Laver Trees in the Generalized Baire Space

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Abstract

Our main result is the following: any suitable generalization of Laver forcing to the space κ^κ for uncountable regular κ, necessarily adds a Cohen κ-real. This is a contribution to the study of generalized Baire spaces which answers a question from [1]. We also prove a sligtly more general result for arbitrary tree-like forcings, and also study a related dichotomy property and the ideal and regularity properties generated by this notion of Laver forcing.

1 Introduction

In set theory of the reals, a basic questions is whether a forcing adds *Cohen reals* and/or *dominating reals*. It is well-known that Cohen forcing adds Cohen but not dominating reals and Laver forcing adds dominating but not Cohen reals. In the language of cardinal characteristics of the continuum, this means that an appropriate iteration of Cohen forcing starting from a model of CH yields a model where $\mathfrak{b} < \text{cov}(\mathcal{M})$ and an appropriate iteration of Laver forcing starting from a model of CH yields a model where $\text{cov}(\mathcal{M}) < \mathfrak{b}$.

In recent years, the study of *generalized Baire spaces* has caught the attention of set theorists. Let κ be a regular, uncountable cardinal and consider elements of the spaces κ^κ or 2^κ as "κ-reals". The concepts *dominating κ-real* and *Cohen κ-real*, as well as the cardinal invariants $\mathfrak{b}_\kappa$ and $\text{cov}(\mathcal{M}_\kappa)$, can be naturally generalized to this setting (see Section 2).

It is not hard to see that κ-Cohen forcing does not add dominating κ-reals, so an appropriate iteration of κ-Cohen forcing, starting from a model of GCH, yields a model in which $\mathfrak{b}_\kappa < \text{cov}(\mathcal{M}_\kappa)$. A natural method for the converse direction, i.e., the consistency of $\text{cov}(\mathcal{M}_\kappa) < \mathfrak{b}_\kappa$, would be to iterate a forcing partial order adding dominating κ-reals but not Cohen κ-reals. The authors of [1, p. 36] asked whether a forcing with such a property exists, and in particular, whether some generalization of Laver forcing has this property.

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In this paper, we show that every generalization of Laver forcing necessarily adds a Cohen κ-real. Using a similar technique we can show that, if $\mathbb{P}$ is any (suitably defined) tree-like forcing on the generalized reals, adding a dominating real which is a continuous image of the generic, then this forcing must add a Cohen κ-real.

At this point we should note that, in an earlier version, we claimed to have a proof that *any* $<\kappa$-closed forcing adding dominating κ-reals adds Cohen κ-reals. The proof contained an irreparable mistake, so for the time being this is still an open question, one which the authors consider rather important for the theory of generalized Baire spaces.

On a related note, a model for $\mathrm{cov}(\mathcal{M}_\kappa) < \mathfrak{b}_\kappa$ was recently constructed by Shelah (private communication). However, the method there is to start from a model of $\mathrm{cov}(\mathcal{M}_\kappa) = \mathfrak{b}_\kappa = 2^\kappa > \kappa^+$ and then add a witness to $\mathrm{cov}(\mathcal{M}_\kappa) = \kappa^+$ by a short forcing iteration. It is therefore still open whether an alternative proof exists by using a forcing iteration starting from a model of GCH which adds dominating κ-reals and no Cohen κ-reals.

The main results are presented in Sections 3 and 4. In section 5 we consider a generalization of a classical dichotomy theorem, and in Section 6 we prove an additional result and end with some questions.

2 Preliminaries and definitions

We work in the setting where κ is an uncountable, regular cardinal, and consider the *generalized Baire space* κ^κ with the topology generated by basic open sets of the form $[\sigma] := \{x \in \kappa^\kappa : \sigma \subseteq x\}$ for $\sigma \in \kappa^{<\kappa}$, as well as the *generalized Cantor spac* 2^κ, with the analogous topology.

A standard cardinal arithmetic assumption in this setting is $\kappa^{<\kappa} = \kappa$, which is sufficient to prove many pleasant properties of generalized Baire spaces, e.g., that it has a topology with base of size κ (without this assumption, the overall theory seems to be less coherent). We refer the reader to [4] for a good introduction to generalized Baire spaces, and to [9] for an overview of the current state of the field and a list of open problems.

Definition 2.1. Let $f, g \in \kappa^\kappa$. We say that g *dominates* f, notation $f \leq^* g$, iff $\exists \alpha < \kappa \, \forall i > \alpha \, (f(i) \leq g(i))$.

Definition 2.2. A set $A \subseteq 2^\kappa$ is *nowhere-dense* if for every basic open $[\sigma]$ there exists a basic open $[\tau] \subseteq [\sigma]$ such that $[\tau] \cap A = \varnothing$. A set $A \subseteq 2^\kappa$ is *κ-meager* if it is contained in the union of κ-many nowhere-dense sets. The ideal of κ-meager sets is denoted by $\mathcal{M}_\kappa$. An analogous definition holds for κ^κ as well.

