+(p) := \{q \in M | p < q\}$.*

Definition 3.3. *We define the chronological future of a compact surface Σ as the union of the chronological future at each point of the surface, $I^+(\Sigma) := \bigcup_{p \in \Sigma} I^+(p)$.*

Definition 3.4. *We define the causal future of a compact surface Σ as the union of the causal future at each point of the surface, $J^+(\Sigma) := \bigcup_{p \in \Sigma} J^+(p)$.*

Remark 3.5. *It is obvious that $I^+(\Sigma)$ is an open set, as a union of open sets, but $J^+(\Sigma)$ is closed. This is something we saw also in the talk a about the Hawking singularity theorem.*

Now we give the next corollary without a proof, since it was proven in the talk about Causality.

Corollary 3.6. *Let (M, g) be a time-oriented spacetime and $p \in M$. If $q \in J^+(p) \backslash I^+(p)$ then any future-directed causal curve connecting p to q must be a reparameterized null geodesic.*

Now we want to prove that even a future-directed null geodesic orthogonal to Σ may eventually enter in the chronological future of Σ, $I^+(\Sigma)$. A sufficient condition for this to happen is given by the next proposition.

Proposition 3.7. *Let (M, g) be a globally hyperbolic spacetime, S a Cauchy hypersurface with future-pointing unit normal vector field n, $\Sigma \subset S$ a compact 2-dimensional submanifold with unit normal vector field ν in S, $p \in \Sigma$, c_p the null geodesic through p with initial condition $n_p + \nu_p$ and $q = c_p(r)$ for some $r > 0$. If c_p has a conjugate point between p and q then $q \in I^+(\Sigma)$.*

Proof. Let s be the first conjugate point along c_p between p and q, there exists another null geodesic γ starting at Σ which (approxiamtely) intersects c_p at s. The piecewise smooth null curve obtained by following γ between Σ and s, and c_p between s and q is causal curve but not a null geodesic. This curve can be easily smoothed while remaining causal and non-geodesic, and by Corollary 3.6 we have $q \in I^+(\Sigma)$.

□

4 Penrose Singularity Theorem

In this section we are going to prove the Penrose singularity theorem, but first we give the definition of a singular manifold and a trapped surface.

Definition 4.1. *A spacetime (M, g) is said to be singular if it is not geodesically complete.*

Definition 4.2. *Let (M, g) be a globally hyperbolic spacetime and S a Cauchy hypersurface with future-pointing unit normal vector field n. A compact 2-dimensional submanifold $\Sigma \subset S$ with unit normal vector field ν in S is said to be trapped if the expansions θ^+ and θ^- of the null geodesics with initial conditions $n + \nu$ and $n - \nu$, respectively are both negative everywhere on Σ.*

So now we have defined and proved everything we needed in order to prove the Penrose singularity theorem.

Theorem 4.3 (Penrose Singularity Theorem)**.** *Let (M, g) be a connected globally hyperbolic spacetime with a non-compact Cauchy hypersurface S, satisfying the null energy condition. If S contains a trapped surface Σ then (M, g) is singular.*

Proof. Let $t : M \longrightarrow \mathbb{R}$ be a global time function such that $S = t^{-1}(0)$. The integral curves of grad$t$, being timelike, intersect S exactly once, and $\partial I^+(\Sigma)$ at most once. This defines a continuous injective map $\pi : \partial I^+(\Sigma) \longrightarrow S$, whose image is open. Indeed, if $q = \pi(p)$, then all points in some neighborhood of q are images of points in $\partial I^+(\Sigma)$, as otherwise there would be a sequence $q_n \longrightarrow q$ such that the integral curves of grad$t$ through q_n would not intersect $\partial I^+(\Sigma)$. If r_n are the intersections of these curves with the Cauchy hypersurface $t^{-1}(t(r))$, for some point r to the future of p along the integral line of grad$t$, we would have $r_n \longrightarrow r$, and so $r_n \in I^+(\Sigma)$ for sufficiently large n (as $I^+(\Sigma)$ is open), leading to a contradiction. Since Σ is trapped, there exists $\theta < 0$ such that the expansions θ^+ and θ^- of the null geodesics orthogonal to Σ both satisfy $\theta^+, \theta^- \leq 0$. We will show that there exists a future-directed null geodesic orthogonal to Σ which cannot be extended to an affine parameter greater than $r_0 = -\frac{2}{\theta_0}$ to the future of Σ.

