Jackson Network in a random environment

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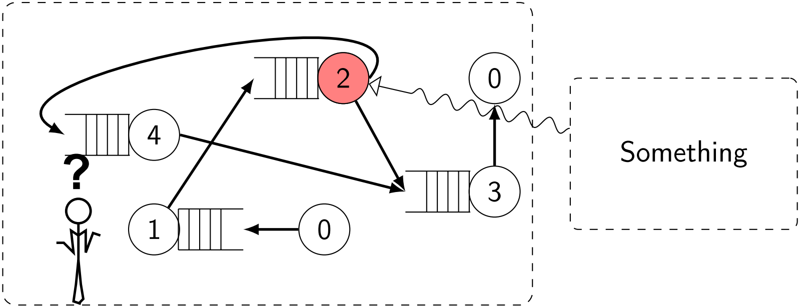
Motivation



Question

What does a served customer do when next node is degraded?

Answer

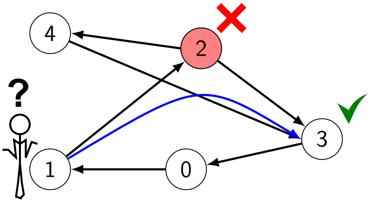
Change the route.

Random walk with random skipping

- Given a routing matrix r on a node set $\{0, 1, ..., J\}$
- Given an acceptance probability vector $\alpha := (\alpha_0, \alpha_1, ..., \alpha_J)$

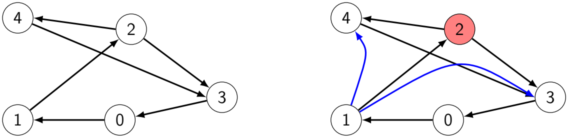
Algorithm how to move from i

1. set $k = i$
2. move to j with probability $r(k, j)$
3. check if new node j will accept the customer with probability α_j
   - 3.1 if accepted, stay in j, END.
   - 3.2 if not accepted, skip: set $k = j$ go to 2



$$\alpha := (1, 1, 0.5, 1, 1)$$

Rerouting, Skipping



(a) original routing r (b) rerouting $r^{(\alpha)}$ by skipping

Figure: Skipping with $\alpha := (\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4) = (1, 1, 0.5, 1, 1)$

rerouting matrix $r^{(\alpha)}$

$$r^{(\alpha)} = (I - r \cdot \mathrm{diag}(1-\alpha))^{-1} r \cdot \mathrm{diag}(\alpha)$$

Properties of $r^{(\alpha)}$

Given:

- Original routing matrix r
- Acceptance probability vector $\alpha := (\alpha_0, \alpha_1, ..., \alpha_J)$
- Rerouting matrix $r^{(\alpha)}$

Nice Property of $r^{(\alpha)}$

If

$$\eta = (\eta_0, ..., \eta_J)$$

solves

$$\eta r = \eta,$$

then

$$\eta^{(\alpha)} = (\alpha_0 \eta_0, \alpha_1 \eta_1, ..., \alpha_J \eta_J)$$

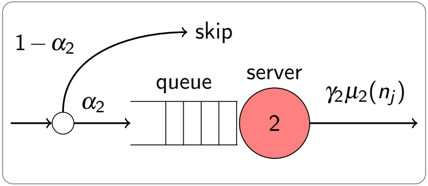
solves

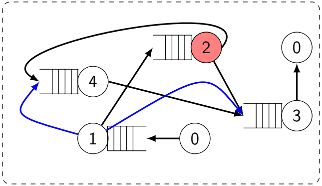
$$\eta^{(\alpha)} r^{(\alpha)} = \eta^{(\alpha)}.$$

Set of nodes $\overline{J} = \{1, ..., J\}$, routing matrix r, service rates $\mu_1(n_1), ..., \mu_J(n_J)$
Process: $\boldsymbol{X}(t) := (X(t) : t \in [0, \infty))$

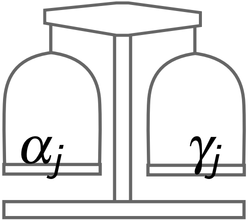
Modification

+ action: service rate changes by factor $\gamma_j \in [0,1]$, $\gamma = (\gamma_1, ..., \gamma_J)$
  $\Longrightarrow$new service rates $\gamma_j \cdot \mu_j(n_j)$
+ reaction: rerouting $r^{(\alpha)}$





acceptance probability $\alpha_j = \gamma_j$



New service rate $\gamma_j \mu_j(n_j)$

	service rate factor γ_j	acceptance α_j
$= 0$	service down	everyone skips over
$= 1$	service up	everyone accepted
$0 < \cdot < 1$	degraded	sometimes accepted