A *tree* in $\kappa^{<\kappa}$ or $2^{<\kappa}$ is a subset closed under initial segments. If T is a tree, we use $[T]$ to denote the set of branches (of length κ) through T, that is $[T] := \{x \in \kappa^\kappa : \forall \alpha \, (x \restriction \alpha \in T)\}$. For $\sigma \in T$ we use the notation $T \uparrow \sigma := \{\tau \in T : \sigma \subseteq \tau \vee \tau \subseteq \sigma\}$. A tree $T \subseteq \kappa^{<\kappa}$ is called *limit-closed*[1] if for any $\subseteq$-increasing sequence $\langle \sigma_i : i < \alpha \rangle$

[1] Alternative terminology used in the literature is "$<\kappa$-closed", and "sequentially closed".

from T of length $\alpha < \kappa$, the limit $\sigma := \bigcup_{\alpha<\kappa} \sigma_\alpha$ is in T. We call a set C *superclosed* if $C = [T]$ for a limit-closed tree T.

Every closed subset of κ^κ is the set of branches through a tree but not necessarily a limit-closed tree, so one could say that being superclosed is a topologically stronger property than being close. We will also need to consider the set of branches of length shorter than κ. For any limit ordinal $\lambda < \kappa$ we use the notation $[T]_\lambda := \{x \in \kappa^\lambda : \forall \alpha < \lambda \ (x{\restriction}\alpha \in T)\}$. Thus T is limit-closed iff $[T]_\lambda \subseteq T$ for all limit ordinals $\lambda < \kappa$.

Definition 2.3. A *Laver tree* is a tree $T \subseteq \omega^{<\omega}$ with the property that such that for every $\sigma \in T$ extending $\mathrm{stem}(T)$, $|\mathrm{Succ}_T(\sigma)| = \omega$. *Laver forcing* $\mathbb{L}$ is the partial order of Laver trees ordered by inclusion.

Laver forcing adds dominating reals while satisfying the so-called *Laver property*, a well-known iterable property implying that no Cohen reals are added. There have been several attempts in the literature to generalize Laver forcing.

Definition 2.4. A *κ-Laver tree* is a tree $T \subseteq \kappa^{<\kappa}$ which is *limit-closed* and such that for every $\sigma \in T$ extending $\mathrm{stem}(T)$, $|\mathrm{Succ}_T(\sigma)| = \kappa$.

In itself, this partial order is not well-suited as a forcing on κ^κ—for example, it is not ω-distributive, see Lemma 6.1. But there have been several attempts to define subtler versions of Laver forcing, for example *club-Laver* $\mathbb{L}_\kappa^{club}$ (see [3]), where the requirement on the trees is strengthened to "$\mathrm{Succ}_T(\sigma)$ contains a club on κ". This is a $<\kappa$-closed forcing adding a dominating κ-real. However, it is also easy to see that it adds a Cohen real: let S be a stationary, co-stationary subset of κ and let $\varphi : \kappa^\kappa \to 2^\kappa$ be given by $\varphi(x)(\alpha) = 1 \iff x(\alpha) \in S$. If x_G is the generic κ-real added by $\mathbb{L}_\kappa^{club}$, then $\varphi(x_G)$ is a Cohen κ-real.

Nevertheless, one could consider other ways of defining a forcing notion $\mathbb{P} \subseteq \mathbb{L}_\kappa$, by *carefully selecting* special types of trees, with the hope that this forcing would not add Cohen κ-reals. Our result, Theorem 3.5, says that such an approach cannot work. A stronger result, Theorem 3.7, shows the same for a slightly wider class of partial orders, and by some generalizations in Section 4, the same conclusion follows for any tree-like forcing adding a dominating κ-real which is a continuous image of the generic κ-real.

3 The Supremum Game

The main ingredient of all the proofs is the following game.

Definition 3.1. Let $S \subseteq \kappa$. The *supremum game* $G^s(S)$ is played by two players, for ω moves, as follows:

I	A_0		A_1		$\dots$	
II		β_0		β_1		$\dots$

where A_n are subsets of κ with $|A_n| = \kappa$ and $\beta_n \in A_n$. Player II wins the game iff $\sup_n \beta_n \in S$.

Lemma 3.2. *Let S be a stationary subset of* $\mathrm{Cof}_\omega(\kappa) = \{\alpha < \kappa \,:\, \mathrm{cf}(\alpha) = \omega\}$. *Then Player I does not have a winning strategy in* $G^s(S)$.