Suppose that this does not hold. Then, according to Proposition 2.2, any null geodesic orthogonal to Σ would have a conjugate point at an affine parameter distance of at most r_0 to the future of Σ, after which it would be in $I^+(\Sigma)$, by Proposition 3.7. Consequently, $\partial I^+(\Sigma)$ would be a (closed) subset of the compact set

$$\exp^+([0, r_0] \times \Sigma) \cup \exp^-([0, r_0] \times \Sigma) \tag{4.1}$$

where $\exp^+$ and $\exp^-$ refer to the exponential map constructed using the unit normals ν and $-\nu$, and therefore $\partial I^+(\Sigma)$ would be compact itself. So since, $\partial I^+(\Sigma)$ is compact its image under π would also be compact, hence closed as well as open. Since M is connected, S would be connected as well, the image of π woulde be S, which then would be homeomorphic to $\partial I^+(\Sigma)$ which is compact, but by hypothesis S is non-compact and we have reached a contradiction.

□

Remark 4.4. *It should be clear that (M, g) is singular if the condition of existence of a trapped surface is replaced by the condition of existence of an anti-trapped surface. A compact surface Σ is anti-trapped if the expansions of null geodesics orthogonal to Σ are both positive.*

Now we would like to conclude the discussion of Penrose singularity theorem by giving some examples of manifods which contain trapped surfaces and some which do not contain trapped surfaces.

Example 4.5 (The Schwarzschild solution)**.**

The region $r < 2m$ of the Schwarzschild solution is globally hyperbolic, as we saw in the talk about the Schwarzschild solution, and also satisfies the null energy condition since $Ric = 0$. Moreover r (or $-r$) is clearly a time function (depending on the time orientation), it must increase (or decrease) along any future-pointing null geodesic, and therefore any sphere of constant (t, r) is anti-trapped (or trapped). Since any Cauchy hypersurface is diffeomorphic to $\mathbb{R} \times S^2$, hence non-compact, by Theorem 4.3 the Schwarzschild solution is singular.

Example 4.6 (The FLRW models)**.**

The FLRW models, as we saw in previous talk, are globally hyperbolic and satisfy the null energy condition (as $\rho > 0$). Moreover, the radial null geodesics satisfy

$$\frac{dr}{dt} = \pm\frac{1}{a}\sqrt{1 - kr^2} \tag{4.2}$$

Therefore, if we start with a sphere Σ of constant (t, r) and follow the orthogonal null geodesics along the direction of increasing or decreasing r we obtain spheres whose radii ar satisfy

$$\frac{d(ar)}{dt} = \dot{a}r + a\dot{r} = \dot{a}r \pm \sqrt{1 - kr^2} \tag{4.3}$$

Assume that the model is expanding, with the big bang at$t = 0$, and spatially non-compact. Then, for sufficiently small $t > 0$, the sphere Σ is anti-trapped, and hence by Theorem 4.3 we get that this model is singular to the past of Σ.

Example 4.7 (Minkowski space)**.**

Minkowski space is a flat spacetime which does not contain any trapped surfaces, its metric is given by the expression

$$\eta = -dt \otimes dt + dx \otimes dx + dy \otimes dy + dz \otimes dz \tag{4.4}$$

The fact that Minkowski spacetime does not contain any trapped surfaces, makes Minkowski space non-singular.

References

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