Boolean algebras

① From $\models_c \phi \Leftrightarrow \vdash_c \phi$

algebraic semantics ⇕ ⇕ algebraic syntax

$Set \models \phi \approx T \Leftrightarrow BA \models \phi \approx T$

② $\Phi = \{p, q, r \dots\}$ propositional letters

$Bool = (\{\bot, \neg, \vee\}, \rho) \quad \rho(\vee) = 2$

$Form(\Phi) = \{p \vee q, \neg r, \dots, \bot, \neg\bot\}$ prop. formulas

$p \wedge q = \neg(\neg p \vee \neg q)$

$x \cdot y = -(-x + -y)$ for $x, y, z \in A$ evaluated in A^*

$A^* = (A, +, -, 0)$ of type Bool is Boolean if ✓

(B0) commut. $x+y = y+x$ $\quad x \cdot y = y \cdot x$

(B1) associative $x+(y+z) = (x+y)+z$ $\quad x(y \cdot z) = (x \cdot y) \cdot z$

(B2) neutral el. $x+0 = x$ $\quad x \cdot 1 = x$

(B3) complement $x+(-x) = 1$ $\quad x \cdot (-x) = 0$

(B4) distributive $x+(y \cdot z) = (x+y) \cdot (x+z)$;

$x \cdot (y+z) = (x \cdot y) + (x \cdot z)$

③ Truth value algebra

$2^* = (\{0,1\}, +, -, 0)$

$1 = -0$

$-a = 1 - a$

$a + b = \max(a, b)$

④ Assign meaning

$\theta : \Phi \to 2^A$ any assignment

$\tilde{\theta} : \text{Form}(\Phi) \to 2^A$ follow truth tables

for $p \in \Phi$: $\tilde{\theta}(p) = \theta(p)$

for $\phi \in \text{Form}(\Phi)$:

$\tilde{\theta}(\bot) = 0$

$\tilde{\theta}(\neg\phi) = -\tilde{\theta}(\phi)$

$\tilde{\theta}(\phi_1 \vee \phi_2) = \max(\tilde{\theta}(\phi_1), \tilde{\theta}(\phi_2))$

⑤ Make a meaning homomorphism

$\text{Form}(\Phi)^A = (\text{Form}(\Phi), +, -, \bot)$

$\tilde{\theta}(\phi_1 \vee \phi_2) = \max(\tilde{\theta}(\phi_1), \tilde{\theta}(\phi_2)) = \tilde{\theta}(\phi_1) + \tilde{\theta}(\phi_2)$

⑥ Evaluate equality of meaning

$\phi, \psi \in \text{form}(\Phi)$

$A^* \models \phi \approx \psi$ validity of $\phi \approx \psi$ in A^*

$2^* \models \phi \approx \top \iff \models_c \phi$

↳ $\tilde{\theta}(\neg\bot) = -\tilde{\theta}(\bot) = -0 = 1$

⑦ Equations are Horn implications

$$\hat{\Theta}(\phi \leftrightarrow \psi) = \begin{cases} 1 & \text{if } \tilde{\Theta}(\phi) = \tilde{\Theta}(\psi) \\ 0 & \text{if } \tilde{\Theta}(\phi) \neq \tilde{\Theta}(\psi) \end{cases}$$

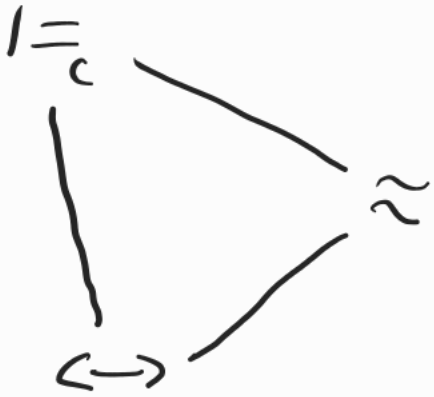
$$1 \models_c \phi \leftrightarrow \psi \iff 2 \, 1 \models \phi \approx \psi$$

⑧ Theorem 5.5 : 2^* algebraises validity

$$1 \models_c \phi \iff 2 \, 1 \models \phi \approx T$$

$$2 \, 1 \models \phi \approx \psi \iff 1 \models_c \phi \leftrightarrow \psi$$

$$1 \models (\phi \leftrightarrow (\phi \leftrightarrow T))$$



⑨ Set algebras from power set algebras

$$\mathcal{P}(A)^* = (\mathcal{P}(A), \cup, -, \emptyset)$$

$$-X = A \setminus X$$

$$X \cap Y = -(-X \cup -Y)$$

$$S^* = (S \subseteq \mathcal{P}(A), \cup, -, \emptyset) \quad \text{closed under } \cup, - \quad \emptyset \in S$$