Proof. Suppose, towards contradiction, that I had a strategy σ in $G^s(S)$. Let θ be sufficiently large and let $M \prec \mathcal{H}(\theta)$ be an elementary submodel such that $\sigma \in M$, $|M| < \kappa$, and $\delta := \sup(M \cap \kappa) \in S$. Note that we can always do that, because the set $\mathcal{M} := \{\sup(M \cap \kappa) \,:\, M \prec \mathcal{H}_\theta,\ \sigma \in M,\ |M| < \kappa\}$ contains a club.

Fix a sequence $\langle \gamma_n \,:\, n < \omega \rangle$ cofinal in δ, such that every $\gamma_n \in M$ (but the sequence itself is not). Inductively, Player II will construct a run of the game according to strategy σ.

At each step n, inductively assume A_k and β_k for $k < n$ have been fixed according to the rules of the game and the strategy σ, and assume they are all in M. Let $A_n := \sigma(A_0, \beta_0, \ldots, A_{n-1}, \beta_{n-1})$. Since the finite sequence was in M and the strategy σ is in M, A_n is also in M. Furthermore, since $|A_n| = \kappa$, the following statement is true:

$$\exists \beta > \gamma_n \; (\beta \in A_n)$$

This statement holds in $\mathcal{H}(\theta)$, so by elementarity, it also holds in M. Thus, there exists $\beta_n \in M$ with $\beta_n > \gamma_n$ and $\beta_n \in A_n$. This completes the construction.

We have produced a sequence $\langle \beta_n \,:\, n < \omega \rangle$ with $\beta_n \in M$ for all n. But clearly $\sup_n \beta_n = \sup_n \gamma_n = \delta \in S$, so Player II wins this game contrary to assumption. $\square$

Definition 3.3. A *short κ-Laver tree* is a tree $L \subseteq \kappa^{<\omega}$ (i.e., depth ω), such that for all $\sigma \in L$ extending $\mathrm{stem}(L)$ we have $|\mathrm{Succ}_L(\sigma)| = \kappa$.

Corollary 3.4. *Let $S \subseteq \kappa$ be a stationary subset of* $\mathrm{Cof}_\omega(\kappa)$. *For every short κ-Laver tree L there exists a branch $\eta \in [L]_\omega$ such that* $\sup_n \eta(n) \in S$.

Proof. The short κ-Laver tree L induces a strategy σ_L for Player I in the supremum game:

$$\sigma_L(A_0, \beta_0, \ldots, A_n, \beta_n) := \mathrm{Succ}_L(\mathrm{stem}(L)^\frown \langle \beta_0, \ldots, \beta_n \rangle).$$

Whenever $\langle A_0, \beta_0, A_1, \beta_1, \ldots \rangle$ is a run of the game according to σ_L, $\mathrm{stem}(L)^\frown \langle \beta_0, \beta_1, \ldots \rangle$ is an element of $[L]_\omega$.

By Lemma 3.2, there exists a run of the game in which Player I follows σ_L but Player II wins. This yields a branch $\eta \in [L]_\omega$ such that $\sup_n \eta(n) \in S$. $\square$

With this, we immediately obtain our first application.

Theorem 3.5. *Let $\mathbb{P} \subseteq \mathbb{L}_\kappa$ be any subset closed under the following condition: if $T \in \mathbb{P}$ and $\sigma \in T$, then $T \uparrow \sigma \in \mathbb{P}$. Then $\mathbb{P}$ adds a Cohen κ-real.*

Proof. We will use the following notation: if $T \in \kappa^{<\kappa}$ is a tree and $\sigma \in T$, then $T{\restriction}^\omega \sigma := \{\tau \in \kappa^{<\omega} : \sigma^\frown \tau \in T\}$. Note that if T is a κ-Laver tree, then for every $\sigma \in T$ extending $\mathrm{stem}(T)$, $T{\restriction}^\omega \sigma$ is a short κ-Laver tree, and moreover $[T{\restriction}^\omega \sigma]_\omega \subseteq T$.

Let $S_0 \cup S_1$ be a stationary/co-stationary partition of $\mathrm{Cof}_\omega(\kappa)$ and consider the mapping $\varphi : \kappa^\kappa \to 2^\kappa$ defined by

$$\varphi(x)(\alpha) = 1 \quad :\Leftrightarrow \quad \sup\{x(\omega{\cdot}\alpha + n) \ : \ n < \omega\} \in S_1.$$

In other words, partition x into κ-many blocks of length ω, and map each piece to 0 or 1 depending on whether its supremum lies in S_0 or S_1. We claim that if x_G is $\mathbb{P}$-generic then $\varphi(x_G)$ is κ-Cohen-generic.

