Queueing systems in a random environment with applications

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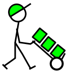
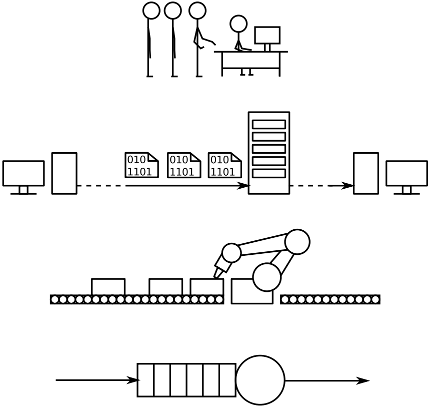
11th German Probability and Statistics Days



Ulm | March 4-7, 2014

Queueing system

Environment



attendance of employee
has a break / is present

finite buffer
packets in buffer

maintenance status
maintained / ready to use

abstract process
countable state space

Figure: Queueing system examples.

Stochastic model

- countable system states $\mathscr{E} = \mathbb{N}_0 \times K$
  - $\mathbb{N}_0$ queue states (number of customers)
  - K environment state space
- time $t \in [0, \infty]$
- stochastic process $(X(t), Y(t)) \in \mathscr{E}$
  - $X(t)$ number of customers at time t
  - $Y(t)$ environment state at time t
- exponential sojourn times
- transition rates
- Find: limiting distribution (long term behavior) $\pi(n,k) := \lim_{t\to\infty} P((X(t), Y(t)) = (n,k))$
- Ansatz: solve $\pi Q = 0$ with generator matrix Q containing the transition rates.

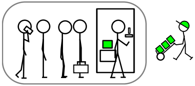
Problem settings & challenges

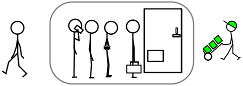
- Given: states $(n,k) \in \mathscr{E}$ and transition rates $Q_{(n,k),(i,m)} \in \mathbb{R}_0^+$
- Find: $\pi(n,k) := \lim_{t\to\infty} P((X(t),Y(t)) = (n,k))$
- Solve: $\pi Q = 0$, $||\pi||_1 = 1$

Challenge

- Problem: matrix Q is large.
  - For a queue with 99 places and 4 environment states we have $Q \in \mathbb{R}^{400\times 400}$.
  - For a queue with ∞ capacity we have $Q \in \mathbb{R}^{\infty\times\infty}$. This system can be easier to solve than one with finite capacity!

Toy problem





Queue at a soft drink vending machine

- Service time is random. Includes: feeding the machine with coins, fetching the can, and so on.
- Service according to FCFS policy.
- Capacity of the machine is limited (maximal three cans).
- As soon as the machine has only 1 can, replenishment is ordered.
- Customer behavior when machine is empty:
  - Customers that were already in the queue, are waiting until replenishment will be finished.
  - New customers go somewhere else $\hat{=}$ **are lost**.

Find:

Limiting distribution of customers and cans in the vending machine.

Mathematical model

- States (n, k): n people in queue, k cans in vending machine. That is $\mathscr{E} = \mathbb{N}_0 \times \{0, 1, 2, 3\}$.

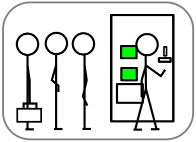


Figure: State (people, cans) $= (n, k) = (4, 2)$

- Stochastic process $(X(t), Y(t) : t \in [0, \infty))$, where $X(t)$ describes the queue and $Y(t)$ describes the environment.
- Customer arrival stream is Poisson with rate λ.
- Service time is exponential with rate μ.
- Replenishment lead time is exponential with rate ν.