Modified Jackson network

- J nodes, input rate λ, routing matrix r, service rates $\mu_j(n_j)$
- n_j is number of customers at the node j

Notation: $\boldsymbol{n} := (n_1, ..., n_J)$, $\boldsymbol{e}_j := (0, ..., 0, \overbrace{1}^{\text{on } j\text{-th pos.}}, 0, ...0)$
Classical, with generator $Q^X = (q^{\boldsymbol{X}}(\boldsymbol{n}, \boldsymbol{n}') : \boldsymbol{n}, \boldsymbol{n}' \in \mathbb{N}_0^J)$

$$q^{\boldsymbol{X}}(\boldsymbol{n}, \boldsymbol{n} + \boldsymbol{e}_i) = \lambda r(0, i)$$
$$q^{\boldsymbol{X}}(\boldsymbol{n}, \boldsymbol{n} - \boldsymbol{e}_j + \boldsymbol{e}_i) = 1_{[n_j > 0]} \mu_j(n_j) r(j, i)$$
$$q^{\boldsymbol{X}}(\boldsymbol{n}, \boldsymbol{n} - \boldsymbol{e}_j) = 1_{[n_j > 0]} \mu_j(n_j) r(j, 0)$$

Modification

+ action: service rate changes by $\gamma = (\gamma_1, ..., \gamma_2)$
+ reaction: rerouting $r^{(\alpha)}$

New system

$$... = \lambda r^{(\alpha)}(0, i)$$
$$... = 1_{[n_j > 0]} \gamma_j \mu_j(n_j) r^{(\alpha)}(j, i)$$
$$... = 1_{[n_j > 0]} \gamma_j \mu_j(n_j) r^{(\alpha)}(j, 0)$$

Modified Jackson network

Theorem (classical)

For the steady state distribution $\xi(\boldsymbol{n}) := \xi(n_1, \ldots, n_J)$ *holds*

$$\xi(n_1, \ldots, n_J) = \prod_{j=1}^{J} \xi(n_j) \qquad \textit{with} \qquad \xi_j(n_j) := \prod_{k=1}^{n_j} \frac{\eta_j}{\mu_j(k)} C(j)^{-1}$$

with η *solution of* $\eta r = \eta$, $\eta_0 = \lambda$.

Blocked nodes $B(\gamma) := \{j \in \overline{J},\ \gamma_j = 0\}$

Theorem (modified)

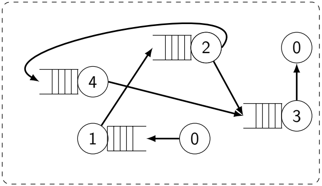
If acceptance probabilities$\equiv \alpha_j = \gamma_j \equiv$ *service rate factors, then*

$$\xi^{(\alpha)}(n_1, \ldots, n_J) = \prod_{j \in J \setminus B(\gamma)} \xi(n_j) \cdot \prod_{i \in B(\gamma)} \varphi_i(n_i)$$

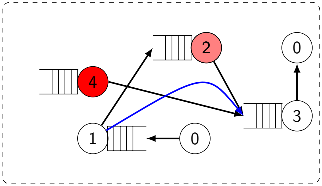
with arbitrary distribution $\varphi_i(n_i)$.

Jackson network with degrading nodes

Example: $\gamma = (1, 0.5, 1, 0) \Longrightarrow B(\gamma) = \{4\}$



(a) classical



(b) modified

If acceptance probabilities$\equiv\alpha_j = \gamma_j\equiv$service rate factors, then steady state distribution ξ is

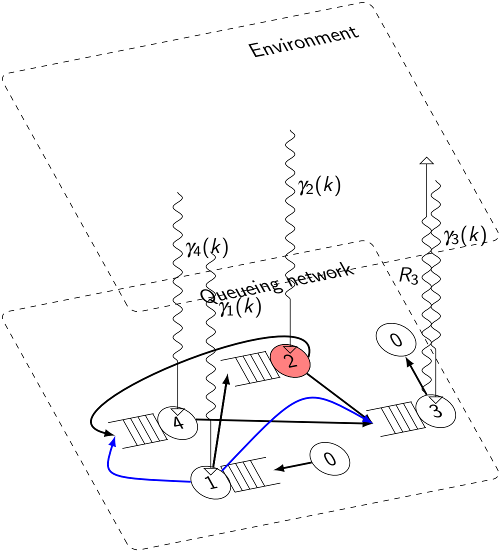
- the same on nodes 1, 2 and 3,
- arbitrary on 4 in the degraded case.