We use $\tilde{\varphi} : \kappa^{<\kappa} \to 2^{<\kappa}$ to denote the approximations of φ (defined as above). Let $T \in \mathbb{P}$ be given and let D be open dense in κ-Cohen forcing. Let $\sigma := \mathrm{stem}(T)$, w.l.o.g. $\mathrm{len}(\sigma)$ is a limit ordinal. Let $t \in D$ extend $\tilde{\varphi}(\sigma)$. Suppose $\tilde{\varphi}(\sigma)^\frown \langle 0 \rangle \subseteq t$. By Corollary 3.4 there is $\eta \in [T{\restriction}^\omega \sigma]_\omega$ such that $\sup_n \eta(n) \in S_0$. If, instead, we have $\tilde{\varphi}(\sigma)^\frown \langle 1 \rangle \subseteq t$, we can again apply Corollary 3.4 and find a branch $\mu \in [T{\restriction}^\omega \sigma]_\omega$ such that $\sup_n \mu(n) \in S_1$. Note that, since T is limit-closed, $\sigma^\frown \eta$ resp. $\sigma^\frown \mu$ are elements of T. Now proceed analogously until reaching τ, such that $\tilde{\varphi}(\tau) = t$. By assumption $T \uparrow \tau \in \mathbb{P}$, and now clearly $T \uparrow \tau \Vdash t \subseteq \varphi(\dot{x}_G)$. Thus $\varphi(x_G)$ is a Cohen κ-real. □

By a slight modification of the above result, we can obtain a stronger theorem.

Definition 3.6. A tree $T \subseteq \kappa^{<\kappa}$ is called a *pseudo-κ-Laver tree* if it is limit-closed and has the following property: $\forall \sigma \in T \ \exists \tau \in T$ s.t. $\sigma \subseteq \tau$ and $T{\restriction}_\omega \tau$ is a short κ-Laver tree. We use $\mathrm{p}\mathbb{L}_\kappa$ to denote the partial order of pseudo-κ-Laver trees ordered by inclusion.

Theorem 3.7. *Let $\mathbb{P} \subseteq \mathrm{p}\mathbb{L}_\kappa$ be any subset closed under the following condition: if $T \in \mathbb{P}$ and $\sigma \in T$, then $T \uparrow \sigma \in \mathbb{P}$. Then $\mathbb{P}$ adds a Cohen κ-real.*

Proof. The method is similar to the above, however here, we let $\{S_t \ : \ t \in \kappa^{<\kappa}\}$ be a disjoint partition of $\mathrm{Cof}_\omega(\kappa)$ into stationary sets, indexed by $\kappa^{<\kappa}$, which is possible by the assumption $\kappa^{<\kappa} = \kappa$. Define the mapping $\pi : \kappa^\kappa \to 2^\kappa$ by $\pi(x) := t_0{}^\frown t_1{}^\frown t_2{}^\frown \dots$, where for all $\alpha < \kappa$, t_α is such that $\sup\{x(\alpha \cdot \omega + n) \ : \ n < \omega\} \in S_{t_\alpha}$. We also use $\tilde{\pi}$ to denote the same operation but from $\kappa^{<\kappa}$ to $2^{<\kappa}$.

Let x_G be the $\mathbb{P}$-generic κ-real, and we show that $\pi(x_G)$ is κ-Cohen. Llet D be κ-Cohen dense, and let $T \in \mathbb{P}$. Find $\sigma \in T$ such that $T{\restriction}^\omega \sigma$ is a short κ-Laver tree. Let $t \in D$ be such that $\tilde{\pi}(\sigma) \subseteq t$. Let u be such that $\tilde{\pi}(\sigma)^\frown u = t$. By Corollary 3.4 there is $\eta \in [T{\restriction}^\omega \sigma]_\omega$ such that $\sup_n \eta(n) \in S_u$. It follows that $\tilde{\pi}(\sigma^\frown \eta) = \tilde{\pi}(\sigma)^\frown u = t$. Therefore $T \uparrow (\sigma^\frown \eta) \Vdash t \subseteq \pi(\dot{x}_G)$. □

4 Tree-forcings and Cohen reals

In this section we generalize the previous results to include a wider class of forcings.

Definition 4.1. A forcing partial order $\mathbb{P}$ is called *tree-like*, if its conditions are limit-closed trees $T \subseteq \kappa^{<\kappa}$, and for every $T \in \mathbb{P}$ and $\sigma \in T$, the restriction $T \uparrow \sigma \in \mathbb{P}$.

Tree-like forcings are $<\kappa$-closed by definition, in particular, they do not change κ or $\kappa^{<\kappa}$ (assuming $\kappa^{<\kappa} = \kappa$). The main result in this section is:

Theorem 4.2. *Let $\mathbb{P}$ be a tree-like forcing adding a dominating κ-real which is the image of the generic κ-real under a continuous function coded in the ground model. Then $\mathbb{P}$ adds a Cohen κ-real.*

We start with the following strengthening of the concept of *dominting reals*, which has also been studied in the classical context, e.g., in [5] (see also [10, 8]).