Construction of Q

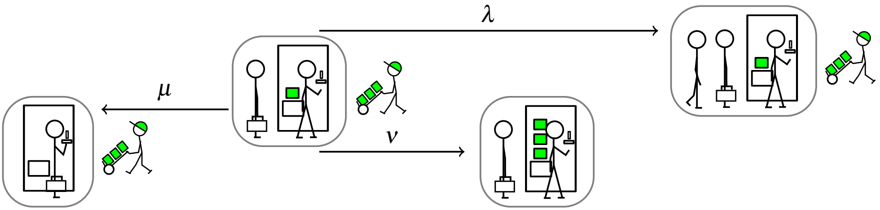


Figure: Possible system changes from (people, cans)$= (X(t), Y(t)) = (2,1)$

	…	(1,0)	(1,1)	(1,2)	(1,3)	(2,0)	(2,1)	(2,2)	(2,3)	(3,0)	(3,1)	(3,2)	(3,3)	…
⋮														
(2,0)														
(2,1)		μ							ν		λ			
(2,2)														
(2,3)														
⋮														

Structure of the Q matrices for $M/M/1/\infty$-queues
with environment states K:

$$Q = \begin{pmatrix} B_0 & B_1 & & & & \\ A_{-1} & A_0 & A_1 & & & \\ & A_{-1} & A_0 & A_1 & & \\ & & A_{-1} & A_0 & A_1 & \\ & & & \ddots & \ddots & \ddots \\ & & & & & \end{pmatrix},$$

$$B_i, A_i \in \mathbb{R}^{K \times K}.$$

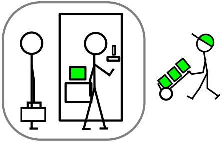


Figure: $(n, k) = (2, 1)$

λ - arrival rate
μ - service rate
ν - replenishment rate

For the limiting distribution $\pi(n,k) := \lim_{n\to\infty} P(X(t) = n, Y(t) = k)$ it holds

Product form!

$$\pi(n,k) = \xi(n)\theta(k)$$

with $\xi(n) = \left(1 - \frac{\lambda}{\mu}\right)\left(\frac{\lambda}{\mu}\right)^n$ and $\theta = C_\theta^{-1}\left(\frac{\lambda}{\nu}, 1, (\frac{\lambda+\nu}{\lambda}), (\frac{\lambda+\nu}{\lambda})\right)$.

Can we keep these properties of π in more general settings?

YES, WE CAN!

	Vending machine	**M/M/1/∞-loss system**
arrival	Poisson(λ)	
service, FCFS	$Exp(\mu)$	$''Exp(\mu(n))''$, $X(t) = n$
environment states	$K = \{0, 1, 2, 3\}$	K - countable
env. states with no service and new customer loss	$\{0\}$ (empty machine)	$K_B \subset K$
env. changes after service $n \geq 1$	$(n,k) \to (n-1, k-1)$ $= \mu,\ k \geq 1$	$(n,k) \to (n-1, m)$ $= \mu R_{km}$, with stochastic matrix R
env. changes independent from queue	$(n,1) \to (n,3) = \nu$ $(n,0) \to (n,3) = \nu$ (replenishment)	$(n,k) \to (n,m) = V_{km}$, with generator matrix V

$M/M/1/\infty$-loss system: solution

Let $(X(t), Y(t))$ be an ergodic $M/M/1/\infty$-loss system with environment states K and system parameters: λ, $\mu(n)$, K_B (resp. I_W), R, V.
Then for the limiting distribution it holds

$$\begin{aligned}\pi(n,k) &:= \lim_{t\to\infty} P(X(t)=n, Y(t)=k)\\ &= \xi(n)\theta(k)\end{aligned}$$

with

$$\xi(n) = C_\xi^{-1} \prod_{i=1}^{n} \left(\frac{\lambda}{\mu(i)}\right), \qquad C_\xi - \text{normalization constant}$$

and θ the unique stochastic solution of

$$\theta \underbrace{\lambda(I_W(R-I)+V)}_{\in\mathbb{R}^{K\times K}} = 0 \qquad \text{(easier to solve than } \pi Q = 0\text{)}$$

Embedded Markov chains (EMC)

What it is:

- Convert a continuous time process (CTP) into a Markov chain in discrete time.

How?

- Observe the system right after customer leaves the queue.
- Calculate transition probabilities $\mathbf{P}$.
- Solve $\hat{\pi}\mathbf{P} = \hat{\pi}$.

Why?

- "Classical method" to analyze $M/G/1/\infty$ queues, which are a superset of $M/M/1/\infty$ queues.
- Without environment the limiting distribution of $M/G/1/\infty$ modeled as EMC is the same as $M/G/1/\infty$ modeled as CTP.