Environment: additional parameter of the system

- $\boldsymbol{Y} = (Y(t) : t \in [0,\infty))$
- Lives on finite state space K
- Can change its state with transition rates $V = (v(k,m) : k,m \in K)$, independent from queueing system
- Can be changed by queueing system with probabilities $R_j = (R(k,m) : k,m \in K)$, each time a customer from j leaves the network

$Y(t)$ is **not** necessarily a Markov process!

Jackson network in a random environment



direction of impact

positive routing entries $r(i,j)$

new positive routing entries of $r^{(\alpha(k))}(i,j)$

j partially degraded node j

Jackson network in a random environment

Given:

- $\boldsymbol{X}$-Jackson network with J nodes and parameters λ, μ, r
- $\boldsymbol{Y}$-environment on a state space K and rates $V = (v(k,m) : k, m \in K)$
- Interactions if environment is in state k:
  - Environment→Queues: by changing of service rates by $\gamma(k)$, and therefore changing of rerouting $r^{(\alpha)}$, $\alpha = \gamma$
  - Queues→Environment: by changing environment states with $R_j = (R(k,m) : k, m \in K)$, when a customer leaves the network from j

Steady state distribution $\pi(\boldsymbol{n}, k)$

$$\pi(\boldsymbol{n}, k) = \xi(\boldsymbol{n}) \cdot \theta(k), \quad \boldsymbol{n} \in \mathbb{N}_0^{\bar{J}}, k \in K,$$

$\xi(\boldsymbol{n})$ Solution for the original Jackson network,
$\theta(k)$ Stochastic solution of $\theta \cdot Q_{red} = 0$,

$$Q_{red} := \left[V + \sum_{j \in \bar{J}} \eta_j \mathrm{diag}\left(r^{(\alpha(\cdot))}(j,0) \right) (R_j - I) \right].$$

General service rate factor $\gamma = (\gamma_1, \dots, \gamma_J)$ with $\gamma_j \in [0, \infty)$

+ action: change of service rates by $\gamma = (\gamma_1, \dots, \gamma_2)$
$\Longrightarrow$ new service rates $\gamma_j \cdot \mu_j(n_j)$
+ reaction: increase overall input $\beta := \begin{cases} 1 & \text{if} \quad ||\gamma||_\infty \le 1 \\ ||\gamma||_\infty & \text{if} \quad ||\gamma||_\infty > 1 \end{cases}$,
+ reaction: rerouting $r^{(\alpha)}$ with $\alpha_j := \begin{cases} \gamma_j & \text{if} \quad ||\gamma||_\infty \le 1 \\ \frac{\gamma_j}{||\gamma||_\infty} & \text{if} \quad ||\gamma||_\infty > 1 \end{cases}$.

$$q^{\mathbf{X}}(\mathbf{n}, \mathbf{n} + \mathbf{e}_i) = \beta \lambda r^{(\alpha)}(0, i)$$
$$q^{\mathbf{X}}(\mathbf{n}, \mathbf{n} - \mathbf{e}_j + \mathbf{e}_i) = 1_{[n_j > 0]} \gamma_j \mu_j(n_j) r^{(\alpha)}(j, i)$$
$$q^{\mathbf{X}}(\mathbf{n}, \mathbf{n} - \mathbf{e}_j) = 1_{[n_j > 0]} \gamma_j \mu_j(n_j) r^{(\alpha)}(j, 0)$$

The same results for the steady state distribution

Other rerouting strategy $r^{(\alpha)}$, for example randomized reflection.
Or any rerouting $r^{(\alpha)}$ for which the **Nice Property** holds:

$$\text{If } \eta = (\eta_0, ..., \eta_J) \text{ solves}$$

$$\eta r = \eta,$$

$$\text{then } \eta^{(\alpha)} = (\alpha_0\eta_0, \alpha_1\eta_1, ..., \alpha_J\eta_J) \text{ solves}$$

$$\eta^{(\alpha)} r^{(\alpha)} = \eta^{(\alpha)}.$$

The same results for the steady state distribution

Nice Property of skipping

$$r^{(\alpha)} := (I - r \cdot \mathrm{diag}(1-\alpha))^{-1} r \cdot \mathrm{diag}(\alpha)$$

If $\eta = (\eta_0, ..., \eta_J)$ solves

$$\eta r = \eta,$$

then $\eta^{(\alpha)} = (\alpha_0 \eta_0, \alpha_1 \eta_1, ..., \alpha_J \eta_J)$ solves

$$\eta^{(\alpha)} r^{(\alpha)} = \eta^{(\alpha)}.$$

- How can we use this property above in importance sampling?
- Applications for the new product form results for Jackson network + Environment.

Thank you for your attention!

Preprints

Ruslan Krenzler, Hans Daduna and Sonja Otten.
Randomization for Markov chains with applications to networks in a random environment.
July 2014.
http://arxiv.org/abs/1407.8378