Definition 4.3. For $f : \kappa^{<\kappa} \to \kappa$ and $x \in \kappa^\kappa$, we say that x *strongly dominates* f if $\exists \alpha_0 \, \forall \alpha > \alpha_0 \, (x(\alpha) > f(x{\restriction}\alpha))$. If M is a model of set theory, then x is called *strongly dominating over* M if for all $f : \kappa^{<\kappa} \to \kappa$ with $f \in M$, x strongly dominates f.

Clearly, if x is strongly dominating, then it is also dominating. The converse is false in general, e.g., let d be dominating over M and let x be defined by $x(\alpha) := d(\alpha)$ for odd α and $x(\alpha) := d(\alpha + 1)$ for even and limit α. Then d is dominating but not strongly dominating. However, the following is true:

Lemma 4.4. *Let M be a model of set theory such that $M \models \kappa^{<\kappa} = \kappa$. Then, if there is a dominating real over M there is also a strongly dominating real over M. Moreover, the map from dominating to strongly dominating reals is given by a continuous function.*

Proof. In M, let $\{\sigma_i : i < \kappa\}$ enumerate $\kappa^{<\kappa}$, writing $\ulcorner\sigma\urcorner = i$ iff $\sigma = \sigma_i$. If d is dominating over M, define inductively $x(\alpha) := d(\ulcorner x{\restriction}\alpha\urcorner)$. If $f : \kappa^{<\kappa} \to \kappa$ is in M, then z defined by $z(i) := f(\sigma_i)$ is also in M (since the enumeration is in M), hence, for all but $<\kappa$-many i we have $z(i) < d(i)$. Hence, for all but $<\kappa$-many α we have $x(\alpha) = d(\ulcorner x{\restriction}\alpha\urcorner) > z(\ulcorner x{\restriction}\alpha\urcorner) = f(x{\restriction}\alpha)$.

If the enumeration used above is such that $\sigma \subseteq \tau$ implies $\ulcorner\sigma\urcorner \leq \ulcorner\tau\urcorner$, then the mapping x defined inductively from d is easily seen to be continuous. □

Definition 4.5. Let $\mathbb{P}$ be any forcing notion, let $\dot{x}$ be a name, and let $p \in \mathbb{P}$ be such that $p \Vdash \dot{x} \in \kappa^\kappa$. Then the *interpretation tree* of $\dot{x}$ below p is:

$$\mathfrak{T}_{\dot{x},p} = \{\sigma \in \kappa^{<\kappa} \ : \ \exists q \leq p \ (q \Vdash \sigma \subseteq \dot{x})\}$$

It is immediate that $\mathfrak{T}_{\dot{x},p}$ is a tree in the ground model. However, in general it need not be a limit-closed tree. We first prove a Lemma showing that if $\dot{d}$ is a strongly dominating real, then the interpretation tree contains a short κ-Laver tree (in the sense of Definition 3.3), i.e., a tree of height ω). This in itself is not sufficient because due to the failure of limit-closure (so the set of branches through the short Laver tree need not be a subset of the interpretation tree). If, in addition, one can prove that the interpretation tree is limit-closed—in fact, it suffices to be σ-closed—then the methods from the previous section can be applied.

Lemma 4.6. *Let $\mathbb{P}$ be any $<\kappa$-closed forcing notion, and suppose $p \Vdash$ "$\dot{d}$ is a strongly dominating real". Then there exists $\tau \in \mathfrak{T}_{\dot{d},p}$ such that $\mathfrak{T}_{\dot{d},p}{\restriction}^\omega\sigma$ contains a short κ-Laver tree.*

Proof. We use a version of the game from [5] (which will be used again in Section 5). Given $A \subseteq \kappa^\omega$ let $G^\star(A)$ be the game defined by:

$$\begin{array}{c||ccccc} \text{I} & \alpha_0 & & \alpha_1 & & \dots \\ \hline \text{II} & & \beta_0 & & \beta_1 & & \dots \end{array}$$

where $\alpha_n, \beta_n < \kappa$, $\alpha_n < \beta_n$ for all $n < \omega$, and Player II wins iff $\langle \beta_n : n < \omega \rangle \in A$.

