The “tree property” for supercompactness

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(Joint work with Matteo Viale)

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Recall that

$$\kappa \text{ is weakly compact} \leftrightarrow \kappa \text{ is inaccessible} + \kappa\text{-TP holds},$$

where κ-TP is the tree property on κ.
Due to Mitchell and Silver we have

$$V \models \kappa \text{ is weakly compact} \Rightarrow V[G] \models \kappa = \omega_2 \text{ and } \kappa\text{-TP holds}$$

for some V-generic G, as well as

$$V \models \kappa\text{-TP holds} \Rightarrow L \models \kappa \text{ is weakly compact.}$$

Furthermore, due to Baumgartner,

$$\text{PFA} \rightarrow \omega_2\text{-TP holds.}$$

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Yes we can!

There is a principle (κ, λ)-ITP such that

$$\kappa \text{ is supercompact} \leftrightarrow \kappa \text{ is inaccessible} + \forall\lambda \geq \kappa\ (\kappa, \lambda)\text{-ITP holds},$$

and such that we get

$$V \models \kappa \text{ is supercompact} \Rightarrow$$
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$$(\kappa, \lambda)\text{-ITP holds} \rightarrow \neg\square(\lambda).$$

(We also get the failure of weaker forms of square.)

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(We also get the failure of weaker forms of square.) And yes, we also get

Theorem

PFA *implies that* (ω_2, λ)-ITP *holds for all* $\lambda \geq \omega_2$.

Definition

Suppose $\langle \mathbb{P}_\alpha \mid \alpha \leq \kappa \rangle$ is a forcing iteration. Let us call it a *standard iteration* iff

- $\mathbb{P}_\alpha$ is the direct limit of $\langle \mathbb{P}_\beta \mid \beta < \alpha \rangle$ for $\alpha = \kappa$ and stationarily many $\alpha < \kappa$,
- $|\mathbb{P}_\alpha| < \kappa$ for all $\alpha < \kappa$.

Note that the usual forcings used to force PFA or MM from a supercompact cardinal are standard iterations.

Conjecture

Suppose κ is inaccessible and $\langle \mathbb{P}_\alpha \mid \alpha \leq \kappa \rangle$ is a standard forcing iteration such that $\Vdash_{\mathbb{P}_\kappa}$ "$\forall \lambda \geq \kappa$ (κ, λ)-ITP holds." Then κ is supercompact.

Note that this would imply the following as a corollary.

Conjecture

Suppose κ *is inaccessible and* $\langle \mathbb{P}_\alpha \mid \alpha \leq \kappa \rangle$ *is a standard forcing iteration such that* $\Vdash_{\mathbb{P}_\kappa}$ "$\kappa = \omega_2 \wedge \mathsf{PFA}$." *Then* κ *is supercompact.*

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While there is a technical problem with proving this, we do have the following.

Theorem

Suppose κ is inaccessible and $\langle \mathbb{P}_\alpha \mid \alpha \leq \kappa \rangle$ is a standard forcing iteration such that $\Vdash_{\mathbb{P}_\kappa}$ "$\forall \lambda \geq \kappa$ (κ, λ)-ITP holds." Then κ is strongly compact.

So in particular the following holds.

Corollary

Suppose κ is inaccessible and $\langle \mathbb{P}_\alpha \mid \alpha \leq \kappa \rangle$ is a standard forcing iteration such that $\Vdash_{\mathbb{P}_\kappa}$ "$\kappa = \omega_2 \wedge \mathsf{PFA}$." Then κ is strongly compact.

So what does (κ, λ)-ITP say?

Definition

$\langle d_a \mid a \in P_\kappa\lambda\rangle$ is called a $P_\kappa\lambda$-*list* iff $d_a \subset a$ for all $a \in P_\kappa\lambda$.

Definition

A $P_\kappa\lambda$-list $\langle d_a \mid a \in P_\kappa\lambda\rangle$ is called *thin* iff there is a club $C \subset P_\kappa\lambda$ such that

$$|\{d_a \cap c \mid c \subset a \in P_\kappa\lambda\}| < \kappa$$

for all $c \in C$.

Definition

(κ, λ)-ITP holds iff for every thin $P_\kappa\lambda$-list $\langle d_a \mid a \in P_\kappa\lambda\rangle$ there are $d \subset \lambda$ and a stationary $S \subset P_\kappa\lambda$ such that $d_a = d \cap a$ for all $a \in S$.

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There is an even better principle (κ, λ)-ISP! It also implies the failure of the approachability property, can be used to prove SCH under PFA!

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There is an even better principle (κ, λ)-ISP! It also implies the failure of the approachability property, can be used to prove SCH under PFA, and is even more complicated!

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There is an even better principle (κ, λ)-ISP! It also implies the failure of the approachability property, can be used to prove SCH under PFA, and is even more complicated!
And since the only thing published on this so far is my thesis, if you are interested you have no choice but to read it:

http://edoc.ub.uni-muenchen.de/11438/