Queueing systems in a random environment with applications

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Queueing system	Environment
	attendance of employee has a break / is present
	finite buffer packets in buffer
	maintenance status maintained / ready to use
	abstract process countable state space

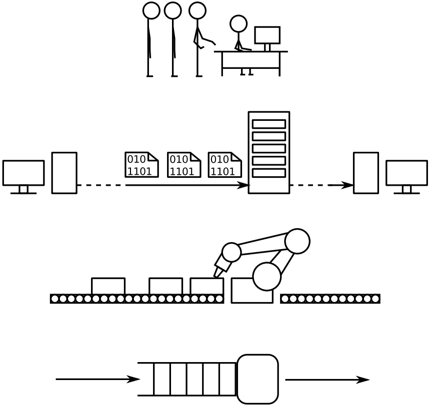

Figure: Queueing system examples.

Stochastic model

- countable system states $\mathscr{E} = \mathbb{N}_0 \times K$
  - $\mathbb{N}_0$ queue states (number of customers)
  - K environment state space
- time $t \in [0, \infty]$
- stochastic process $(X(t), Y(t)) \in \mathscr{E}$
  - $X(t)$ number of customers at time t
  - $Y(t)$ environment state at time t
- exponential sojourn times
- transition rates
- Find: limiting distribution (long term behavior) $\pi(n,k) := \lim_{t\to\infty} P((X(t), Y(t)) = (n,k))$
- Ansatz: solve $\pi Q = 0$ with generator matrix Q containing the transition rates.

Problem settings

- Given: states $(n,k) \in \mathscr{E}$ and transition rates $Q_{(n,k),(i,m)} \in \mathbb{R}_0^+$
- Find: $\pi(n,k) := \lim_{t\to\infty} P((X(t),Y(t)) = (n,k))$
- Solve: $\pi Q = 0$, $||\pi||_1 = 1$

Challenges

Solve: $\pi Q = 0$, $||\pi||_1 = 1$

- Problem: matrix Q is large.
  - For a queue with 99 places and 7 environment states (state space $\{0, ..., 99\} \times \{1, ..., 7\}$) we have $Q \in \mathbb{R}^{700x700}$.
  - For a queue with ∞ capacity we have $Q \in \mathbb{R}^{\infty \times \infty}$. Analytically it can be easier to solve than one with finite capacity!
- Help through special structure of Q.

Toy problem

Queue at a soft drink vending machine

- Service time is stochastic. Includes: feeding the machine with coins, fetching the can, and so on.
- Service according to FCFS policy.
- Capacity of the machine is limited (maximal two cans).
- As soon as the machine is empty, replenishment is ordered.
- Customer behavior during replenishment period:
  - Customers that were already in the queue, are waiting until replenishment will be finished.
  - New customers go somewhere else $\hat{=}$ **are lost**.

Find:

Limiting distribution of customers and cans in the vending machine.

Mathematical model

- States (n,k): n persons in queue, k cans in vending machine. That is $\mathscr{E} = \mathbb{N}_0 \times \{0,1,2\}$

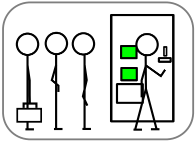

Figure: State (persons, cans) $= (n,k) = (4,2)$

- Stochastic process $(X(t), Y(t) : t \in [0,\infty))$, where $X(t)$ describes the queue and $Y(t)$ describes the environment.
- Customer arrival stream is Poisson with rate λ.
- Service time is exponential with rate μ.
- Replenishment lead time is exponential with rate ν.