Note that this game is only played for ω-many moves. It is easy to see that if Player II wins $G^\star(A)$ then there exists a short κ-Laver tree L such that $[L]_\omega \subseteq A$. Also it is clear that if A is closed (in the topology on κ^ω) then $G^\star(A)$ is determined. We have two cases:

- Case 1: for some $\tau \in \mathfrak{T}_{\dot{d},p}$, Player II has a winning strategy in $G^\star([\mathfrak{T}_{\dot{d},p}{\restriction}^\omega\tau]_\omega)$. In this case we are done, since there is a short κ-Laver tree L with $[L]_\omega \subseteq [\mathfrak{T}_{\dot{d},p}{\restriction}^\omega\tau]_\omega$, hence $L \subseteq \mathfrak{T}_{\dot{d},p}{\restriction}^\omega\tau$.
- Case 2: for all $\tau \in \mathfrak{T}_{\dot{d},p}$, Player II does not have a winning strategy in $G^\star([\mathfrak{T}_{\dot{d},p}{\restriction}^\omega\tau]_\omega)$. Then for all τ, Player I has a winning strategy, say, f_τ. By glueing together these winning strategies we define a function $f : \kappa^{<\kappa} \to \kappa$, as follows: for every $t \in \mathfrak{T}_{\dot{d},p}$ let $\tau \subseteq t$ be the maximal node of limit length, let u be such that $t = \tau^\frown u$, and let $f(t) := f_\tau(u)$ (where, as usual, f_τ takes as input only the opponents' moves).

  Since p forces $\dot{d}$ to be strongly dominating, in particular

  $$p \Vdash \exists\beta\, \forall\alpha > \beta\ (\dot{d}(\alpha) > f(\dot{d}{\restriction}\alpha)).$$

  Let $q \leq p$ decide β, i.e., let γ be a (wlog. limit) ordinal such that

  $$q \Vdash \forall\alpha > \gamma\ (\dot{d}(\alpha) > f(\dot{d}{\restriction}\alpha)).$$

  Now let $r \leq q$ decide $\dot{x}{\restriction}(\gamma{+}\omega)$, which is possible since $\mathbb{P}$ is $<\kappa$-closed. Say that r decides $\dot{d}{\restriction}\gamma = \tau$ and $\dot{d}{\restriction}(\gamma{+}\omega) = \tau^\frown z$. Then $z \in \kappa^\omega$ and $\tau^\frown z \in \mathfrak{T}_{\dot{d},p}$. Moreover, by what was forced by q, for every n we have $z(n) > f(\tau^\frown(z{\restriction}n)) = f_\tau(z{\restriction}n)$. Therefore, $z \in [\mathfrak{T}_{\dot{d},p}{\restriction}^\omega\tau]_\omega$ satisfies the winning conditions for Player II in the game $G^\star([T{\restriction}^\omega\tau]_\omega)$, contradicting the assumption that f_τ was a winning strategy for Player I. □

Lemma 4.7. *Suppose $\mathbb{P}$ is a $<\kappa$-closed forcing, $p \Vdash$ ($\dot{d}$ is strongly dominating), and every interpretation trees of $\dot{d}$ is limit-closed, or even just closed under ω-sequences. Then $\mathbb{P}$ adds a Cohen κ-real.*

Proof. Assume $p \Vdash$ ($\dot{d}$ is strongly dominating), recall the mapping π from Theorem 3.7, and we will show that $p \Vdash \pi(\dot{d}))$ is κ-Cohen. Let $q \leq p$ be arbitrary and let D be κ-Cohen dense. Since q also forces that $\dot{d}$ is strongly dominating, by Lemma 4.6 there is $\tau \in \mathfrak{T}_{\dot{d},q}$ such that $\mathfrak{T}_{\dot{d},q}{\restriction}^\omega\tau$ contains a short κ-Laver tree L. As in the proof of Theorem 3.7 let t extend $\tilde{\pi}(\tau)$ such that $t \in D$, and let u be such that $t = \tilde{\pi}(\tau)^\frown u$. Then find $\eta \in [L]_\omega$ such that $\tilde{\pi}(\tau^\frown\eta) = \tilde{\pi}(\tau)^\frown u = t$. Since, by assumption, $\mathfrak{T}_{\dot{d},p}$ is closed under ω-sequences, we know that in fact $\tau^\frown\eta \in \mathfrak{T}_{\dot{d},q}$. So there exists $r \leq q$ such that $r \Vdash \tau^\frown\eta \subseteq \dot{d}$, and therefore $r \Vdash \tilde{\pi}(\tau^\frown\eta) = t \subseteq \pi(\dot{d})$. Hence p forces that $\pi(\dot{d})$ is κ-Cohen. □

To complete the proof of the main result, it suffices to show the following:

Lemma 4.8. *Let $\mathbb{P}$ be a tree-like forcing, $\dot{y}$ any name for a κ-real, g a ground-model continuous function and $T \in \mathbb{P}$ a condition such that $T \Vdash \dot{y} = g(\dot{x}_G)$. Then $\mathfrak{T}_{\dot{y},T}$ is limit-closed.*

Proof. This follows by adapting the game from above in a similar way to Solovay's "unfolding trick", see [11]. The details are left to the reader. □

This completes the proof of Theorem 4.2. Note that if $\dot{d}$ is a name for a dominating real, then the strongly dominating real is given the image of $\dot{d}$ under a ground-model coded continuous function, so it is enough for either the dominating or the strongly dominating real to be the continuous image of the generic.