Embedded Markov chains

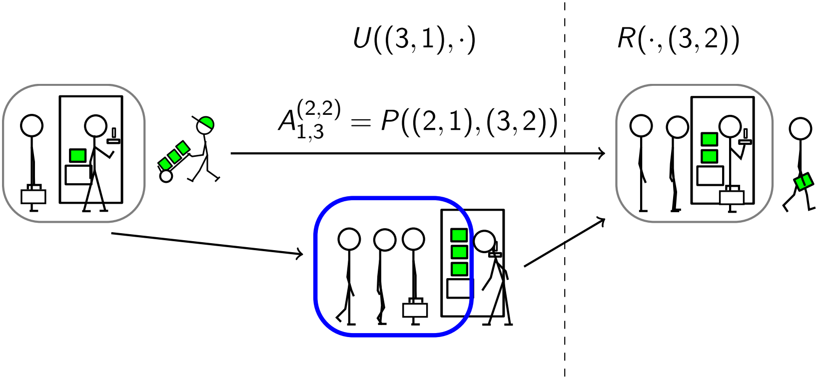


Figure: Probability to change from (2,1) to (3,2).

$$U_{km}^{(i,n)} := P\left(\left(X(\tau_1), Y(\tau_1^-)\right) = (n+i-1, m) | Z(0) = (i,k)\right).$$

$$A^{(i,n)} = U^{(i,n)} R$$

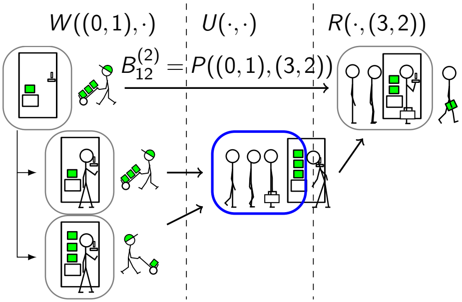


Figure: Probability to change from $(0,1)$ to $(3,2)$.

$$W_{km} := P(Z(\sigma_1) = (1,m) | Z(0) = (1,k))$$
$$B^{(n)} = WU^{(1,n)}R$$

$$\mathbf{P} = \begin{pmatrix} WU^{(1,0)}R & WU^{(1,1)}R & WU^{(1,2)}R & WU^{(1,3)}R & \dots \\ U^{(1,0)}R & U^{(1,1)}R & U^{(1,2)}R & U^{(1,3)}R & \dots \\ 0 & U^{(2,0)}R & U^{(2,1)}R & U^{(2,2)}R & \dots \\ 0 & 0 & U^{(3,0)}R & U^{(3,1)}R & \dots \\ \vdots & \vdots & \vdots & \vdots & \end{pmatrix}$$

- before arrival:

$$W = \lambda(\lambda I_W - V)^{-1} I_W$$

- before departure:

$$U^{(i,0)} = ((\lambda + \mu(i))I_W - V)^{-1}\mu(i)I_W$$

$$U^{(i,n+1)} = U^{(i,n)}\left(\frac{\lambda}{\mu(n+i)}\right)\mu(n+1+i)(\lambda I_W + \mu(n+1+i)I_W - V)^{-1}$$

Solve $\hat{\pi}\mathbf{P} = \hat{\pi}$

Results

Let $(X(t), Y(t) : t \in \mathbb{R}_0)$ be an ergodic $M/M/1/\infty$-loss system with states K and system parameters: λ, $\mu(n)$, K_B (resp. I_W), R, V. And let $(\hat{X}(t), \hat{Y}(t)) : t \in \mathbb{N}_0)$ be the appropriate Markov chain.
Then for the limiting distribution it holds

$$\begin{aligned}\hat{\pi}(n,k) &:= \lim_{t\to\infty} P(\hat{X}(t) = n, \hat{Y}(t) = k) \\ &= \xi(n)\hat{\theta}(k)\end{aligned}$$

with

$$\xi(n) = C_\xi^{-1} \prod_{i=1}^{n} \left(\frac{\lambda}{\mu(i)}\right), \qquad C_\xi - \text{normalization constant}$$

and $\hat{\theta}$ the unique stochastic solution of

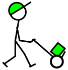
$$\hat{\theta} \underbrace{\left(I_W - \frac{1}{\lambda} V\right)^{-1} I_W R}_{\in \mathbb{R}^{K\times K}} = \hat{\theta} \qquad (\text{easier to solve than } \hat{\pi}\mathbf{P} = \hat{\pi})$$