Construction of Q

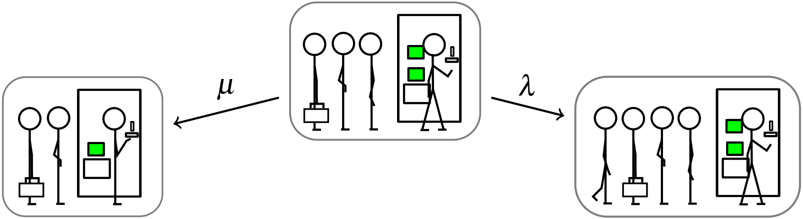

Figure: Possible system changes from (persons, cans)$= (X(t), Y(t)) = (4, 2)$

$$
\begin{array}{c||c|ccc|ccc|ccc|c}
 & \dots & (3,0) & {\color{blue}(3,1)} & (3,2) & (4,0) & (4,1) & (4,2) & (5,0) & (5,1) & {\color{blue}(5,2)} & \dots \\
\hline\hline
\vdots & & & & & & & & & & & \\
\hline
(4,0) & & & & & & & & & & & \\
(4,1) & & & & & & & & & & & \\
{\color{blue}(4,2)} & & & {\color{blue}\mu} & & & & & & & {\color{blue}\lambda} & \\
\hline
\vdots & & & & & & & & & & &
\end{array}
$$

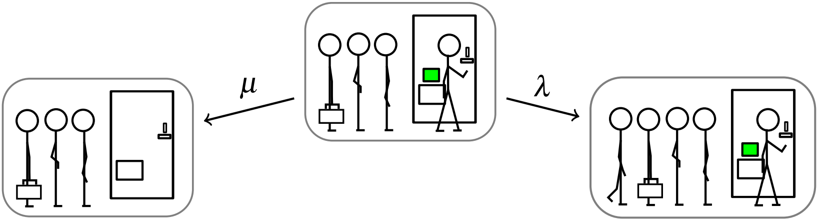

Figure: Changes from $(X(t), Y(t)) = (4,1)$

	…	(3,0)	(3,1)	(3,2)	(4,0)	(4,1)	(4,2)	(5,0)	(5,1)	(5,2)	…
⋮											
(4,0)											
(4,1)		μ							λ		
(4,2)			μ							λ	
⋮											

Construction of Q

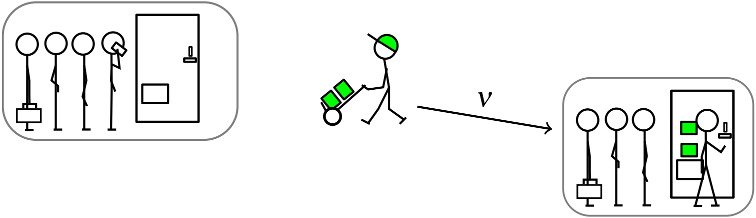

Figure: Changes from $(X(t), Y(t)) = (4,0)$

	$\dots$	$(3,0)$	$(3,1)$	$(3,2)$	$(4,0)$	$(4,1)$	$(4,2)$	$(5,0)$	$(5,1)$	$(5,2)$	$\dots$
$\vdots$											
$(4,0)$							ν				
$(4,1)$		μ							λ		
$(4,2)$			μ							λ	
$\vdots$											

Diagonal of Q

$$\left(\begin{array}{c||c|ccc|ccc|ccc|c}
 & \cdots & (3,0) & (3,1) & (3,2) & (4,0) & (4,1) & (4,2) & (5,0) & (5,1) & (5,2) & \cdots \\
\hline\hline
\vdots & & & & & & & & & & & \\
\hline
(4,0) & & & & & {\color{blue}-\nu} & & \nu & & & & \\
(4,1) & & \mu & & & & {\color{blue}-(\mu+\lambda)} & & & \lambda & & \\
(4,2) & & & \mu & & & & {\color{blue}-(\mu+\lambda)} & & & \lambda & \\
\hline
\vdots & & & & & & & & & & &
\end{array}\right)$$

Structure of the Q matrices for $M/M/1/\infty$-queues with environment states K.

$$Q = \begin{pmatrix} B_0 & B_1 & & & & \\ A_{-1} & A_0 & A_1 & & & \\ & A_{-1} & A_0 & A_1 & & \\ & & A_{-1} & A_0 & A_1 & \\ & & & \ddots & \ddots & \ddots \\ & & & & & \end{pmatrix}$$

$$B_i, A_i \in \mathbb{R}^{K \times K}$$

See M.F. Neuts. *Matrix Geometric Solutions in Stochastic Models - An Algorithmic Approach.* 1981.