A worlds
S propositions true in subset of worlds

⑩ $\mathcal{P}(A)^* \cong 2^{*A}$

$$2^{*A} = (F, +, -, 0)$$

$F = \{ f \mid f : A \to \{0,1\} \}$ all assignments in 2^*

$0 = f : A \to 0$

for $a \in A$ $(f_1 + f_2)(a) = \max(f_1(a), f_2(a))$

$-f(a) = 1 - f(a)$

Make isomorphic $\chi : P(A) \to F$

for $Y \subseteq A$, $\chi(Y)(a) = \begin{cases} 1 & \text{if } a \in Y \\ 0 & \text{if } a \notin Y \end{cases}$

$\chi(Y)$ = characteristic function of Y

if $\chi(Y) = \chi(Z)$ $\forall a \in A$ $\chi(Y)(a) = \chi(Z)(a)$

$Y = Z$; χ is injective

if $f \in 2^{*A}$ take $Y = \{a \in A \mid f(a) = 1\}$

$\chi(Y)(a) = 1 \iff f(a) = 1$; χ is surjective

χ preserves $\cup$: $\chi(Y \cup Z)(a) = 1 \iff$

$a \in Y \cup Z \iff$

$a \in Y \cup a \in Z \iff$

$\max(\chi(Y)(a), \chi(Z)(a)) = 1 \iff$

$(\chi(Y) + \chi(Z))(a) = 1$

for $-$: $\chi(-Y)(a) = 1 \iff$

$a \in -Y \iff$

$a \notin Y \iff$

$\chi(Y)(a) = 0 \iff$

$-\chi(Y)(a) = -0 = 1$

for $\emptyset$: $\chi(\emptyset) = f : A \to 0$

(11) Theorem 5.9: Set algebraises validity

for $\phi, \psi \in Form(\Phi)$ $\vdash_c \phi \iff Set \models \phi \approx T$

$\vdash_c \phi \iff$

$2^A \models \phi \approx T \iff$

for any extended $\tilde{\theta}: Form(\Phi) \to \{0,1\}$:

$\tilde{\theta}(\phi) = \tilde{\theta}(T) = 1 \iff$

for $\tilde{\theta}: Form(\Phi) \to 2^A$ and $a \in A$:

$\tilde{\theta}(\phi)(a) = \tilde{\theta}(T)(a) = 1 \iff$

for $\tilde{\theta}: Form(\Phi) \to S \subseteq P(A) : \tilde{\theta}(\phi) = A$

(12) Theorem 5.11: Boolean algebras algebraise theorem hood

$\vdash_c \phi \iff BA \models \phi \approx T$

(Induction on the length of proofs)

"$\Leftarrow$" Find assignment to a BA with $\phi \not\approx 1$ if $\not\vdash_c \phi$

(13) Provable equivalence is a classical congruence on $Form(\Phi)$

$\phi \equiv_c \psi \iff \vdash_c \phi \leftrightarrow \psi$

the Bool operations preserve congruence

for $\bot$: $\bot \equiv_c \bot$

for $-$: $\phi \equiv_c \psi \Rightarrow \vdash_c \phi \leftrightarrow \psi$

$$\dfrac{\dfrac{[\psi]^1 \quad \phi \leftrightarrow \psi}{\neg\phi \quad \phi}\leftrightarrow E}{\dfrac{\bot}{\neg\psi}\neg I\,1}\neg E\ (\text{elim. } \phi \to \bot)$$

for $\vdash$: $\phi_1 \equiv_c \psi_1$

$\phi_2 \equiv \psi_2$

$$\dfrac{\dfrac{\dfrac{[\phi_1]^1 \quad \phi_1 \leftrightarrow \psi_1}{\psi_1}\leftrightarrow E \qquad \dfrac{[\phi_2]^1 \quad \phi_2 \leftrightarrow \psi_2}{\psi_2}\leftrightarrow E}{\psi_1 \vee \psi_2}\vee I \qquad \dfrac{\psi_1 \vee \psi_2}{\psi_1 \vee \psi_2}\vee I}{\psi_1 \vee \psi_2}\vee E\,1}{(\phi_1 \vee \phi_2) \to (\psi_1 \vee \psi_2)}\to I\,2$$