5 The generalized Laver dichotomy

In this section, we consider a generalization of the dominating dichotomy. In Remark 5.3 we will explain how this is connected to the previous results.

Definition 5.1. A collection $X \subseteq \kappa^\kappa$ is a *strongly dominating family* if for every $f : \kappa^{<\kappa} \to \kappa$ there exists $x \in X$ which strongly dominates f. $\mathcal{D}_\kappa$ denotes the ideal of non-strongly dominating families.

For $\kappa = \omega$, this ideal $\mathcal{D}_\omega = \mathcal{D}$ is the well-known *non-strongly-dominating ideal*, introduced in [5] and independently in [14], and studied among others in [2]. If T is a Laver tree then $[T] \notin \mathcal{D}$ and by [5, Lemma 2.3], any analytic set $A \subseteq \omega^\omega$ is either in $\mathcal{D}$ or contains $[T]$ for some Laver tree T. This, in turn, implies that classical Laver forcing densely embeds into the algebra of Borel subsets of ω^ω modulo $\mathcal{D}$.

Dichotomies as above are common in classical descriptive set theory, the most notable example being the perfect set property and the closely related K_σ-dichomoty ([7]) all of which are false for arbitrary sets of reals but true for analytic sets. Interest in generalizing such dichotomies to the κ^κ-context was recently spurred by a result of Schlicht [13] showing that the generalized perfect set property for generalized projective sets is consistent, and Motto Ros-Lücke-Schlicht [12] showing that the generalized Hurewicz dichotomy for generalized projective sets is consistent. Thus, it might initially come as a surprise that the generalized Laver dichotomy fails for closed sets, provably in ZFC.

Lemma 5.2. *There is a closed subset of κ^κ which is neither in $\mathcal{D}_\kappa$ nor contains the branches of a generalized Laver tree.*

Proof. Let φ be as in the proof of Theorem 3.5. Let z be the constant 0 function (or any other element of 2^ω). We show that $C := \varphi^{-1}\{z\}$ is a counterexample to the dichotomy. Given any $T \in \mathbb{L}_\kappa$, we can easily find $x \in [T]$ such that $\varphi(x) \neq z$, therefore $[T] \not\subseteq C$. We claim that C is strongly dominating. Let $f : \kappa^{<\kappa} \to \kappa$ be given. Let

$$T_f := \{\sigma \in \kappa^{<\kappa} \ : \ \forall \beta < \mathrm{len}(\sigma) \ (\sigma(\beta) > f(\sigma \restriction \beta))\}.$$

It is not hard to see that T_f is a generalized Laver tree and $\mathrm{stem}(T_f) = \varnothing$. Therefore, we can find an $x \in [T]$ such that $\varphi(x) = z$. But then x strongly dominates f and $x \in C$, completing the argument. □

Remark 5.3. The relevance of this lemma is that it explains why our results to not (as one may initially assume) yield a ZFC-proof of $\mathfrak{b}_\kappa \leq \mathrm{cov}(\mathcal{M}_\kappa)$. Indeed, it is not hard to verify that $\mathrm{cov}(\mathcal{D}_\kappa) = \mathfrak{b}_\kappa$ and that if $X \in \mathcal{M}_\kappa$ then $\varphi^{-1}[X]$ does not contain a κ-Laver tree. Thus, if the dichotomy would hold for generalized Borel (or just F_σ) sets then one could have concluded $\mathfrak{b}_\kappa \leq \mathrm{cov}(\mathcal{M}_\kappa)$.

One could wonder whether there is *any* dichotomy for the ideal $\mathcal{D}_\kappa$, i.e., whether there is any collection $\mathbb{P}$ of limit-closed trees, such that for every $T \in \mathbb{P}$, $[T] \notin \mathcal{D}_\kappa$, and every analytic (or at least closed) set not in $\mathcal{D}_\kappa$ contains $[T]$ for some $T \in \mathbb{P}$. In fact, this is not the case either.

Lemma 5.4. *Let T be a tree such that $[T]$ is strongly dominating. Then there exists $s \in T$ such that $T \restriction^\omega s$ contains a short κ-Laver tree.*

Proof. Generalizing the game argument from [5], given $A \subseteq \kappa^\omega$ let $G^{\mathcal{D}_\kappa}(A)$ be the game defined by:

I	α_0		α_1		$\dots$	
II		β_0		β_1		$\dots$

where $\alpha_n, \beta_n < \kappa$, $\alpha_n < \beta_n$ for all n, and Player II wins iff $\langle \beta_n \ : \ n < \omega \rangle \in A$.