Continuous time vs embedded Markov chains

Continuous time	**Embedded Markov chains**
transition rates Q	transition probabilities $\mathbf{P}$
solve $\pi Q = 0$	solve $\hat{\pi}\mathbf{P} = \hat{\pi}$
$\pi(n,k) = \xi(n)\theta(k)$	$\hat{\pi}(n,k) = \xi(n)\hat{\theta}(k)$
$\xi(n) = C_{\xi}^{-1}\prod_{i=1}^{n}\left(\frac{\lambda}{\mu(i)}\right)$	
$\theta\lambda(I_W(R-I)+V) = 0$	$\hat{\theta}\, C_{\theta}^{-1}\left(I_W - \frac{1}{\lambda}V\right)^{-1} = \hat{\theta}$
in general $\pi \neq \hat{\pi}$ (different from just a queue)	
$\theta = \left(\hat{\theta}\left(I_W - \frac{1}{\lambda}V\right)^{-1}\mathbf{e}\right)^{-1}\hat{\theta}\left(I_W - \frac{1}{\lambda}V\right)^{-1}$	
$\hat{\theta} = (\theta I_W \mathbf{e})^{-1}\cdot\theta I_W R$	

Ongoing research

- Why no product form for non-constant arrival rate?
- Is exponential distribution necessary for the product form?
- Extend results to networks
- Link to similar problems: boundaries, starting points



Thank you for your attention!

$M/M/1/\infty$-loss system

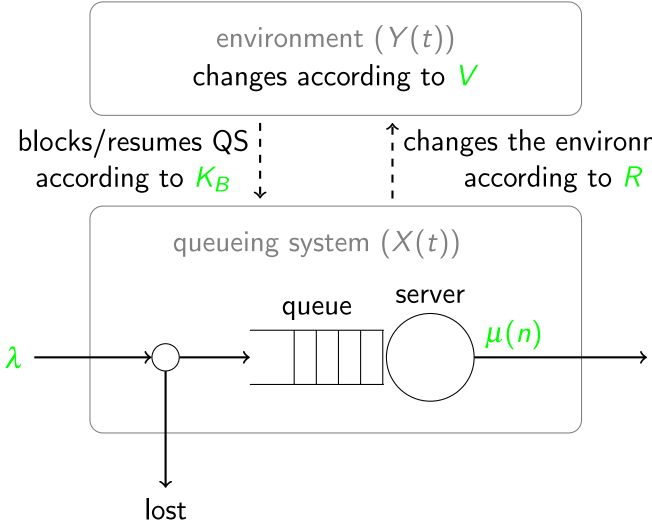


Figure: Loss systems with parameters λ, $\mu(n)$, K_B (resp. I_W), R, V.

The matrix $I_W \in \{0,1\}^{K\times K}$ is a special way to write the blocking states K_B in a matrix form.

$$(I_W)_{km} := \delta_{km} 1_{[k \notin K_B]}$$

Example $K = \{0,1,2,3\}$, $K_B = \{0\}$

$$I_W = \left(\begin{array}{c|cccc} & 0 & 1 & 2 & 3 \\ \hline 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 2 & 0 & 0 & 1 & 0 \\ 3 & 0 & 0 & 0 & 1 \end{array}\right)$$

$$I_W = \left(\begin{array}{c|cccc} & 0 & 1 & 2 & 3 \\ \hline 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 2 & 0 & 0 & 1 & 0 \\ 3 & 0 & 0 & 0 & 1 \end{array}\right),\ R = \left(\begin{array}{c|cccc} & 0 & 1 & 2 & 3 \\ \hline 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 2 & 0 & 1 & 0 & 0 \\ 3 & 0 & 0 & 1 & 0 \end{array}\right),\ V = \left(\begin{array}{c|cccc} & 0 & 1 & 2 & 3 \\ \hline 0 & -\nu & 0 & 0 & \nu \\ 1 & 0 & -\nu & 0 & \nu \\ 2 & 0 & 0 & 0 & 0 \\ 3 & 0 & 0 & 0 & 0 \end{array}\right)$$