Solution

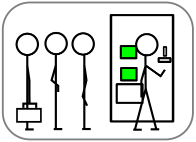

Figure: $(n,k) = (4,2)$

λ - arrival rate
μ - service rate
ν - replenishment rate

For the limiting distribution $\pi(n,k) := \lim_{n\to\infty} P(X(t) = n, Y(t) = k)$ it holds

Product form!

$$\pi(n,k) = \xi(n)\theta(k)$$

with $\xi(n) = \left(1 - \frac{\lambda}{\mu}\right)\left(\frac{\lambda}{\mu}\right)^n$ and $\theta(k) = \begin{cases} \frac{1}{2+\frac{\lambda}{\nu}}\left(\frac{\lambda}{\nu}\right), & k = 0 \\ \frac{1}{2+\frac{\lambda}{\nu}}, & k \in \{1,2\} \end{cases}$

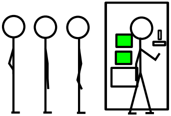

Figure: $(n,k)=(4,2)$

The limiting distribution $\pi(n,k) := \lim_{n\to\infty} P(X(t)=n, Y(t)=k)$

Nice properties of π

- product form $\pi(n,k)=\xi(n)\theta(k)$
- $\xi(n)=\left(1-\frac{\lambda}{\mu}\right)\left(\frac{\lambda}{\mu}\right)^n$ stochastically independent from environment
- $\theta(k)$ easy to solve and independent from service intensity μ

Can we keep these properties in more general settings?

YES, WE CAN

Loss system

	Vending machine	$M/M/1/\infty$**-loss system**
arrival	Poisson(λ)	Poisson(λ)
service, FCFS	$Exp(\mu)$	$Exp(\mu(n))$, $X(t) = n$
environment states	$K = \{0,1,2\}$	K - countable
env. states with no service and new customer loss	$\{0\}$ (empty machine)	$K_B \subset K$
env. changes after service $n \geq 1$	$(n,k) \to (n-1,k-1)$ $= \mu,\ k \geq 1$	$(n,k) \to (n-1,m)$ $= \mu R_{km}$, with stochastic matrix R
env. change independent from queue	$(n,0) \to (n,2) = \nu$ (replenishment)	$(n,k) \to (n,m) = V_{km}$, with generator matrix V

$M/M/1/\infty$-loss system

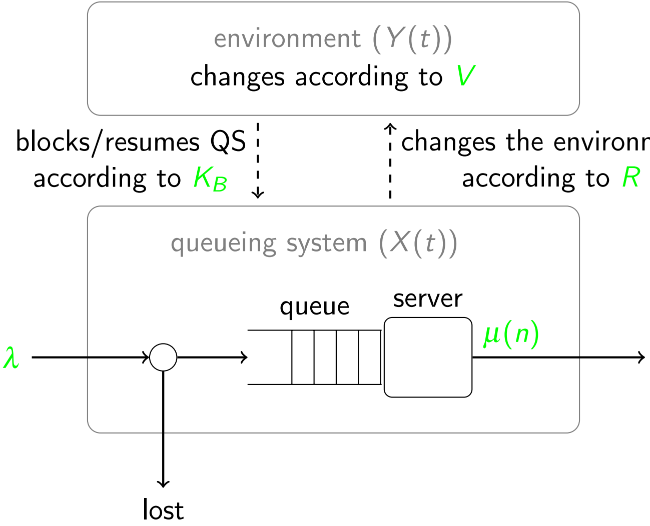

Figure: Loss systems with parameters λ, $\mu(n)$, K_B (resp. I_W), R, V.