$[\phi_1 \vee \phi_2]^2$

analogously $\vdash_c \psi_1 \vee \psi_2 \to \phi_1 \vee \phi_2$

$\vdash_c \psi_1 \vee \psi_2 \leftrightarrow \phi_1 \vee \phi_2$

$\psi_1 \vee \psi_2 \equiv_c \phi_1 \vee \phi_2$

(14) The Lindenbaum-Tarski algebra

$$\mathcal{L}_c(\Phi)^* = (\mathrm{Form}(\Phi)/\equiv_c, +, -, 0)$$

$0 = [\bot]$

$-[\phi] = [\neg\phi]$

$[\phi] + [\psi] = [\phi \vee \psi]$

$[\phi] + (-[\phi]) = [\phi \vee \neg\phi] = [\top] = 1$

(15) Prop. 5.14 $\vdash_c \phi \Leftrightarrow \mathcal{L}_c^{A}(\Phi) \models \phi \approx \top$

„$\Leftarrow$" $\theta : \Phi \to Form(\Phi)/\equiv_c$

$\theta(p) = [p]$

$\tilde{\theta} : Form(\Phi) \to Form(\Phi)/\equiv_c$

$\tilde{\theta}(p) = [p]$

$\tilde{\theta}(\neg\phi) = -\tilde{\theta}(\phi)$

$\tilde{\theta}(\phi \vee \psi) = \tilde{\theta}(\phi) + \tilde{\theta}(\psi)$

$\tilde{\theta}(\phi) = [\phi]$: induction start $\tilde{\theta}(\bot) = [\bot]$

$\tilde{\theta}(\neg\phi) = -\tilde{\theta}(\phi) = -[\phi] = [\neg\phi]$

$\tilde{\theta}(\phi \vee \psi) = \tilde{\theta}(\phi) + \tilde{\theta}(\psi) = [\phi] + [\psi] = [\phi \vee \psi]$

$\not\vdash_c \phi \Rightarrow \not\vdash_c \phi \leftrightarrow \top \Rightarrow \phi \not\equiv_c \top \Rightarrow [\phi] \neq [\top]$

$\tilde{\theta}(\phi) = [\phi] \neq [\top] = \tilde{\theta}(\top)$

$\mathcal{L}_c^{A}(\Phi) \not\models \phi \approx \top$

(16) $\mathcal{L}_c(\Phi)^*$ is a boolean algebra

(B_0) $[\phi] + [\psi] = [\phi \vee \psi] = [\psi] + [\phi]$

$[\phi] \cdot [\psi] = [\phi \wedge \psi] = [\psi] \cdot [\phi]$

(B_1) $[\phi] + ([\psi] + [\chi]) = [\phi \vee (\psi \vee \chi)]$

$= [(\phi \vee \psi) \vee \chi)] = ([\phi] + [\psi]) + [\chi]$

$[\phi] \cdot ([\psi] \cdot [\chi]) = [\phi \wedge (\psi \wedge \chi)] = [(\phi \wedge \psi) \wedge \chi]$

$= ([\phi] \cdot [\psi]) \cdot [\chi]$

(B_2) $[\phi] + 0 = [\phi \vee \bot] = [\phi]$

$[\phi] \cdot 1 = [\phi \wedge \top] = [\phi]$

(B_3) $[\phi] + (-[\phi]) = [\phi \vee \neg\phi] = [\top]$

$[\phi] \cdot (-[\phi]) = [\phi \wedge \neg\phi] = [\bot]$

(B_4) $[\phi] + ([\psi] \cdot [\chi]) = [\phi \vee (\psi \wedge \chi)]$

$= [(\phi \vee \psi) \wedge (\phi \vee \chi)] = ([\phi] + [\psi]) \cdot ([\phi] + [\chi])$

$[\phi] \cdot ([\psi] + [\chi]) = [\phi \wedge (\psi \vee \chi)]$

$= [(\phi \wedge \psi) \vee (\phi \wedge \chi)] = ([\phi] \cdot [\psi]) + ([\phi] \cdot [\chi])$

Theorem 5.11 $\vdash_c \phi \iff BA \models \phi \approx \top$

Theorem 5.9 $\models_c \phi \iff Set \models \phi \approx \top$

Stone representation theorem:

Every BA is isomorphic to a Set Algebra

Corollary 5.17 $\vdash_c \phi \Leftrightarrow \models_c \phi$