It is easy to see that if Player II wins $G^{\mathcal{D}_\kappa}(A)$ then there exists a short κ-Laver tree L such that $[L]_\omega \subseteq A$. Also it is clear that if A is closed (in the topology on κ^ω) then $G^{\mathcal{D}_\kappa}(A)$ is determined.

Suppose, towards contradiction, that there is no $s \in T$ such that $T \restriction^\omega s$ contains a short κ-Laver tree. Then Player II does not have a winning strategy in $G^{\mathcal{D}_\kappa}([T \restriction^\omega s]_\omega)$ for any $s \in T$, and therefore Player I has a winning strategy, call it σ_s. Define $f : \kappa^{<\kappa} \to \kappa$ as follows: for every $t \in T$, let $s \subseteq t$ be the maximal node of limit length, let u be such that $t = s^\frown u$, and define $f(t) := \sigma_s(u)$. Since $[T]$ is strongly dominating there is $x \in [T]$ and α such that $x(\beta) > f(x \restriction \beta)$ for all $\beta > \alpha$. In particular, there is $s \subseteq x$, of limit length, such that $x(|s| + n) > f(x \restriction (|s| + n))$ for all n. Letting $z \in \kappa^\omega$ be such that $s^\frown z = x \restriction (|s| + \omega)$, we see that $z(n) > f(s^\frown z \restriction n) = \sigma_s(z \restriction n)$, for every n. Therefore, $z \in [T \restriction^\omega s]_\omega$ satisfies the winning conditions for Player II in the game $G^{\mathcal{D}_\kappa}([T \restriction^\omega s]_\omega)$, contradicting the assumption that σ_s was a winning strategy for Player I. □

Corollary 5.5. *There exists a closed strongly dominating set without a super-closed strongly dominating subset.*

Proof. If the set $C := \varphi^{-1}\{z\}$ from above contains a strongly dominating $[T]$ for some limit-closed T, then by Lemma 5.4 there is $s \in T$ such that $T{\restriction}^{\omega} s$ contains a short κ-Laver tree L. By Corollary 3.4 there is $\eta \in [L]_\omega$ such that $\sup_n \eta(n) \in S_1$, and by limit-closure, there is $x \in [T]$ such that $s^\frown \eta \subseteq x$. But then $\varphi(x)$ contains a "1" and thus is not equal to z, the constant 0-function. $\square$

6 Additional results and questions

Lemma 6.1. *$\mathbb{L}_\kappa$ is not ω-distributive.*

Proof. Consider the following game:

$$\begin{array}{c||cccccc}
\text{I} & T_0 & & T_2 & & \dots \\
\hline
\text{II} & & T_1 & & T_3 &
\end{array}$$

where $T_n \in \mathbb{L}_\kappa$, $T_{n+1} \leq T_n$ for all n, and Player II wins if there exists $T \in \mathbb{L}_\kappa$ such that $T \leq T_n$ for all n.

A forcing is not ω-distributive iff Player I has a winning strategy in the above game [6, Lemma 30.23]. We define the strategy as follows: $T_0 \in \mathbb{L}_\kappa$ is arbitrary. Let T_n be the last move of Player II. For every splitting node $\sigma \in T_n$, let $\xi_{\sigma,n}$ be the increasing enumeration of $\mathrm{Succ}_{T_n}(\sigma)$. Let T_{n+1} be obtained by inductively pruning T_n so that $\mathrm{Succ}_{T_{n+1}}(\sigma) = \mathrm{Succ}_{T_n}(\sigma) \setminus \{\alpha : \xi_{\sigma,n}(\alpha) \text{ is a limit ordinal or } 0\}$. Also, $\xi_{\sigma,n+1}$ is an enumeration of $\mathrm{Succ}_{T_{n+1}}(\sigma)$.

Notice that if σ is splitting in T_n and in T_{n+1}, for n odd, then for every $\alpha \in \mathrm{Succ}_{T_n}(\sigma) \cap \mathrm{Succ}_{T_{n+1}}(\sigma)$ we have $\xi_{\sigma,n}(\alpha) > \xi_{\sigma,n+1}(\alpha)$. If the same thing holds for n even, then $\xi_{\sigma,n}(\alpha) \geq \xi_{\sigma,n+1}(\alpha)$.

It follows that if Player I plays according to this strategy, then $T := \bigcap_n T_n$ does not contain any splitting node, since if $\sigma \in T$ was splitting, it would be splitting in every T_n, and then for $\alpha \in \mathrm{Succ}_T(\sigma)$ we would have a strictly decreasing sequence of ordinals $\{\xi_{\sigma,n}(\alpha) : n \text{ odd}\}$. $\square$

Question 6.2. *Is it true that every $<\kappa$-distributive forcing notion adding dominating κ-reals adds Cohen κ-reals?*

Question 6.3. *Does $\mathbb{L}_\kappa$ preserve or collapse κ?*

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