$M/M/1/\infty$-loss system: Solution

Let $(X(t), Y(t))$ be an ergodic $M/M/1/\infty$-loss system with environment states K and system parameters: λ, $\mu(n)$, K_B (resp. I_W), R, V.
Then for the limiting distribution it holds

$$\pi(n,k) := \lim_{t\to\infty} P(X(t)=n, Y(t)=k)$$
$$\pi(n,k) = \xi(n)\theta(k)$$

with

$$\xi(n) = C^{-1}\prod_{i=1}^{n}\left(\frac{\lambda}{\mu(i)}\right), \qquad C := \sum_{n=0}^{\infty}\left(\prod_{i=1}^{n}\left(\frac{\lambda}{\mu(i)}\right)\right)$$

and θ the unique stochastic solution of

$$\theta\underbrace{(\lambda I_W(R-I)+V)}_{\in\mathbb{R}^{K\times K}} = 0 \qquad \text{(easier to solve than } \pi Q = 0\text{)}$$

Applications

- inventory systems
- unreliable systems
- sensor networks
- tests of simulations

Ongoing research

- queueing systems without lost customer in random environment
- network of queues in random environment
- mathematical properties of the stationary distribution of the systems
- modeling of specific systems

Thank you for your attention!

Relation between I_W and K_B

The matrix $I_W \in \{0,1\}^{K \times K}$ is a special way to write the blocking states K_B in a matrix form.

$$(I_W)_{km} := \delta_{km} 1_{[k \notin K_B]}$$

Example $K = \{0,1,2\}$, $K_B = \{0\}$

$$I_W = \left(\begin{array}{c|ccc} & 0 & 1 & 2 \\ \hline 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 2 & 0 & 0 & 1 \end{array} \right)$$

$$\lambda,\ \mu,\ I_W = \begin{array}{cc} & \begin{matrix} 0 & 1 & 2 \end{matrix} \\ \begin{matrix} 0 \\ 1 \\ 2 \end{matrix} & \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \end{array},\ R = \begin{array}{cc} & \begin{matrix} 0 & 1 & 2 \end{matrix} \\ \begin{matrix} 0 \\ 1 \\ 2 \end{matrix} & \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \end{array},\ V = \begin{array}{cc} & \begin{matrix} 0 & 1 & 2 \end{matrix} \\ \begin{matrix} 0 \\ 1 \\ 2 \end{matrix} & \begin{pmatrix} -\nu 0 & 0 & \nu \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \end{array}$$

$$\theta(\lambda I_W(R-I)+V)=0$$

$$\iff (\theta(0),\theta(1),\theta(2)) \begin{array}{cc} & \begin{matrix} 0 & 1 & 2 \end{matrix} \\ \begin{matrix} 0 \\ 1 \\ 2 \end{matrix} & \begin{pmatrix} -\nu & 0 & \nu \\ \lambda & -\lambda & 0 \\ 0 & \lambda & -\lambda \end{pmatrix} \end{array} = 0$$

$$\Longrightarrow \theta(0)\nu = \theta(1)\lambda \Longrightarrow \theta(0) = \frac{\lambda}{\nu}\theta(1)$$

$$\Longrightarrow \theta(1)\lambda = \theta(2)\lambda \Longrightarrow \theta(1) = \theta(1)$$

Normalization:

$$1 = \sum_{k=0}^{2} \theta(k) = \left(\frac{\lambda}{\nu}+2\right)\theta(1) \Longrightarrow \theta(1) = \frac{1}{\frac{\lambda}{\nu}+2}$$

If we observe a loss system $(X(t), Y(t) : t \in [0, \infty])$ at departure times, we obtain a Markov-chain $(\hat{X}(i), \hat{Y}(i) : i \in \mathbb{N}_0)$.
It is a well known result, that for an $M/M/1/\infty$ queue without environment, the limiting distributions

$$\xi(n) := \lim_{t\to\infty} P(X(t) = n), \qquad \hat{\xi}(n) = \lim_{i\to\infty} P(\hat{X}(i) = n)$$

are the same

$$\xi = \hat{\xi}$$

In contrast to this fact, we could show that for loss systems, the limiting distribution may differ

$$\pi(n,k) := \lim_{t\to\infty} P(X(t) = n, Y(t) = k), \hat{\pi}(n,k) = \lim_{i\to\infty} P(\hat{X}(i) = n, \hat{Y}(i) = k)$$

$$\pi \neq \hat{\pi} \qquad \text{(possible)}$$

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