$$\theta(\lambda I_W(R-I)+V)=0$$

$$\iff (\theta(0),\theta(1),\theta(2),\theta(3)) \left(\begin{array}{c|cccc} & 0 & 1 & 2 & 3 \\ \hline 0 & -\nu & 0 & 0 & \nu \\ 1 & \lambda & -(\nu+\lambda) & 0 & \nu \\ 2 & 0 & \lambda & -\lambda & 0 \\ 3 & 0 & 0 & \lambda & -\lambda \end{array}\right) = 0$$

$$\Longrightarrow \theta(0)\nu = \theta(1)\lambda \Longrightarrow \theta(0) = \frac{\lambda}{\nu}\theta(1)$$

$$\Longrightarrow \theta(1)(\nu+\lambda) = \theta(2)\lambda \Longrightarrow \theta(2) = \frac{(\nu+\lambda)}{\lambda}\theta(1)$$

$$\Longrightarrow \theta(2)\lambda = \theta(3)\lambda \Longrightarrow \theta(2) = \theta(3)$$

Normalization: $C_\theta = \sum_{k=0}^{3} \theta(k) = \left(\frac{\lambda}{\nu} + \frac{2\nu}{\lambda} + 3\right)\theta(1) \Longrightarrow \theta(1) = \frac{1}{\left(\frac{\lambda}{\nu} + \frac{2\nu}{\lambda} + 3\right)}$

Stopping times

The paths of Z are cadlag. With $\tau_0 = \sigma_0 = \zeta_0 = 0$ and

$$\tau_{n+1} := \inf(t > \tau_n : X(t) < X(\tau_n)), \quad n \in \mathbb{N}.$$

denote the sequence of departure times of customers by $\tau = (\tau_0, \tau_1, \tau_2, \dots)$, and with

$$\sigma_{n+1} := \inf(t > \sigma_n : X(t) > X(\sigma_n)), \quad n \in \mathbb{N},$$

denote by $\sigma = (\sigma_0, \sigma_1, \sigma_2, \dots)$ the sequence of instants when arrivals are admitted to the system (because the environment is in states of K_W, i.e., not blocking)
and with

$$\zeta_{n+1} := \inf(t > \zeta_n : Z(t) \neq Z(\zeta_n)), \quad n \in \mathbb{N},$$

denote by $\zeta = (\zeta_0, \zeta_1, \zeta_2, \dots)$ the sequence of jump times of Z.

Matrix invertible?

$$\hat{\theta}\Big(\underbrace{I_W - \frac{1}{\lambda}V}_{\text{invertible?}}\Big)^{-1} I_W R = \hat{\theta}$$

Properties of $I_W - \frac{1}{\lambda}V$ (in general):

- ''to some extent'' diagonally dominant
- ''to some extent'' irreducible

Known facts for finite dimensional matrices:

- diagonally dominant $\Longrightarrow$ invertible
- irreducible weakly diagonally dominant $\Longrightarrow$ invertible

More general condition for irreversibility (finite dimensional)

Combine and extend: ''to some extent'' diagonally dominant and ''to some extent'' irreducible are sufficient.

Matrix Invertible?

Theorem

Let $M \in \mathbb{R}^{K \times K}$, where the set of indices is partitioned according to $K = K_W + K_B$, $K_W \neq \emptyset$, and $|K| < \infty$, whose diagonal elements have following properties:

$$|M_{kk}| = \sum_{m \in K \setminus \{k\}} |M_{km}|, \qquad \forall k \in K_B \tag{1}$$

$$|M_{kk}| > \sum_{m \in K \setminus \{k\}} |M_{km}|, \qquad \forall k \in K_W \tag{2}$$

and it holds the flow condition

$$\forall \tilde{K}_B \subset K_B,\ \tilde{K}_B \neq \emptyset : \quad \exists \quad k \in \tilde{K}_B,\ \ m \in \tilde{K}_B^c : \quad M_{km} \neq 0\,. \tag{3}$$

Then M is invertible.

Preprints

Ruslan Krenzler and Hans Daduna.
Loss systems in a random environment.
December 2013.
http://arxiv.org/abs/1312.0539

Errata

23 March 2014: Corrected expression for the environment equation to $\theta(\lambda I_W(R-I)+V)=0$